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ZORN'S LEMMA AND COMPLETE BOOLEAN ALGEBRAS IN INTUITIONISTIC TYPE THEORIES

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Abstract. We analyze Zorn's Lemma and some of its consequences for Boolean algebras in a constructive setting. We show that Zorn's Lemma is persistent in the sense that, if it holds in the underlying set theory, in a properly stated form it continues to hold in all intuitionistic type theories of a certain natural kind. (Observe that the axiom of choice cannot be persistent in this sense since it implies the law of excluded middle.) We also establish the persistence of some familiar results in the theory of (complete) Boolean algebras—notably, the proposition that every complete Boolean algebra is an absolute subretract. This (almost) resolves a question of Banaschewski and Bhutani as to whether the Sikorski extension theorem for Boolean algebras is persistent.

§1. Introduction. Typically, applications of Zorn's Lemma (**ZL**) take the following form. Suppose, for example, one wishes to show that a function possessing a certain property P exists with domain a certain set A. To do this one proves first that the collection $\mathscr{F}$ of functions with property P and domain a subset of A is closed under unions of chains and then infers from Zorn's Lemma that $\mathscr{F}$ has a maximal element m. Finally, a "one-step extension" argument is formulated so as to yield the conclusion that the domain of m is A itself. This "one-step" argument may be distilled into what I shall call the *extension principle for* $\mathscr{F}$, *viz.*

$$\mathbf{EP}(\mathscr{F}) \qquad \forall f \forall x \in A[f \in \mathscr{F} \rightarrow \exists g \in \mathscr{F}[f \subseteq g \wedge x \in \text{domain}(g)]].$$

Applying this to the maximal m immediately yields the desired conclusion $A = \text{domain}(m)$.

More specifically, let us examine the standard derivation from **ZL** of the axiom of choice (**AC**) in the form: *each relation R contains a function with the same domain.* Here A is domain(R), $\mathscr{F}$ is the set $R^{\#}$ of subfunctions of R, and the extended function g figuring in $\mathbf{EP}(\mathscr{F})$ is obtained from the given function f and the given element $x \in A$ by means of a classical definition by cases:

$$g = f \text{ if } x \in \text{dom}(f)$$
$$g = f \cup \langle x, y\rangle \text{ for some } y \text{ such that } \langle x, y\rangle \in R \text{ if } x \notin \text{domain}(f).$$

Now we shall see (in §3) that **ZL** is perfectly compatible with constructive reasoning (i.e., with reasoning formulable within the intuitionistic type theories to be introduced presently). Moreover, if we write **EP** for the statement

$$\forall R(R \text{ is a relation} \longrightarrow \mathbf{EP}(R^{\#}),$$

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then the implication **ZL** + **EP** $\longrightarrow$ **AC** is evidently constructively valid. But it is well known (see, e.g., [6, 4.31]) that **AC** implies the law of excluded middle. It follows that **EP** must be nonconstructive. And indeed, we have

PROPOSITION 1.1. **EP** *implies excluded middle.*

PROOF. Write $2 = \{0, 1\}$ and, given any proposition φ, let $U = \{x \in 2 : x = 0 \vee \varphi\}$, $V = \{x \in 2 : x = 1 \vee \varphi\}$, and $R = (\{U\} \times U) \cup (\{V\} \times V)$. Then the function $f_0 = \{\langle U, 0\rangle\}$ is in $R^\#$ and so **EP** yields a function g in $R^\#$ extending f_0 such that $\text{domain}(g) = \{U, V\}$. Thus $g(U) = 0$ and $g(V) \in V$. This latter conjunct means that

$$(*) \qquad g(V) = 1 \text{ or } \varphi.$$

But clearly $\varphi \longrightarrow V = U \longrightarrow g(V) = g(U) = 0$. Thus $g(V) \neq 0 \longrightarrow \neg\varphi$, whence $g(V) = 1 \longrightarrow \neg\varphi$. We conclude from this and (*) that $\neg\varphi$ or φ. Since φ was arbitrary, excluded middle follows. ⊣

We are going to show, by contrast, that **ZL** is consistent with a certain form of constructive reasoning, namely, that embodied within the intuitionistic type theories associated with toposes (see, e.g., [6]), and that this is also the case for certain of its consequences in the theory of Boolean algebras. This will be achieved by establishing the *persistence* of these assertions in the sense that, if any of them holds in the underlying set theory, then it continues to hold in all of the above type theories. Since each one of these assertions is known to be consistent with classical set theory, it will then follow, in particular, that each is consistent with constructive reasoning in the sense we have specified. In other words, unlike the axiom of choice, neither **ZL** nor any of the other assertions which we show to be persistent has "nonconstructive" logical consequences: thus they are what might be called *constructively neutral*. We take up the theme of persistence again, briefly, at the end of the next section.

§2. Intuitionistic type theories. We summarize briefly the system presented in [6]. A *language* L *for intuitionistic type theory,* or a *language* for short, has the following ingredients:

Basic symbols. **1** (unit type), Ω (truth value type), **S**, **T**, **U**, ... (ground types), **f**, **g**, **h**, ... (function symbols).

Types. These are members of the smallest class containing **1**, **Ω** and the ground types and closed under the *product* and *power* operations: here the product of two types **A**, **B** is denoted by **A** × **B** and the power of a type **A** is denoted by **PA**.

Signatures. Each function symbol is assigned a pair of types called its *signature.* Notation: **f** : **A** $\longrightarrow$ **B**.

Terms and associated types. These are specified as follows:

(i) # is a term of type **1**, and for each type **A** there is a list of variables x, y, z, ... of that type;

(ii) the collection of terms is closed under the following operations (where $\tau : A$ indicates that the term τ has type **A**):

$$
\begin{array}{ll}
\langle \sigma, \tau \rangle : \mathbf{A} \times \mathbf{B} & \text{for } \sigma : \mathbf{A} \text{ and } \tau : \mathbf{B}, \\
f(\tau) : \mathbf{B} & \text{for } \tau : \mathbf{A} \text{ and } f : \mathbf{A} \longrightarrow \mathbf{B} \\
\{x : \alpha\} : \mathbf{PA} & \text{for } x : \mathbf{A} \text{ and } \alpha : \mathbf{\Omega} \\
\sigma = \tau : \mathbf{\Omega} & \text{for } \sigma : \mathbf{A} \text{ and } \tau : \mathbf{A} \\
\sigma \in \tau : \mathbf{\Omega} & \text{for } \sigma : \mathbf{A} \text{ and } \tau : \mathbf{PA}.
\end{array}
$$

Closed terms are, as usual, terms without free variables, i.e., variables x not appearing in a context of the form $\{x : \alpha\}$. A closed term of power type will be called an $\mathscr{L}$*-set*: it is easily seen that each $\mathscr{L}$-set is of the form $\{x : \alpha\}$. For any type $\mathbf{A}$, we write A for the $\mathscr{L}$-set $\{x : x = x\}$, where $x : A$. A term of type $\mathbf{\Omega}$ is called a *formula*, and a closed formula is called a *sentence*. We use the letters α, β, γ to denote formulas and the letters ω, ω' to denote variables of type $\mathbf{\Omega}$. We write $\tau(x/\sigma)$ for the result of substituting σ for x at each of the latter's free occurrences in τ. We shall write $\tau(x, x', \dots)$ to indicate that the term τ has at most the free variables x, $x', \dots$; in this situation we shall usually write $\tau(\sigma, \sigma', \dots)$ for $\tau(x/\sigma, x'/\sigma', \dots)$.

Axioms and rules of inference. We adopt a sequent notation, writing $\Gamma|\alpha$ for the sequent composed of a finite set Γ of formulas and a formula α, and $|\alpha$ for $\emptyset|\alpha$. The (basic) *axioms* in $\mathscr{L}$ are, writing $\alpha \leftrightarrow \beta$ for $\alpha = \beta$, the sequents

$$
\begin{array}{ll}
|x = \# & (\text{with } x : 1) \\
x = y, \alpha(z/x)|\alpha(z/y) & (\text{with } x, y \text{ free for } z \text{ in } \alpha) \\
\langle x, y \rangle = \langle x', y' \rangle | x = x' & \\
\langle x, y \rangle = \langle x', y' \rangle | y = y' & \\
|x \in \{x : \alpha\} \leftrightarrow \alpha. &
\end{array}
$$

The *rules of inference* in $\mathscr{L}$ are

$$
\begin{array}{ll}
\dfrac{\Gamma|\alpha \quad \alpha, \Gamma|\beta}{\Gamma|\beta} & (\text{all free variables of } \alpha \text{ free in conclusion}) \\
\dfrac{\Gamma|\alpha}{\beta, \Gamma|\alpha} & \\
\dfrac{\Gamma|\alpha}{\Gamma(x/\tau)|\alpha(x/\tau)} & (\tau \text{ free for } x \text{ in } \alpha \text{ and all members of } \Gamma) \\
\dfrac{\Gamma|x \in \sigma \leftrightarrow x \in \tau}{\Gamma|\sigma = \tau} & (x \text{ not free in conclusion}) \\
\dfrac{\alpha, \Gamma|\beta \quad \beta, \Gamma|\alpha}{\Gamma|\alpha \leftrightarrow \beta}. &
\end{array}
$$

These axioms and rules of inference yield a system of *natural deduction* in $\mathscr{L}$. If S is any collection of sequents in $\mathscr{L}$, we say that the sequent $\Gamma|\alpha$ is S*-derivable*, and write $\Gamma \vdash_S \alpha$, provided there is a derivation of $\Gamma|\alpha$ using the basic axioms, the sequents in S, and the rules of inference. For $\emptyset \vdash_S \alpha$ we write simply $\vdash_S \alpha$. A *theory* in $\mathscr{L}$ is a collection of sequents closed under derivability. A theory in some typed intuitionistic language will be called a *type theory* (in [6], type theories are called *local set theories*).

Logical operators in $\mathscr{L}$ are defined as follows:

$$\text{true} =_{\text{df}} \# = \# \qquad (\text{Note that } \vdash_S \forall\omega(\omega \leftrightarrow \omega = \text{true}).)$$
$$\alpha \wedge \beta =_{\text{df}} \langle \alpha, \beta \rangle = \langle \text{true}, \text{true} \rangle$$
$$\alpha \rightarrow \beta =_{\text{df}} (\alpha \wedge \beta) \leftrightarrow \alpha$$
$$\forall x\alpha =_{\text{df}} \{x : \alpha\} = \{x : \text{true}\}$$
$$\text{false} =_{\text{df}} \forall\omega.\,\omega = \text{true}$$
$$\neg\alpha =_{\text{df}} \alpha \rightarrow \text{false}$$
$$\alpha \vee \beta =_{\text{df}} \forall\omega[(\alpha \rightarrow \omega \wedge \beta \rightarrow \omega) \rightarrow \omega]$$
$$\exists x\alpha =_{\text{df}} \forall\omega[\forall x(\alpha \rightarrow \omega) \rightarrow \omega].$$

Other logical operators such as $\exists!x$ (there is a unique x) and standard set-theoretic terms such as $\{x\}$ and $f(x)$ can be introduced into $\mathscr{L}$ in the usual way: we shall use such terms freely. The theory S is said to be *well-termed* if whenever $\vdash_S \exists!x\alpha$ there is some term τ for which $\vdash_S \alpha(x/\tau)$.

It can be shown (see [6, Ch. 3]) that the theorems of (free) higher-order intuitionistic logic are S-derivable for any theory S: in particular, it is S-derivable that $\langle \Omega, \rightarrow \rangle$ is a complete Heyting algebra, and that its associated algebra $\Omega_{\neg\neg} =_{\text{df}} \{\omega : \neg\neg\omega = \omega\}$ of regular elements is a complete Boolean algebra. S is said to be *classical* if $\vdash_S \forall\omega.\ \omega \vee \neg\omega$, or, equivalently, if $\vdash_S \Omega = \Omega_{\neg\neg}$.

Let $\Omega(S)$ be the collection of sentences of $\mathscr{L}$, in which we identify two sentences when their equivalence is S-derivable. Define the relation $\leq$ on $\Omega(S)$ by $\alpha \leq \beta$ iff $\vdash_S \alpha \rightarrow \beta$. Then $(\Omega(S), \leq)$ is a Heyting algebra.

If S is a theory in a language $\mathscr{L}$, an *S-set* is an equivalence class of $\mathscr{L}$-sets under the equivalence relation of S-derivable equality: we shall often use the symbol for an $\mathscr{L}$-set to denote its associated S-set. By an *element* of an S-set E we shall mean a closed term e of $\mathscr{L}$ for which $\vdash_S e \in E$. We write $E^\sim$ for the set of all elements of E, where we agree to identify two such elements if their identity is S-derivable. By a *subset* of an S-set E we shall mean an S-set U such that $\vdash_S U \subseteq E$; we write $\text{Pow}(E)$ for the collection of all subsets of E. If we define the relation $\sqsubseteq$ on $\text{Pow}(E)$ by $U \sqsubseteq V$ iff $\vdash_S U \subseteq V$, then $(\text{Pow}(X), \sqsubseteq)$ is a Heyting algebra.

An *S-map* is an S-set which is in addition an S-derivably functional relation. The collection $\mathscr{C}(S)$ of S-sets and S-maps naturally possesses the structure of a Cartesian closed category with a subobject classifier; in short, $\mathscr{C}(S)$ is a *topos*. Conversely, given a topos $\mathscr{E}$, we can associate with it a language $\mathscr{L}(\mathscr{E})$ called its *internal language* having its objects as types and its arrows as function symbols. There is a natural *interpretation* of $\mathscr{L}(\mathscr{E})$ in $\mathscr{E}$ yielding the notion of *validity* of sequents of $\mathscr{L}(\mathscr{E})$ in $\mathscr{E}$. The *theory* $\text{Th}(\mathscr{E})$ of $\mathscr{E}$ is the collection of sequents of $\mathscr{L}(\mathscr{E})$ valid under this interpretation. It can be shown that $\text{Th}(\mathscr{E})$ is well-termed and that $\mathscr{C}(\text{Th}(\mathscr{E}))$ is equivalent (in the category-theoretic sense) to $\mathscr{E}$.

If S is a theory in a language $\mathscr{L}$, and X is an $\mathscr{L}$-set, an *S-singleton in X* is an $\mathscr{L}$-set U such that $\vdash_S U \subseteq X \wedge \forall xy \in U.\ x = y$. The $\mathscr{L}$-set X is said to be *S-coverable* if, for any formula $\alpha(x)$, $x : A$, whenever $\vdash_S \forall x \in U\alpha$ for all S-singletons

U in X, then $\vdash_S \forall x \in X\alpha$. (This means, intuitively, that X is "covered" by its singletons.) The theory S is itself said to be *coverable* if every $\mathscr{L}$-set is S-coverable.

The theory S is said to satisfy the *axiom of choice* (**AC**) if for any S-sets X, Y and any formula $\alpha(x, y)$, whenever $\vdash_S \forall x \in X \exists y \in Y\alpha(x, y)$, there is an S-map $f : X \longrightarrow Y$ for which $\vdash_S \forall x \in X\alpha(x, f(x))$.

The theory S is said to be *full* if for each (intuitive) set I there is a type symbol $I^\frown$ of $\mathscr{L}$ and for each $i \in I$ a closed term $i^\frown : I^\frown$ such that the following *universal condition* is satisfied.

- For any I-indexed family $\{\tau_i : i \in I\}$ of closed terms of common type $\mathbf{A}$ there is a term $\tau(x) : \mathbf{A}$, $x : I^\frown$, such that $\vdash_S \tau_i = \tau(i^\frown)$ for all $i \in I$, and, for any term $\sigma(x) : \mathbf{A}$, $\vdash_S \tau_i = \sigma(i^\frown)$ for all $i \in I$ implies $\vdash_S \tau = \sigma$.

The following basic facts concerning full type theories can now be established (see [6, Ch. 4]). Let S be a full type theory. Then we have:

GENERALIZATION PRINCIPLE. Let $\alpha(x_1, \ldots, x_n)$ be a formula with $x_1 : \mathbf{I}_1^\frown, \ldots, x_n : \mathbf{I}_n^\frown$. If $\vdash_S \alpha(i_1^\frown, \ldots, i_n^\frown)$ for all $i_1 \in I_1, \ldots, i_n \in I_n$, then $\vdash_S \forall x_1 \cdots \forall x_n \alpha$.

ISOMORPHISM PRINCIPLE. Suppose that S is well-termed. Then for any sets I, J there is a term $\rho(v) : (\mathbf{I} \times \mathbf{J})^\frown$, $v : \mathbf{I}^\frown \times \mathbf{J}^\frown$ such that $\vdash_S \rho(\langle i^\frown, j^\frown\rangle) = (i, j)^\frown$ for any $i \in I$, $j \in J$ and $\vdash_S$ *ρ is a bijection* $I^\frown \times J^\frown \longrightarrow (I \times J)^\frown$. In view of this we shall identify $\langle i^\frown, j^\frown\rangle$ and $(i, j)^\frown$, thus also identifying $I^\frown \times J^\frown$ and $(I \times J)^\frown$.

COMPLETENESS PRINCIPLE. $\Omega(S)$ and $\mathrm{Pow}(E)$, for any S-set E, are complete Heyting algebras. In $\Omega(S)$, the supremum of a subset $A = \{\alpha_i : i \in I\}$ is the sentence $\exists x \alpha(x)$, where $\alpha(x)$ satisfies $\vdash_S \alpha_i = \alpha(i^\frown)$ for all $i \in I$. In $\mathrm{Pow}(E)$, the supremum of a subset $\{A_i : i \in I\}$ is the S-set $\{y : \exists x.\ y \in \tau(x)\}$, where $\tau(x)$ satisfies $\vdash_S \tau(i^\frown) = A_i$ for all $i \in I$.

Some noteworthy facts concerning coverability and fullness are the following. Let $\mathscr{E}$ be a topos. Then $\mathrm{Th}(\mathscr{E})$ is full if and only if $\mathscr{E}$ is *defined* over the topos $\mathscr{S}et$ of sets, i.e., iff $\mathscr{E}$ admits arbitrary copowers of its terminal object. Also, the following are equivalent: (i) $\mathrm{Th}(\mathscr{E})$ is coverable and full, (ii) for some complete Heyting algebra H, $\mathscr{E}$ is (equivalent to) the topos $\mathscr{S}h(H)$ of sheaves on H. If either of these conditions hold, $\mathscr{E}$ is called *localic*, and if in addition $\mathrm{Th}(\mathscr{E})$ is classical, $\mathscr{E}$ is called *Boolean localic*.

Let Φ be a property of type theories (e.g., satisfying **AC**). We say that Φ is *strongly persistent* (resp. *persistent, classically persistent*) if, whenever Φ holds in the underlying set theory, it holds in any full well-termed type theory (resp. full coverable well-termed type theory, full coverable classical well-termed type theory). In topos-theoretic terms, the meaning of strong persistence (resp. persistence, classical persistence) of a given property is that, whenever it holds in $\mathscr{S}et$, it continues to hold in any topos defined over $\mathscr{S}et$ (resp. localic topos, Boolean localic topos).

Finally, a few remarks on the concept of persistence. By abuse of language, let us call a proposition P persistent when the property "P holds" is persistent. Apart from the trivial cases of the absurd proposition and the theorems of intuitionistic type theory, persistent propositions seem to be rare: indeed the only nontrivial examples known to me are **ZL**, a certain weak version of the axiom of choice (see Corollary 3.3), and the variants of the Sikorski extension theorem for Boolean algebras to be investigated here. One can often show that a proposition is *nonpersistent* by noting that it follows from the axiom of choice and also has a nonconstructive logical consequence (or is clearly of a nonconstructive nature itself). In that event, it is usually quite easy to construct a sheaf model (i.e., of the form $\mathcal{SA}(H)$ for some complete Heyting algebra H) in which the logical consequence, and hence also the proposition itself, fails, even when the axiom of choice is assumed in the underlying set theory: nonpersistence follows immediately. Such is the case, for example, for the axiom of choice itself, the Stone Representation Theorem for Boolean algebras, the completeness of the 2 element Boolean algebra, and, at a purely logical level, Markov's principle. (In [3] it is shown that the Stone Representation Theorem implies the law of excluded middle in sheaf models and in [8] that the completeness of the 2 element Boolean algebra implies the nonconstructive instance of De Morgan's law.) However, this technique will not work for such statements as the Boolean Ultrafilter Theorem which, as a constructive corollary of the persistent **ZL**, cannot have nonconstructive logical consequences. To demonstrate the nonpersistence of propositions of this sort more complex arguments are required, such as those to be found in [9], by means of which one establishes the stronger property of *classical* nonpersistence using a generic extension of a model of set theory in which the statement in question, but not the axiom of choice, holds.

§3. **Maximal principles in type theories.** Let S be a well-termed type theory in a language $\mathcal{L}$. A *partially ordered S-set* is a pair $(E, \leq)$ of S-sets such that $\vdash_S \leq$ *is a partial ordering of* E. (In accordance with the customary abuse of notation, we shall frequently identify $(E, \leq)$ with E.) A *chain* in E is a subset C of E such that $\vdash_S \forall xy \in C.\ x \leq y \vee y \leq x$. A *supremum* of a subset A of E is an element a of E such that $\vdash_S$ *a is the least upper bound of A in* $(E, \leq)$. Note that, as usual, A has at most one supremum, which, if it exists, we shall denote by $\bigvee A$. (In introducing $\bigvee A$ as a closed term of $\mathcal{L}$, we are implicitly appealing to the well-termedness of S.) An element m of E is called *maximal* if, for any element e of E, we have $\vdash_S m \leq e$ implies $\vdash_S m = e$. E is *inductive* if any chain in E has a *supremum* in E (not merely an upper bound: more on this below). *Zorn's Lemma* (**ZL**) is said to hold in S if any inductive partially ordered S-set has a maximal element. Finally **ZL** is said to hold in a topos $\mathcal{E}$ if it holds in $\mathrm{Th}(\mathcal{E})$.

THEOREM 3.1. **ZL** *is strongly persistent.*

PROOF. Let $(E, \leq)$ be an inductive partially ordered S-set in a full well-termed type theory S. Since $\emptyset$ is a chain in E, $\bigvee \emptyset \in E^{\sim}$ and so $E^{\sim}$ is nonempty. Partially order $E^{\sim}$ by stipulating that $a \leq^* b$ iff $\vdash_S a \leq b$. We claim that $(E^{\sim}, \leq^*)$ is inductive. To this end, let $C = \{c_i : i \in I\}$ be a chain in $E^{\sim}$. Since S is full, there is a term $\tau(x)$ such that $\vdash_S \tau(i^{\frown}) = c_i$ for all $i \in I$. Since $\vdash_S c_i \leq c_j \vee c_j \leq c_i$ so that $\vdash_S \tau(i^{\frown}) \leq \tau(j^{\frown}) \vee \tau(j^{\frown}) \leq \tau(i^{\frown})$ for every $i, j \in I$, it follows that, by the Generalization Principle, $\vdash_S \forall xy[\tau(x) \leq \tau(y) \vee \tau(y) \leq \tau(x)]$. Therefore the S-set $\{z : \exists x.\ z = \tau(x)\}$ is a chain in E and accordingly has a supremum c. We claim that c is the supremum of C in $E^{\sim}$. First, c is obviously an upper bound for C. And it is the least upper bound since, if $e \in E^{\sim}$ satisfies $\vdash_S c_i \leq e$ for all $i \in I$, then $\vdash_S \tau(i^{\frown}) \leq e$ for all $i \in I$, so that $\vdash_S \forall x.\ \tau(x) \leq e$ by the Generalization Principle. Therefore $\vdash_S c \leq e$ and so $c \leq^* e$. Thus $E^{\sim}$ is inductive and so, applying **ZL** in the underlying set theory, has a maximal element, which is automatically a maximal element of E. ⊣

REMARKS. (1) In Theorem 3.1 we cannot infer that the maximal element—call it m—is *internally maximal*, i.e., satisfies the stronger condition $m \leq e \vdash_S m = e$ for any $e \in E^{\sim}$. However, if S is *classical* (and well-termed), it is easy to show that maximality implies internal maximality. In this case, for any $e \in E^{\sim}$ there is $e' \in E^{\sim}$ for which $\vdash_S (m \leq e \wedge e' = e) \vee (m \not\leq e \wedge e' = m)$. Clearly $\vdash_S m \leq e'$, whence $\vdash_S m = e'$, and $m \leq e \vdash_S m = e$ easily follows.

(2) The term "inductive" in the statement of Zorn's Lemma is often construed in the weaker sense of chains merely possessing *upper bounds* rather than suprema. The resulting ostensibly stronger form $\mathbf{ZL}^+$ of **ZL** is, as is well known, classically equivalent to **ZL**. However, the proof of Theorem 3.1 (appropriately modified) shows only that $\mathbf{ZL}^+$ persists in type theories S which, in addition to being full and well-termed, are also *witnessed* in the sense that, for any formula $\alpha(x)$, if $\vdash_S \exists x\alpha$, then $\vdash_S \alpha(\tau)$ for some (closed) term τ. In topos-theoretic terms, $\mathbf{ZL}^+$ persists in any topos which, in addition to being defined over $\mathcal{S}et$, has the property that its terminal object is projective.

Although, as we now see, **AC**, not being persistent, cannot be a constructive consequence of **ZL**, there is a weaker version which is. This weaker version may be stated as follows. The theory S is said to satisfy *weak* **AC** (**WAC**) if whenever $\vdash_S \forall x \exists y \alpha(x, y)$, $x : \mathbf{A}$, $y : \mathbf{B}$, there is an S-set M of type $\mathbf{P}(\mathbf{A} \times \mathbf{B})$ such that, writing $\mathrm{Fun}(X)$ for the formula expressing "X is a function",

$$\vdash_S \mathrm{Fun}(M) \wedge \forall xy(\langle x, y\rangle \in M \to \alpha(x, y)) \wedge A - \mathrm{domain}(M) = \emptyset.$$

An S-set satisfying this condition is "almost" a choice function for α in that its domain is $\neg\neg$*-dense* in A: cf. [6, Ch. 5].

THEOREM 3.2. *If S is coverable and* **ZL** *holds in S, then S satisfies* **WAC**.

PROOF. Assume the hypotheses and

$$(*) \qquad \vdash_S \forall x \exists y \alpha(x, y) \quad x : \mathbf{A}\ y : \mathbf{B}.$$

Let E be the S-set $\{X : X \subseteq R \wedge \mathrm{Fun}(X)\}$. Then $(E, \subseteq)$ is a partially ordered S-set; if C is a chain in E it is easily shown in the usual way that its union is the supremum of C in E. So $(E, \subseteq)$ is inductive and accordingly has, by **ZL**, a maximal element M. Clearly,

$$\vdash_S \mathrm{Fun}(M) \wedge \forall xy(\langle x, y\rangle \in M \to \alpha(x, y)).$$

To complete the proof we need to show that

$$(**) \qquad \vdash_S A - \mathrm{domain}(M) = \emptyset.$$

To this end define $D = R \cap [(A - \mathrm{domain}(M)) \times B]$. Let U be any S- singleton in $A \times B$ and let $V = U \cap D$. Evidently the S-set $M \cup V$ is an element of E, so since M is maximal in E it follows that $\vdash_S V \subseteq M$. Obviously, however, $\vdash_S V \cap M = \emptyset$, whence $\vdash_S V = \emptyset$. Thus $\vdash_S \forall u \in U.\ u \notin D$, so, since S is coverable, $\vdash_S \forall u.\ u \notin D$, i.e., $\vdash_S D = \emptyset$. Therefore $\vdash_S \text{domain}(R) \cap [A - \text{domain}(M)] = \emptyset$. This, together with (*), immediately yields (**). ⊣

This result and Theorem 3.1 quickly give the

COROLLARY 3.3. **WAC** *is persistent.* ⊣

It is interesting to note that, if we attempt to apply **WAC** in the same manner as **AC** is applied to derive the law of excluded middle (cf. [6, 4.31]), we find that, for a given formula α, we can only infer $\neg\neg(\alpha \vee \neg\alpha)$ which is of course already derivable constructively.

§4. Complete Boolean algebras in type theories.

A well-known consequence of **ZL** in the theory of Boolean algebras is the *Sikorski Extension Theorem* [10] which states that, for any Boolean algebra B and any *complete* Boolean algebra C, any homomorphism of a subalgebra of B to C can be extended to the whole of B. That is, in classical set theory, **ZL** implies

(**INJ**) Any complete Boolean algebra is injective in the category $\mathscr{Bool}$ of Boolean algebras.

(It is, incidentally, still unknown whether **INJ** is classically equivalent to **ZL**: see [4] and [5].) In [3] it is shown that, if **ZL** holds in $\mathscr{Set}$, then **INJ** holds in any localic topos. Our next task will be to show that this is the case because **INJ** can in fact be constructively derived from **ZL**.

We shall employ the standard notation and terminology for Boolean algebras, both in the underlying set theory and, analogously, in type theories. If $(B, \wedge_B, \vee_B, *_B, \leq_B, 0_B, 1_B)$ is a Boolean algebra (we shall usually, but not invariably, omit the subscript "B"), we write $a \Rightarrow b$ for $a^* \vee b$, and $a \iff b$ for $(a \Rightarrow b) \wedge (b \Rightarrow a)$. We write 2 for the initial Boolean algebra $\{0, 1\}$.

If S is a type theory in a language $\mathscr{L}$, we say that **INJ** *holds in* S if, whenever A, B, C, h are $\mathscr{L}$-sets such that

(†) $\vdash_S$ *A, B, C are Boolean algebras, A is a subalgebra of B, C is complete, and h is a homomorphism $A \longrightarrow C$,*

then there is an $\mathscr{L}$-set f such that

$$\vdash_S f \textit{ is a homomorphism } B \longrightarrow C \textit{ extending } h.$$

THEOREM 4.1. *Let S be a well-termed coverable type theory in which* **ZL** *holds. Then* **INJ** *also holds in S.*

PROOF. Suppose that the premise (†) of **INJ** is satisfied by A, B, C, h. Let H be the $\mathscr{L}$-set $\{f : f$ is a homomorphism of a subalgebra of B to C extending $h\}$. A standard argument shows that $(H, \subseteq)$ is inductive and so by **ZL** H has a maximal element m. Writing $M = \text{domain}(m)$, to derive the conclusion of **INJ** it clearly suffices to show that $\vdash_S M = B$. And since S is coverable, for this it is enough to show that, for any S-singleton U in B,

(*) $$\vdash_S U \subseteq M.$$

To this end let $\mathrm{Alg}(X)$ be the term defining the subalgebra of B generated by X, for $X \subseteq B$. Then the standard argument (cf. [7, Proof of Lemma 2, p. 142]) enables a term $m^+(x)$ to be constructed in $\mathscr{L}$ in such a way that $\vdash_S \forall x \in B.\ m^+(x)$ *is a homomorphism* $\mathrm{Alg}(M \cup \{x\}) \longrightarrow C$ *extending m for which*

$$m^+(x)(x) = \bigvee\{m(y) : y \in M \wedge y \leq x\}.$$

If U is an S-singleton in B let m_U be the term $\bigcap\{m^+(x) : x \in U\}$. Clearly m_U is an element of H and $\vdash_S m \subseteq m_U$. Since m is maximal, $\vdash_S m = m_U$, and (*) instantly follows. ⊣

From Theorem 4.1 and Theorem 3.1 we immediately obtain the

COROLLARY 4.2. *If S is coverable, well-termed and full, and* **ZL** *holds in the underlying set theory, then* **INJ** *holds in S.* ⊣

Applying this corollary to $\mathrm{Th}(\mathscr{E})$ for a localic topos $\mathscr{E}$ yields (the hard implication of) Proposition 1.9 of [3], viz. that a Boolean algebra in a localic topos is injective if it is complete.

Observe that the instance of **EP** needed for the proof of Theorem 4.1 to go through was obtainable constructively via the term $m^+(x)$, whose definition was made possible by the *completeness* of C. Now *classically*, the initial Boolean algebra 2 is complete, so, again classically, the assertion $\mathbf{INJ}(2)$ that 2 is injective is a special case of **INJ**. Moreover, since—classically—homomorphisms to 2 correspond to ultrafilters, $\mathbf{INJ}(2)$ is equivalent to the *Boolean Ultrafilter Theorem* (**BUT**) that every Boolean algebra contains an ultrafilter. That is, the implications $\mathbf{INJ} \rightarrow \mathbf{INJ}(2) \leftrightarrow \mathbf{BUT}$ are classically provable. Also, it is observed in [3] that, if $\mathbf{INJ}(2)$ holds in a localic topos $\mathscr{S}\!h(H)$, then H must be a *Stone algebra*, i.e., satisfies the identity $x^* \vee x^{**} = 1$. In fact it is easy to establish an analogous assertion for arbitrary type theories: for, arguing within a given type theory S, if 2 is injective in $\mathscr{B}\!\mathit{ool}$, then there is a homomorphism $\Omega_{\neg\neg} \longrightarrow 2$; since $\Omega_{\neg\neg}$ is complete, the existence of such a homomorphism implies that 2 is complete, and the completeness of 2 in S is known [8] to be equivalent to the assertion that, in S, Ω is a Stone algebra.

It follows that, because 2 is not necessarily complete in a constructive sense, $\mathbf{INJ}(2)$ is not a constructive consequence of **INJ**. Nevertheless, **BUT** *is* a constructive consequence of **INJ** in view of the following

PROPOSITION 4.3. *Suppose that* F, B *are* $\mathscr{L}$*-sets satisfying* $\vdash_S$ B is a Boolean algebra and F is a filter in B. *Then the following statements are derivably equivalent in* S:

(i) *F is an ultrafilter;*

(ii) *for all* $b \in B$, $b \notin F \longleftrightarrow b^* \in F$;

(iii) *there is a homomorphism* $\varphi : B \longrightarrow \Omega_{\neg\neg}$ *such that* $F = \varphi^{-1}(\mathrm{true})$.

PROOF. We argue in S throughout.

(i) $\rightarrow$ (ii). Clearly $b^* \in F \rightarrow b \notin F$ since $0 \notin F$. Conversely, assume (i) and $b \notin F$. For any $x \in F$, $x \wedge b^* = 0$ implies $x \leq b$, so that $b \in F$. It follows that $x \wedge b^* \neq 0$ for any $x \in F$. Therefore the filter $\{y : \exists x \in F.\ x \wedge b^* \leq y\}$ generated by $F \cup \{b^*\}$ is proper; since F is an ultrafilter, $b^* \in F$.

(ii) $\rightarrow$ (iii). Define $\varphi : B \longrightarrow \Omega$ by $\varphi(x) = (x \in F)$. Assuming (ii), we have $\varphi(x) = \neg\varphi(x^*)$, so that $\varphi(x) \in \Omega_{\neg\neg}$. It is easy to see, using (ii), that f is a homomorphism $B \longrightarrow \Omega_{\neg\neg}$.

(iii) $\longrightarrow$ (i). Assume (iii) and suppose that G is a proper filter in B such that $F \subseteq G$. Suppose now that $b \in G$ but $b \notin F$. Then $\varphi(b) \neq$ true, i.e., $\neg\varphi(b)$ whence $\varphi(b^*)$ since f is a homomorphism. Thus $b^* \in F$, so that $b^* \in G$. Then $0 = b \wedge b^* \in G$, a contradiction. We conclude that, if $b \in G$, then $\neg(b \notin F)$. But $\neg(b \notin F) \leftrightarrow \neg\neg\varphi(b) \leftrightarrow \varphi(b) \leftrightarrow b \in F$ since $\varphi(b) \in \Omega_{\neg\neg}$. Therefore, $b \in G \rightarrow b \in F$, so that $G = F$ and F is an ultrafilter. $\dashv$

COROLLARY 4.4. *If* **INJ** *holds in* S, *then so does* **BUT**.

PROOF. If **INJ** holds in S, then, arguing in S, since $\Omega_{\neg\neg}$ is a complete Boolean algebra, it is injective. So if, for any Boolean algebra B, we identify 2 with the unique two element subalgebra of B, then the canonical homomorphism $2 \longrightarrow \Omega_{\neg\neg}$ can be extended to a homomorphism $B \longrightarrow \Omega_{\neg\neg}$, which, by Theorem 4.3, gives rise to an ultrafilter in B. $\dashv$

REMARK. It is (essentially) shown in [9] that **BUT** fails to be classically persistent.

In [3] the question is raised as to whether **INJ** is persistent. In [2] it is shown that **INJ** is classically persistent: a related result was earlier established in [4]. We will now show that the answer to the original question is "almost" affirmative. To be precise, we formulate a proposition **RET**—a special case of **INJ** known to be classically equivalent to it—and show that **RET** is persistent.

A Boolean algebra B is said to be an *absolute subretract* if, for any monomorphism $m : B \rightarrowtail A$ to a Boolean algebra A, there is a homomorphism $h : A \longrightarrow B$ such that $h \circ m = \mathrm{id}_B$. The principle **RET** is said to *hold in* a type theory S if, for any $\mathscr{L}$-sets A, B, m such that

$$\vdash_S A, B \textit{ are Boolean algebras, } B \textit{ is complete, and } m : B \rightarrowtail A \textit{ is a monomorphism,}$$

there is an $\mathscr{L}$-set h such that

$$\vdash_S h : A \longrightarrow B \textit{ is a homomorphism and } h \circ m = \mathrm{id}_B.$$

REMARK. It is easy to see that, if **INJ** holds in S, so does **RET**. Observe that this implication can be reversed when S is *classical*. For in this case 2 is complete and so, assuming **RET**, injective. Therefore **BUT** holds in S, and so, since S is classical, we have available in S the (nonconstructive) machinery of Stone duality for Boolean algebras: that is, an equivalence St between $\mathscr{B}\mathit{ool}^{\mathrm{op}}$ and the category $\mathscr{S}\mathit{tone}$ of Stone (or Boolean) spaces. Now suppose that C is an absolute subretract and that we are given a diagram

$$\begin{tikzcd} B \arrow[r, tail, "m"] \arrow[d, "f"'] & A \\ C & \end{tikzcd}$$

in $\mathscr{B}ool$. Form the pushout

$$\begin{tikzcd} B \arrow[r, tail, "m"] \arrow[d, "f"'] & A \arrow[d, "g"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

in $\mathscr{B}ool$. Then the corresponding dual diagram

$$\begin{tikzcd} \mathrm{St}(B) & \mathrm{St}(A) \arrow[l, two heads, "\mathrm{St}(m)"'] \\ \mathrm{St}(C) \arrow[u, "\mathrm{St}(f)"] & \mathrm{St}(D) \arrow[l, "\mathrm{St}(k)"'] \arrow[u, "\mathrm{St}(g)"'] \end{tikzcd}$$

in $\mathscr{S}tone$ is a pullback and the map $\mathrm{St}(m)$ is epic. It is easily shown that, in $\mathscr{S}tone$, pullbacks of epics are epic, and so $\mathrm{St}(k)$ is epic. It follows that its dual k is monic. Assuming **RET**, C is an absolute subretract, so there is $l : D \longrightarrow C$ such that $l \circ k = \mathrm{id}_C$. Then $l \circ g : A \longrightarrow C$ and $l \circ g \circ m = l \circ k \circ f = \mathrm{id}_C \circ f = f$. Hence C is injective, and **INJ** follows. Thus, if it could be *constructively* demonstrated that, in $\mathscr{B}ool$, pushouts of monics are monic, then **INJ** would be constructively equivalent to **RET**, and so persistent. I have not, however, been able to formulate a constructive proof of this former assertion.

We will now prove

THEOREM 4.5. **RET** *is persistent.*

PROOF. Let S be a coverable full type theory in a language $\mathscr{L}$. Suppose that B is an $\mathscr{L}$-set for which $\vdash_S$ *B is a Boolean algebra.* We endow $B^\sim$ with the partial ordering $\leq$ inherited from B in the natural way: i.e., for $b, b' \in B^\sim$ we define $b \leq b'$ to mean $\vdash_S b \leq_B b'$. Using the fullness of S, it is easy to to show (along the lines of the Completeness Principle) that with this ordering $B^\sim$ is a complete Boolean algebra. To simplify notation write $B^\diamond$ for $B^{\sim\frown}$. We again employ the fullness of S to yield a term $\tau_B(x)$, $x : B^\diamond$ such that $\vdash_S \tau(b^\frown) = b$ for all $b \in B^\sim$. We now claim that

(A) $\quad \vdash_S (B^\diamond, \leq^\frown)$ *is a Boolean algebra and τ_B is an epimorphism $B^\diamond \longrightarrow B$.*

To prove (A) we argue in S. Deriving the first conjunct of (A) and the fact that τ_B is a homomorphism $B^\diamond \longrightarrow B$ is a straightforward matter using the Generalization and Isomorphism Principles for full type theories: we omit the details. The crucial thing is to show that τ_B is S-derivably epic. To do this we need to prove $\vdash_S \forall y \in B \exists x.\ \tau_B(x) = y$. Since S is coverable, this will follow if it can be shown that, for each S-singleton U in B,

$$(*) \qquad \vdash_S \forall y \in U \exists x.\ \tau_B(x) = y.$$

To this end let u be the supremum of U in B. Then clearly $\vdash_S \forall y \in U.\ u = y$ so that $\vdash_S \forall y \in U.\ \tau_B(u^\frown) = y$, and (*) follows.

Thus (A) is proved. Now suppose that A, B, m are $\mathscr{L}$-sets such that

$\vdash_S$ *A, B are Boolean algebras, B is complete, and $m : B \rightarrowtail A$ is a monomorphism.*

It is easily checked that the usual argument establishing the existence of the *normal completion* of an arbitrary Boolean algebra can be carried out constructively (this fact is mentioned in [3, 1.6]). Thus the assertion *any Boolean algebra can be embedded in a complete Boolean algebra* is S-derivable. So let C, n be $\mathscr{L}$-sets for which

$$\vdash_S C \text{ is a complete Boolean algebra and } n : A \rightarrowtail C \text{ is a monomorphism.}$$

Write $p = n \circ m$. Thus it is S-derivable that *p is a monomorphism $B \longrightarrow C$*. Let $p^\sim : B^\sim \longrightarrow C^\sim$ be the natural map induced by p, i.e., the unique map $B^\sim \longrightarrow C^\sim$ satisfying $\vdash_S p^\sim(b) = p(b)$ for all $b \in B^\sim$. It is readily checked that, since p is S-provably a monomorphism, $p^\sim$ is itself a monomorphism. Now, assuming **RET** holds in the underlying set theory, there is a homomorphism $h : C^\sim \longrightarrow B^\sim$ such that $h \circ p^\sim = \mathrm{id}_{B^\sim}$. Note that, since $\vdash_S h(1_C) = 1_B$, we have, for $c \in C^\sim$,

$$(**) \qquad c = 1_C \vdash_S c = 1_C \wedge h(1_C) = 1_B \vdash_S h(c) = 1_B.$$

Using the fullness of S, we get terms λ, κ such that $\vdash_S \lambda(c^\frown) = h(c)^\frown$ for $c \in C^\sim$ and $\vdash_S \kappa(b^\frown) = p^\sim(b)^\frown$ for $b \in B^\sim$. It is then readily checked, using the Universality Principle, that (writing $C^\diamond$ for $C^{\sim\frown}$), the following statement is S-derivable:

$\lambda : C^\diamond \longrightarrow B^\diamond$ and $\kappa : B^\diamond \longrightarrow C^\diamond$ are homomorphisms and $\lambda \circ \kappa$ is the identity on $B^\diamond$.

Now define the $\mathscr{L}$-set

$$h' =_{\mathrm{df}} \{\langle y, z\rangle \in C \times B : \exists x \in C^\diamond [y = \tau_C(x) \wedge z = \tau_B(\lambda(x))]\}.$$

Since $\vdash_S \tau_C$ is epic by (A), it follows that $\vdash_S \forall y \in C \exists z.\ \langle y, z\rangle \in h'$. Moreover, for $c, d \in C^\sim$, we have

$$\begin{aligned}\vdash_S \tau_C(c^\frown) = \tau_C(d^\frown) \to c = d \to (c \Longleftrightarrow d) = 1_C &\to h(c \Longleftrightarrow d) = 1_B (\text{by } (**)) \\ &\to h(c) = h(d) \\ &\to \tau_B(h(c)^\frown) = \tau_B(h(d)^\frown) \\ &\to \tau_B(\lambda(c^\frown)) = \tau_B(\lambda(d^\frown)).\end{aligned}$$

Therefore, by the Universality Principle,

$$\vdash_S \forall xy[\tau_C(x) = \tau_C(y) \to \tau_B(\lambda(x)) = \tau_B(\lambda(y))],$$

and it follows that $\vdash_S$ *h' is a map from C to B*. And since, S- derivably, λ, τ_C and τ_B are all homomorphisms, it is easy to verify that h' is also.

We claim finally that $\vdash_S h' \circ p = \mathrm{id}_B$. For clearly the diagram of maps in S

$$\begin{tikzcd} B^\diamond \arrow[r, "\kappa"] \arrow[d, "\tau_B"'] & C^\diamond \arrow[r, "\lambda"] \arrow[d, "\tau_C"'] & B^\diamond \arrow[d, "\tau_B"] \\ B \arrow[r, "p"] & C \arrow[r, "h'"] & B \end{tikzcd}$$

commutes (S-derivably) and the upper composite is the identity on B°. So $\vdash_S h' \circ p \circ \tau_B = \tau_B$; since τ_B is S-derivably epic it follows that $\vdash_S h' \circ p = \mathrm{id}_B$ as claimed.

Accordingly $h' \circ n : A \longrightarrow B$ satisfies $\vdash_S (h' \circ n) \circ m = \mathrm{id}_B$, which shows that **RET** holds in S. $\dashv$

In [5] it is shown, in a classical setting, that **INJ**, and hence also **RET**, is equivalent to what I shall call the *maximal filter principle*:

(**MFP**) For any Boolean algebras A, B such that B is a subalgebra of A, there is a filter F in A maximal with respect to the property $F \cap B = \{1\}$.

(It is mentioned in [2] that this equivalence is also due independently to W. A. J. Luxemburg.) We conclude by establishing, using means quite different from those employed in [5], the *constructive* equivalence of **RET** with a certain naturally weakened (but classically equivalent) form of **MFP**, thus producing another example of a persistent statement.

All of the definitions, proofs, etc. to follow will be constructive.

Let B be a subalgebra of a Boolean algebra A. A filter F in A is called B-*maximal* if it is maximal with respect to the property $F \cap B = \{1\}$. Using the fact that, for any $a \in A$, the filter generated by $F \cup \{a\}$ is $\{y \in A : \exists x \in F.\ x \wedge a \leq y\}$, it is easily shown that, writing B_x for $\{y \in B : x \leq y\}$,

(4.6) F is B-maximal iff, for all $a \in A$, $F \cap B = \{1\}$ and $(\forall x \in F.\ B_{x \wedge a} = \{1\}) \rightarrow a \in F$.

We now make the following

ASSUMPTION. B is a subalgebra of A, and B is complete. We write $\bigvee$, $\bigwedge$ for the join and meet operations in B.

LEMMA 4.7. *Suppose that F is a B-maximal filter in A. Let $a \in A$, $b \in B$, $X \subseteq B$, and suppose that $\bigwedge X \leq b$. If $a \Rightarrow x \in F$ for all $x \in X$, then $a \Rightarrow b \in F$.*

PROOF. Suppose that $a \Rightarrow x \in F$ for all $x \in X$. To show that $a \Rightarrow b \in F$ it suffices, by (4.6), to show that, if $y \in F$, $c \in B$ and $y \wedge (a \Rightarrow b) \leq c$, then $c = 1$. Assuming these hypotheses, we have $y \wedge (a^* \vee b) \leq c$, so that

$$y = [y \wedge (a^* \vee b)] \vee [y \wedge (a \wedge b^*)] \leq c \vee (a \wedge b^*) = (c \vee a) \wedge (c \vee b^*),$$

whence $(c \vee a) \wedge (c \vee b^*) \in F$. Therefore (1) $c \vee a \in F$, and (2) $c \vee b^* \in F$. From (1) we get $c^* \Rightarrow a \in F$; since $a \Rightarrow x \in F$ for all $x \in X$ by assumption, we deduce that $c^* \Rightarrow x \in F$ for all $x \in X$. Since $c^* \Rightarrow x \in B$, it follows that $c^* \Rightarrow x = 1$, whence $c^* \leq x$ for all $x \in X$. Therefore $c^* \leq \bigwedge X \leq b$, so that $b^* \leq c$. From (2) we get, since $c \vee b^* \in B$, $c \vee b^* = 1$ whence $b \leq c$. This, together with $b^* \leq c$, yields $c = 1$ as required. $\dashv$

We shall call a homomorphism $f : A \longrightarrow B$ a B-*retraction* if the restriction of f to B is the identity.

THEOREM 4.8. *Let F be a B-maximal filter in A. Then the map $f : A \longrightarrow B$ defined by*

$$f(a) = \bigwedge\{x \in B : a \Rightarrow x \in F\}$$

is a B-retraction.

Conversely, if $g : A \longrightarrow B$ is a B-retraction, then $g^{-1}(1)$ is a B-maximal filter in A.

These correspondences establish a bijection between B-maximal filters in A and B-retractions $A \longrightarrow B$.

PROOF. Let F be B-maximal in A. We show first that, for $a \in B, b \in B$,

(i) $f(a) \leq b \longleftrightarrow a \Rightarrow b \in F$.

(ii) $b \leq f(a) \longleftrightarrow b \Rightarrow a \in F$.

For (i), we need only prove the "$\rightarrow$" direction, the reverse being obvious. If $f(a) \leq b$, then, writing $X = \{x \in B : a \Rightarrow x \in F\}$, we have $\bigwedge X = f(a) \leq b$. It now follows from Lemma 4.7 that $a \Rightarrow b \in F$.

For (ii) suppose that $b \Rightarrow a \in F$ and $x \in B$ satisfies $a \Rightarrow x \in F$. Then $b \Rightarrow x \in F$, so $b \Rightarrow x = 1$ (since $b \Rightarrow x \in B$), whence $b \leq x$. It now follows from the definition of $f(a)$ that $b \leq f(a)$.

Conversely, suppose that $b \leq f(a)$. By (4.6), to derive $b \Rightarrow a \in F$ it suffices to show that, for, $x \in F$, $c \in B$, if $x \wedge (b \Rightarrow a) \leq c$, then $c = 1$. Assuming these hypotheses, we have $x \wedge (b^* \vee a) \leq c$ so that

$$x = [x \wedge (b^* \vee a)] \vee [x \wedge (b \wedge a^*)] \leq c \vee (b \wedge a^*) = (c \vee b) \wedge (c \vee a^*).$$

Therefore (1) $c \vee b \in F$ and (2) $c \vee a^* \in F$. From (1), since $c \vee b \in B$, it follows that $c \vee b = 1$, whence $b^* \leq c$. From (2) we get $a \Rightarrow c \in F$, so that $b \leq f(a) \leq c$. This and $b^* \leq c$ give $c = 1$ as required.

It is easy to see that f is order preserving and, using $F \cap B = \{1\}$, that it is the identity on B. So to show that it is a B-retraction, it suffices to show that, for $a, a' \in A$,

(iii) $f(a \vee a') \leq f(a) \vee f(a')$

(iv) $f(a) \wedge f(a^*) = 0$.

For (iii), we observe that, using (i), for $b \in B$,

$$\begin{aligned} f(a) \vee f(a') \leq b \longrightarrow a \Rightarrow b \in F \ \&\ a' \Rightarrow b \in F &\longrightarrow (a \vee a') \Rightarrow b \in F \\ &\longrightarrow f(a \vee a') \leq b. \end{aligned}$$

For (iv), we observe that, using (i) and (ii), for $b \in B$,

$$b \leq f(a) \longrightarrow b \Rightarrow a \in F \longrightarrow a^* \Rightarrow b^* \in F \longrightarrow f(a^*) \leq b^* \longrightarrow b \wedge f(a^*) = 0.$$

We have therefore shown that f is a B-retraction.

Now suppose that we are given a B-retraction $g : A \longrightarrow B$, and let $F = g^{-1}(1)$. If $b \in B$, then $b \in F \longrightarrow b = g(b) = 1$, so $B \cap F = \{1\}$. To show that F is B-maximal we use (4.6). Thus suppose that $a \in A$ and, for any $x \in F$, $B_{x \wedge a} = \{1\}$. We need to conclude that $a \in F$, i.e., $g(a) = 1$. Now $g(a) = g(g(a))$ so that $g(a \Rightarrow g(a)) = g(a) \Rightarrow g(g(a)) = 1$. Therefore $a \Rightarrow g(a) \in F$. But $(a \Rightarrow g(a)) \wedge a \leq g(a)$ so since $B_{x \wedge a} = \{1\}$ for any $x \in F$, in particular for $x = a \Rightarrow g(a)$, it follows that $g(a) = 1$, as required. So F is B-maximal.

Finally, it is easily established that the correspondences between B-maximal filters and B-retractions thus formulated are mutually inverse. ⊣

Let **MFP**$^{\sim}$ be the weakened (but in fact classically equivalent) form of **MFP** in which B is required to be *complete*. We assume that **MFP**$^{\sim}$ has been provided with a type-theoretic formulation similar to those for **INJ** and **RET**. In view of the constructive equivalence of **MFP**$^{\sim}$ and **RET** establshed in Theorem 4.8, and the persistence of **RET** established in Theorem 4.5, we may finally state the

COROLLARY 4.9. **MFP**$^{\sim}$ *is persistent.* ⊣

REFERENCES

[1] B. BANASCHEWSKI, *Compact regular frames and the Sikorski theorem*, ***Kyungpook Mathematical Journal***, vol. 28 (1988), no. 1, pp. 1–14.

[2] ———, *On the injectivity of Boolean algebras*, ***Commentationes Mathematiae Universitatis Carolinae***, vol. 34 (1993), pp. 501–511.

[3] B. BANASCHEWSKI and K. R. BHUTANI, *Boolean algebras in a localic topos*, ***Mathematical Proceedings of the Cambridge Philosophical Society***, vol. 100 (1986), pp. 43–55.

[4] J. L. BELL, *On the strength of the Sikorski extension theorem for Boolean algebras*, this JOURNAL, vol. 48 (1983), no. 3, pp. 841–846.

[5] ———, *Some propositions equivalent to the Sikorski extension theorem for Boolean algebras*, ***Fundamenta Mathematicae***, vol. 130 (1988), pp. 51–55.

[6] ———, ***Toposes and local set theories: An introduction***, Oxford University Press, 1988.

[7] P. R. HALMOS, ***Lectures on Boolean algebras***, Van Nostrand, 1963.

[8] P. T. JOHNSTONE, *Conditions related to de Morgan's law*, ***Applications of sheaves***, Springer Lecture Notes in Mathematics, no. 753, Springer-Verlag, 1979, pp. 479–91.

[9] G. P. MONRO, *On generic extensions without the axiom of choice*, this JOURNAL, vol. 48 (1983), no. 1, pp. 39–52.

[10] R. SIKORSKI, *A theorem on extensions of homomorphisms*, ***Annales de la Societé Pol. de Mathématiques***, vol. 21 (1948), pp. 332–35.

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Theorem 3.1. can be strengthened. Let S be a (well-termed) local set theory and $(E, \leq)$ a partially ordered S-*set.* An element m of E is *internally maximal* if it satisfies

$$\vdash_S \forall x \in E\, [m \leq x \to m = x].$$

We can then prove the

Theorem. Suppose Zorn's Lemma holds in the underlying set theory, and let S be a full, well-termed, coverable local set theory. Then, in S, any inductive partially ordered set has an internally maximal element. In short, Zorn's Lemma holds internally in S.

Proof. Suppose $(E, \leq)$ is an inductive partially ordered S-set. Then, by the argument in the proof of Thm. 3.1. of my JSL paper, the partially ordered set $E^{\sim}$ of global elements of E has a maximal element m. We claim that m is internally maximal. Since S is coverable, to establish this it suffices to show that, for any S-singleton U in E, we have

$$(*) \qquad \vdash_S \forall x \in U\, [m \leq x \rightarrow m = x].$$

Defining V to be the S-set $\{x \in U\colon m \leq x\}$, it is easily seen that (*) is equivalent to

$$(**) \qquad \vdash_S V \subseteq \{m\}.$$

1. Now consider $V' = V \cup \{m\}$. This is a chain in E (recall that V is a singleton), and so has a supremum v. Clearly $\vdash_S m \leq v$, so the maximality of m gives $\vdash_S m = v$. It follows that

$$(***) \qquad \vdash_S x \in V \rightarrow x \leq v \rightarrow x \leq m.$$

But since $\vdash_S x \in V \rightarrow m \leq x$, (***) yields

$$\vdash_S x \in V \rightarrow x = m,$$

i.e. (**). ■

This result can also be stated as follows: if Zorn's Lemma holds in the base topos $\mathcal{S}et$ of sets, it continues to hold internally in any localic lopos defined over $\mathcal{S}et$.