Research Design - - Topic 20
Multivariate Analysis of Variance

1

Purpose and General Rationale

The purpose of multivariate analysis of variance in any of its forms is to perform an analysis of variance when there is more than one dependent variable. A multivariate analysis of variance can be performed on a single factor completely randomized design (see Gardner & Tremblay, 2007), a completely randomized factorial design, a single factor repeated measures design, a split plot factorial design, a randomized blocks factorial design, etc. It is basically the equivalent of the corresponding univariate analysis of variance, but it has more than one dependent variable for each subject. The objective is to determine whether the main effects (and interactions where appropriate) are significant considering the variables as a set. This section will consider the single factor completely randomized design.

2

The general rationale is comparable to that for principal components analysis. That is, we can form a weighted aggregate that accounts for as much of the between groups variation relative to the within groups variation as possible. Then, once we have done this we can form another weighted aggregate to account for as much of the relative variation as possible, and we can continue this until we have accounted for all of the relative variation. For the single factor design, we will find that the number of weighted aggregates that can be formed is equal to the number of groups minus 1 (i.e., the degrees of freedom associated with the univariate effect), or the number of dependent variables, whichever is less.

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Multivariate analysis of variance (MANOVA) has been used for two (seemingly contradictory) reasons.

(1.) To control Type I error.

Since only one test of significance is made, the Type I error for the collection of tests is set at α. Thus by performing MANOVA on a series of dependent variables, the Type I error experimentwise is equal to α (but see Slide 12).

(2.) To maximize power.

MANOVA maximally weights the various dependent measures to obtain the greatest differentiation among the groups. Thus, assuming Ho false, MANOVA is most likely to detect the difference.

An additional advantage is that MANOVA might identify an effect due to a pattern of means for specific variables that would not be detected if the variables were analyzed one at a time.

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Basic Mathematics

The weighted aggregate is:

$$L_{ai} = w_1 X_{1i} + w_2 X_{2i} + \ldots$$

where:

L_{ai} = the aggregate score for an individual in group *a*
$w_1, w_2, \ldots$ = the weights for variables 1, 2, …
$X_{1i}, X_{2i}, \ldots$ = the scores for variables 1,2, etc…in each group

The weights are determined such that the Sum of Squares Between Groups divided by the Sum of Squares Within Groups is as large as possible if one had done a single factor analysis of variance on the L_{ai} scores . The weights are computed using the notion of the Determinantal Equation.

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The Determinantal Equation is formed on the matrix equivalent of the Sum of Squares Between groups (SSB) divided by the Sum of Squares within groups (SSW). This is a matrix formed by multiplying the inverse of the SSW matrix by the SSB matrix. This will produce eigenvalues which account for the largest amount of relative variation. That is:

$$\left| SSW^{-1}SSB - \lambda I \right| = 0$$

$$[SSW^{-1}SSB - \lambda_p I][W] = 0$$

With the side condition that $W^T MS_W W = 1$

this produces a unique solution of the weights such that the MS_{within} for each set of L scores (i.e., each Discriminant Function) = 1.

The eigenvalue is the ratio of 2 Sums of Squares of the aggregates

$$\lambda = \frac{SSB_L}{SSW_L}$$

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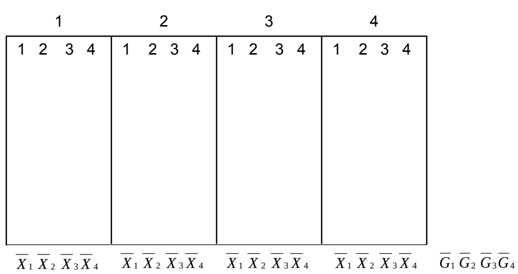
Schematic for a Single Factor Design



In this example, there are a = 4 groups, with p = 4 variables for each subject in each group. Note we can compute a mean for each variable for each group, and a grand mean for each variable. We could also compute a set of weighted aggregate scores for each individual as described above.

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An Example Using SPSS GLM Multivariate

To perform a single factor multivariate analysis of variance, use SPSS GLM Multivariate, input the data for the variables, and click on the operations desired. The next slide shows the input data. The example to be discussed produced the following Syntax file.

```
GLM
  x1 x2 x3 x4  BY a
  /METHOD = SSTYPE(3)
  /INTERCEPT = INCLUDE
  /EMMEANS = TABLES(a) COMPARE ADJ(BONFERRONI)
  /PRINT = ETASQ OPOWER HOMOGENEITY
  /CRITERIA = ALPHA(.05)
  /DESIGN = a .
```

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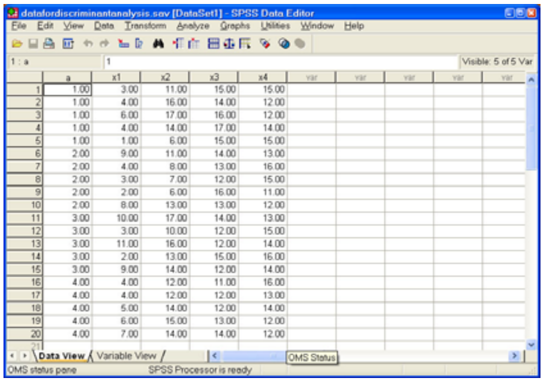
SPSS Data input for GLM Multivariate a = Groups, X1-X4 = Variables



	a	x1	x2	x3	x4
1	1.00	3.00	11.00	15.00	15.00
2	1.00	4.00	16.00	14.00	12.00
3	1.00	6.00	17.00	16.00	12.00
4	1.00	4.00	14.00	17.00	14.00
5	1.00	1.00	6.00	15.00	15.00
6	2.00	9.00	11.00	14.00	13.00
7	2.00	4.00	8.00	13.00	16.00
8	2.00	3.00	7.00	12.00	15.00
9	2.00	2.00	6.00	16.00	11.00
10	2.00	8.00	13.00	13.00	12.00
11	3.00	10.00	17.00	14.00	13.00
12	3.00	3.00	10.00	12.00	15.00
13	3.00	11.00	16.00	12.00	14.00
14	3.00	2.00	13.00	15.00	16.00
15	3.00	9.00	14.00	12.00	14.00
16	4.00	4.00	12.00	11.00	16.00
17	4.00	4.00	12.00	12.00	13.00
18	4.00	5.00	14.00	12.00	14.00
19	4.00	6.00	15.00	13.00	12.00
20	4.00	7.00	14.00	14.00	12.00

Multivariate Output

Multivariate Tests[d]

Effect		Value	F	Hypothesis df	Error df	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[a]
Intercept	Pillai's Trace	.998	1902.588[b]	4.000	13.000	.000	.998	7610.353	1.000
	Wilks' Lambda	.002	1902.588[b]	4.000	13.000	.000	.998	7610.353	1.000
	Hotelling's Trace	585.412	1902.588[b]	4.000	13.000	.000	.998	7610.353	1.000
	Roy's Largest Root	585.412	1902.588[b]	4.000	13.000	.000	.998	7610.353	1.000
a	Pillai's Trace	1.396	3.266	12.000	45.000	.002	.465	39.187	.982
	Wilks' Lambda	.141	3.170	12.000	34.686	.004	.479	31.949	.932
	Hotelling's Trace	2.911	2.830	12.000	35.000	.008	.492	33.962	.949
	Roy's Largest Root	1.472	5.522[c]	4.000	15.000	.006	.596	22.087	.913

a. Computed using alpha = .05

b. Exact statistic

c. The statistic is an upper bound on F that yields a lower bound on the significance level.

d. Design: Intercept+a

There are four output statistics. Generally, the Pillais statistic is recommended. Using Pillai's trace, the effect for Group is significant, P' = 1.396, F(12, 45) = 3.276, p < .002, suggesting that considering the variables as a set of four, there is evidence for an effect of group.

The null hypothesis is:

$$H_o \quad \begin{bmatrix} \mu_{11} \\ \mu_{21} \\ \mu_{31} \\ \mu_{41} \end{bmatrix} = \begin{bmatrix} \mu_{12} \\ \mu_{22} \\ \mu_{32} \\ \mu_{42} \end{bmatrix} = \begin{bmatrix} \mu_{13} \\ \mu_{23} \\ \mu_{33} \\ \mu_{43} \end{bmatrix} = \begin{bmatrix} \mu_{14} \\ \mu_{24} \\ \mu_{34} \\ \mu_{44} \end{bmatrix}$$

It states that:

The vector of 4 population means is the same across all 4 populations. Note, this doesn't mean that $\mu_{11} = \mu_{21} = \mu_{31}$, etc., but instead that $\mu_{11} = \mu_{12}, = \ldots = \mu_{14}$, and $\mu_{21} = \mu_{22} = \ldots = \mu_{24}$

Follow-up Tests

1. Evaluate the univariate analyses of variance with α = .05. If so, do follow up tests of means as for a single dependent variable. This is the most common procedure that is used. For example, Kieffer,Reese & Thompson (2001), report that more than 70% of published articles use this approach even though it has been shown to depend on a misconception of Type I error (see, Huberty & Petoskey, 2001).

2. Evaluate the univariate analyses of variance with α = .05/p where *p* = the number of dependent variables. This approach is recommended by Tabachnick & Fidell (2001), though it seems then that the multivariate analysis is unnecessary (cf., Huberty & Morris, 1989). Marascuilo and Levin (1983) recommend post hoc contrasts with α = .05/*p.*

3. Consider the treatment conditions two-at-a-time and perform Hotelling's T^2 for each pair, with α = .15/m, where m = the number of such tests performed (i.e., a(a-1)/2). Stevens (1996) recommends doing follow-up pairwise contrasts with α = .05.

4. Focus attention on the aggregate scores. These are known as discriminant functions and/or canonical variates, and though the weights are not output from SPSS GLM Multivariate, the weights and all other information needed to interpret these scores are output by SPSS Discriminant. Furthermore, if p < (a-1), you can perform a univariate analysis of variance on the discriminant functions, interpret the discriminant function as an aggregate variable and perform relevant post hoc tests of the means (cf., Enders, 2003).

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The Univariate Results

As with univariate analysis of variance, tests of homogeneity of variance can be tested. The results of the Levene tests suggest that there is violation of the assumption of homogeneity of variance for X1 and X2 (p< .25).

Levene's Test of Equality of Error Variances[a]

	F	df1	df2	Sig.
x1	7.427	3	16	.002
x2	2.071	3	16	.144
x3	.341	3	16	.796
x4	1.012	3	16	.413

Tests the null hypothesis that the error variance of the dependent variable is equal across groups.

a. Design: Intercept+a

The following slide shows the results of the univariate analyses of variance. These are exactly the same as those obtained if each variable was tested separately. Note, that only the F-ratio for Variable X3 is indicated as significant (but this assumes the use of the first alternative in slide 12). Using the logic of alternative 2, it would not be judged as significant. If it is judged significant, tests of means could be conducted.

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Tests of Between-Subjects Effects

Source	Dependent Variable	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[a]
Corrected Model	x1	28.950[b]	3	9.650	1.199	.342	.184	3.596	.261
	x2	76.200[c]	3	25.400	2.709	.080	.337	8.128	.546
	x3	25.200[d]	3	8.400	4.870	.014	.477	14.609	.819
	x4	3.400[e]	3	1.133	.424	.739	.074	1.271	.116
Intercept	x1	551.250	1	551.250	68.478	.000	.811	68.478	1.000
	x2	3025.800	1	3025.800	322.752	.000	.953	322.752	1.000
	x3	3699.200	1	3699.200	2144.464	.000	.993	2144.464	1.000
	x4	3753.800	1	3753.800	1403.290	.000	.989	1403.290	1.000
a	x1	28.950	3	9.650	1.199	.342	.184	3.596	.261
	x2	76.200	3	25.400	2.709	.080	.337	8.128	.546
	x3	25.200	3	8.400	4.870	.014	.477	14.609	.819
	x4	3.400	3	1.133	.424	.739	.074	1.271	.116
Error	x1	128.800	16	8.050					
	x2	150.000	16	9.375					
	x3	27.600	16	1.725					
	x4	42.800	16	2.675					
Total	x1	709.000	20						
	x2	3252.000	20						
	x3	3752.000	20						
	x4	3800.000	20						
Corrected Total	x1	157.750	19						
	x2	226.200	19						
	x3	52.800	19						
	x4	46.200	19						

a. Computed using alpha = .05

b. R Squared = .184 (Adjusted R Squared = .030)

c. R Squared = .337 (Adjusted R Squared = .213)

d. R Squared = .477 (Adjusted R Squared = .379)

e. R Squared = .074 (Adjusted R Squared = -.100)

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Means for Each of the Dependent Variables

Estimates

				95% Confidence Interval	
Dependent Variable	a	Mean	Std. Error	Lower Bound	Upper Bound
x1	1.00	3.600	1.269	.910	6.290
	2.00	5.200	1.269	2.510	7.890
	3.00	7.000	1.269	4.310	9.690
	4.00	5.200	1.269	2.510	7.890
x2	1.00	12.800	1.369	9.897	15.703
	2.00	9.000	1.369	6.097	11.903
	3.00	14.000	1.369	11.097	16.903
	4.00	13.400	1.369	10.497	16.303
x3	1.00	15.400	.587	14.155	16.645
	2.00	13.600	.587	12.355	14.845
	3.00	13.000	.587	11.755	14.245
	4.00	12.400	.587	11.155	13.645
x4	1.00	13.600	.731	12.049	15.151
	2.00	13.400	.731	11.849	14.951
	3.00	14.400	.731	12.849	15.951
	4.00	13.400	.731	11.849	14.951

The following table presents the Bonferroni tests of means for all variables. Note that only one contrast 1 vs 4 for X3 is considered significant (but see Slides 12 and 13).

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Pairwise Comparisons

Dependent Variable	(I) a	(J) a	Mean Difference (I-J)	Std. Error	Sig. [a]	95% Confidence Interval for Difference [a]	
						Lower Bound	Upper Bound
x1	1.000	2.000	-1.60000	1.79444	1.00000	-6.99826	3.79826
		3.000	-3.40000	1.79444	.45807	-8.79826	1.99826
		4.000	-1.60000	1.79444	1.00000	-6.99826	3.79826
	2.000	1.000	1.60000	1.79444	1.00000	-3.79826	6.99826
		3.000	-1.80000	1.79444	1.00000	-7.19826	3.59826
		4.000	.00000	1.79444	1.00000	-5.39826	5.39826
	3.000	1.000	3.40000	1.79444	.45807	-1.99826	8.79826
		2.000	1.80000	1.79444	1.00000	-3.59826	7.19826
		4.000	1.80000	1.79444	1.00000	-3.59826	7.19826
	4.000	1.000	1.60000	1.79444	1.00000	-3.79826	6.99826
		2.000	.00000	1.79444	1.00000	-5.39826	5.39826
		3.000	-1.80000	1.79444	1.00000	-7.19826	3.59826
x2	1.000	2.000	3.80000	1.93649	.40416	-2.02561	9.62561
		3.000	-1.20000	1.93649	1.00000	-7.02561	4.62561
		4.000	-.60000	1.93649	1.00000	-6.42561	5.22561
	2.000	1.000	-3.80000	1.93649	.40416	-9.62561	2.02561
		3.000	-5.00000	1.93649	.12037	-10.82561	.82561
		4.000	-4.40000	1.93649	.22335	-10.22561	1.42561
	3.000	1.000	1.20000	1.93649	1.00000	-4.62561	7.02561
		2.000	5.00000	1.93649	.12037	-.82561	10.82561
		4.000	.60000	1.93649	1.00000	-5.22561	6.42561
	4.000	1.000	.60000	1.93649	1.00000	-5.22561	6.42561
		2.000	4.40000	1.93649	.22335	-1.42561	10.22561
		3.000	-.60000	1.93649	1.00000	-6.42561	5.22561
x3	1.000	2.000	1.80000	.83066	.27405	-.69891	4.29891
		3.000	2.40000	.83066	.06406	-.09891	4.89891
		4.000	3.00000*	.83066	.01405	.50109	5.49891
	2.000	1.000	-1.80000	.83066	.27405	-4.29891	.69891
		3.000	.60000	.83066	1.00000	-1.89891	3.09891
		4.000	1.20000	.83066	1.00000	-1.29891	3.69891
	3.000	1.000	-2.40000	.83066	.06406	-4.89891	.09891
		2.000	-.60000	.83066	1.00000	-3.09891	1.89891
		4.000	.60000	.83066	1.00000	-1.89891	3.09891
	4.000	1.000	-3.00000*	.83066	.01405	-5.49891	-.50109
		2.000	-1.20000	.83066	1.00000	-3.69891	1.29891
		3.000	-.60000	.83066	1.00000	-3.09891	1.89891
x4	1.000	2.000	.20000	1.03441	1.00000	-2.91184	3.31184
		3.000	-.80000	1.03441	1.00000	-3.91184	2.31184
		4.000	.20000	1.03441	1.00000	-2.91184	3.31184
	2.000	1.000	-.20000	1.03441	1.00000	-3.31184	2.91184
		3.000	-1.00000	1.03441	1.00000	-4.11184	2.11184
		4.000	.00000	1.03441	1.00000	-3.11184	3.11184
	3.000	1.000	.80000	1.03441	1.00000	-2.31184	3.91184
		2.000	1.00000	1.03441	1.00000	-2.11184	4.11184
		4.000	1.00000	1.03441	1.00000	-2.11184	4.11184
	4.000	1.000	-.20000	1.03441	1.00000	-3.31184	2.91184
		2.000	.00000	1.03441	1.00000	-3.11184	3.11184
		3.000	-1.00000	1.03441	1.00000	-4.11184	2.11184

Based on estimated marginal means

*. The mean difference is significant at the .05 level.

a. Adjustment for multiple comparisons: Bonferroni.

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Multivariate Test Statistics and Tests of Significance

There are four multivariate tests. Each tests the same null hypothesis, but uses slightly different logic, and they do not always lead to comparable conclusions. Generally, the Pillais statistic is recommended because it is robust with respect to violations of assumptions. Following are definitions of needed parameters and the statistics.

Basic Parameters

S = min{df_1, p} where: min = minimum value.
df_1 = degrees of freedom for the numerator for the univariate effect.
p = the number of dependent variables.

$m = \frac{|p - df_1| - 1}{2}$ where: | | = absolute value

r = max { p, df_1 } where: max = maximum value

$n = \frac{df_e - p - 1}{2}$ where: df_e = degrees of freedom for the error term for the corresponding univariate analysis.

a = Number of levels of the treatment factor N = Total number of subjects in the study

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Multivariate Tests of Significance

1. Pillais Trace

$$P' = \sum \frac{\lambda_j}{1+\lambda_j}$$

2. Hotelling-Lawley Trace

$$H = \sum \lambda_j$$

3. Wilks Lambda

$$\Lambda = \prod \left(\frac{1}{1+\lambda_j} \right)$$

4. Roy's Largest Root

$$R = \lambda_{\max}$$

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F-ratios and degrees of freedom

1. Pillais Trace

$$F = \frac{(N-a-p+S)P'}{r(S-P')} \qquad v_1 = p(df_1) \quad v_2 = S(df_e - p + S)$$

2. Hotelling-Lawley Trace

$$F = \frac{2(Sn+1)H}{S^2(2m+S+1)} \qquad v_1 = S(2m+S+1) \quad v_2 = 2(Sn+1)$$

3. Wilks Lambda

$$F = \frac{v_2(1-\Lambda^{1/b})}{v_1 \Lambda^{1/b}}$$

where

$$a' = (N-1) - (p+a)/2$$

$$b = \sqrt{\frac{p^2 df_1^2 - 4}{p^2 + df_1^2 - 5}}$$

$$v_1 = p(df_1) \quad v_2 = a'b - (p(df_1)/2) + 1$$

4. Roy's Largest Root

$$F = \frac{v_2 R}{v_1} \qquad v_1 = r \quad v_2 = df_e - r + S$$

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Assumptions for Multivariate Analysis of Variance

1. Multivariate Normality.
   The observations on the *p* dependent variables have a multivariate normal distribution in the population. Not only does this require each variable to be normally distributed, but also that variables in pairs and sets are mutually normally distributed. Violations of multivariate normality have minimal effects of Type I error for the collection of variables, though it has a real effect on power (i.e., Type II errors increase).

2. Independence of Observations.
   The observations (i.e., subjects) in each group are independent of each other and independent of those in other groups. If this assumption is violated, the Type I error rate is large.

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3. Equivalence of the Covariance Matrices.
   The population covariance matrices for the *p* dependent variables in each group are equal. That is, given 4 groups and 4 variables, each covariance matrix would have the same general form as follows:

σ_1^2	$\rho_{12}\sigma_1\sigma_2$	$\rho_{13}\sigma_1\sigma_3$	$\rho_{14}\sigma_1\sigma_4$
$\rho_{12}\sigma_1\sigma_2$	σ_2^2	$\rho_{23}\sigma_2\sigma_3$	$\rho_{24}\sigma_2\sigma_4$
$\rho_{13}\sigma_1\sigma_3$	$\rho_{23}\sigma_2\sigma_3$	σ_3^2	$\rho_{34}\sigma_3\sigma_4$
$\rho_{14}\sigma_1\sigma_4$	$\rho_{24}\sigma_2\sigma_4$	$\rho_{34}\sigma_3\sigma_4$	σ_4^2

If sample sizes are equal, the analysis is robust with respect to reasonable violations of this assumption. With unequal sample sizes, large n's associated with small variances will result in increased Type I error; large n's associated with large variances results in decreased Type I error. The assumption is tested in SPSS by the Box M Multivariate test for Homogeneity of Dispersion Matrices.

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Sums of Squares Matrices

Given that the sums of squares Between groups for a single variable is:

$$SS_{Between} = n\sum(\overline{X}_a - \overline{G})^2$$

and that the sums of cross products Between Groups for two variables is:

$$SCP_{Between} = n\sum(\overline{X}_{1a} - \overline{G}_1)(\overline{X}_{2a} - \overline{G}_2)$$

The Sums of Squares Between Matrix is:

SSB_1	$SCPB_{12}$	$SCPB_{13}$	$SCPB_{14}$
$SCPB_{12}$	SSB_2	$SCPB_{23}$	$SCPB_{24}$
$SCPB_{13}$	$SCPB_{23}$	SSB_3	$SCPB_{34}$
$SCPB_{14}$	$SCPB_{24}$	$SCPB_{34}$	SSB_4

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Given that the sums of squares Within groups for a single variable is:

$$SS_{Within} = \sum\sum(X_{ai} - \overline{X}_a)^2$$

and that the sums of cross products Within Groups for two variables is:

$$SCP_{Within} = \sum\sum(X_{1ai} - \overline{X}_{1a})(X_{2ai} - \overline{X}_{2a})$$

The Sums of Squares Within Matrix is:

SSW_1	$SCPW_{12}$	$SCPW_{13}$	$SCPW_{14}$
$SCPW_{12}$	SSW_2	$SCPW_{23}$	$SCPW_{24}$
$SCPW_{13}$	$SCPW_{23}$	SSW_3	$SCPW_{34}$
$SCPW_{14}$	$SCPW_{24}$	$SCPW_{34}$	SSW_4

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