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Newton's Principia

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1.1 Introduction

In the Preface to his *Mathematical Principles of Natural Philosophy* (hereafter, *Principia*) Newton announces a striking new aim for natural philosophy and expresses optimism that the aim can be achieved (Newton, 1726, pp. 382-83):

> For the basic problem of philosophy seems to be the discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces. It is to these ends that the general propositions in books 1 and 2 are directed, while in book 3 our explanation of the system of the world illustrates these propositions. For in book 3, by means of propositions demonstrated mathematically in books 1 and 2, we derive from the celestial phenomena the gravitational forces by which bodies tend toward the sun and toward the individual planets. Then the motions of the planets, the comets, the moon, and the sea are deduced from these forces by propositions that are also mathematical. If only we could derive the other phenomena of nature from mechanical principles by the same kind of reasoning!

Often pronouncements like this reveal a contrast between the goals of a given scientist, however astute, and the subsequent historical development of the field. But this is not so in Newton's case: Newton's work effectively reoriented natural philosophy for generations. The *Principia*, which appeared in three editions (1687, 1713, 1726) clarified the concept of force used in physical reasoning regarding motion and marshaled evidence for one such force, gravity. We read the *Principia* to be guided by Newton's evolving recognition of various challenges to evidential reasoning regarding forces, and his development of the tools needed to respond to these challenges. The mathematical results Newton achieved provided an initial framework in which to pursue the project of discovering forces and finding further evidence in favor of gravity. He called attention to the role gravity played in a wide variety of natural phenomena (i.e., "the motions of the planets, the comets, the moon, and the sea"). He gave a precise, quantitative treatment of phenomena that had previously been the subject of inchoate speculation, such as the perturbing effects of planets on each other. He decisively settled the great

[1] We thank Bill Harper and George Smith for helpful discussions, and especially Erik Curiel and Niccolo Guicciardini for detailed comments on earlier drafts. The usual caveats apply. Sections 1.3.6 and 1.4.1 overlap with forthcoming publications by one of us: Eric Schliesser "Newton and Newtonianism" in the *Oxford Handbook to Eighteenth Century British Philosophy*, edited by James Harris, Oxford: Oxford University Press, and Eric Schliesser "Newton and European Philosophy" in *The Routledge Guide to Eighteenth Century Philosophy*, edited by Aaron Garrett, London: Routledge.

unresolved cosmological question of his era, the status of the Copernican hypothesis. He defended a Copernican-Keplerian account of planetary motions, and showed on the basis of universal gravity that the the sun itself moves, albeit not far from the common center of gravity of the solar system (Prop. 3.12).[2] The impact of universal gravity on the subsequent study of celestial mechanics is hard to overstate: Newton's conception of gravity remains a durable part of celestial mechanics, even as it was augmented in the eighteenth century and corrected by Einstein's theory of general relativity in the twentieth century.

Alongside this achievement in laying the mathematical foundations for mechanics, the *Principia* also exemplifies a "new way of inquiry." In its mathematical style and approach to mechanics, the *Principia* most closely resembles Christian Huygens's *Horologium Oscillatorium*, which Newton greatly admired. Just as Huygens had generalized and considerably enriched Galileo's results in mechanics, Newton generalized Huygens treatment of uniform circular motion to an account of forces applicable to arbitrary curvilinear trajectories. Newton then used this enriched Galilean-Huygensian approach to mechanics to describe the motions of the planets and their satellites rather than just bodies near the earth's surface. Newton characterized his methodology as offering more secure conclusions than those reached via the hypothetical methods of his contemporaries. Newton's claim to achieve *greater* certainty, given the centrality of a theoretically defined entity like force to his approach, was controversial at the time and has remained so.

Newton introduced a striking new goal for natural philosophy — "to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces" —, articulated the challenges to achieving this goal, and developed an innovative and sophisticated methodology for overcoming the challenges. Newton's contemporaries and later readers have had great difficulty comprehending Newton's new way of inquiry. Newton's own attempts to clarify his new "experimental philosophy" late in his life, in response to controversy, are often too cryptic to provide much illumination. The growing authority of Newtonian science, and the stunning reach and apparent certainty of the claims it makes, made Newton's methodology one of the most contested areas in eighteenth century philosophy. Below we will draw on recent scholarship on the *Principia* and emphasize three interwoven aspects of Newton's approach. First, Newton establishes mathematical results that license inferences regarding forces that are robust in the sense that they do not require that claims regarding phenomena hold *exactly.* Second, Newton identifies the various assumptions needed to define a tractable mathematical model, and then assesses the consequences of relaxing these idealizations. And, finally, the initial idealized model serves as the first step towards a more sophisticated model, with the deviations between the idealized case and observed phenomena providing further empirical input.

This chapter discusses the historical context of the *Principia*, its contents, and its impact, with a primary focus on the issues just described. We should acknowledge at the outset two of the important topics that we do not have space to give the

[2]This refers to Proposition 12 from the *Principia*'s Book 3; we will generally refer to propositions in this way, unless the relevant book is clear from context. All page references and quotations are from (Newton, 1726).

treatment they deserve. First, and most important from the standpoint of the history of physics, we will leave aside entirely Newton's optical works. The *Opticks* differs from the *Principia* in much more than subject matter; in it Newton elucidates a sophisticated experimental approach, with an expository style that is accessible and free from the *Principia*'s daunting mathematics. The *Opticks* engendered a research tradition of "Newtonian philosophy" of a different character than that produced by the *Principia.* Of course, many of Newton's readers interpreted the *Principia* in light of the *Opticks*, especially the more speculative "Queries," as we will occasionally do below.

Second, we also do not aim to give an account of Newton's overall philosophical views or to place the *Principia* in the context of his other intellectual pursuits. The last half century has seen a renaissance in Newton scholarship due in part to the assimilation of newly available manuscript sources. These manuscripts reveal that his published work occupied only a fraction of Newton's prodigious intellectual energy. He pursued alchemy, biblical chronology, and theology with the same seriousness of purpose as the work culminating in the *Principia.* Newton did not regard these pursuits as completely isolated from one another, and neither do we. Various scholars have undertaken the ambitious project of understanding the genesis of the *Principia* in relation to these other pursuits. Earlier generations of historians have often dismissed these other aspects of Newton's thought out of hand. But Newton himself took the opportunity to publicly announce some of the connections between his work in natural philosophy and broader questions in philosophy and theology in the General Scholium added to the second edition of the *Principia* and the Queries in the *Optics.* He also considered making these views more explicit in substantial revisions to parts of the *Principia.* He held his cards close to his chest with regard to his dangerously heterodox theological views and his work in alchemy. As a result, aside from Newton's closest colleagues, few of his contemporaries or successors had access to Newton's views on these other topics. There are a number of fascinating and difficult questions that face any attempt to give a systematic treatment of Newton's philosophical and theological views. But given our aim of elucidating the impact of the *Principia* on the development of physics, we focus below on the *Principia* itself and related published texts that would have been widely available to Newton's contemporaries and successors.

1.2 Historical Context

The seventeenth century saw the emergence of a new systematic approach to natural philosophy, called the "mechanical philosophy," which offered explanations of natural phenomena in terms of matter in motion without appeal to Aristotelian forms. By mid-century Hobbes, Gassendi, and Descartes had elaborated and defended significantly different versions of this view. Descartes' *Principles of Philosophy* (1644) was by far the most influential on Newton, and thus sets the context for his own distinctive natural philosophy. Newton's very title alludes to the earlier *Principia.* And as his title offers an implicit correction to Descartes, so too throughout the *Principia* Newton is often at pains to distinguish his views from those of Descartes and his followers. One crucial part of Newton's intellectual context was his critical engagement with a Cartesian version of the mechanical philosophy.

The ambition and scope of Descartes' *Principles* is breathtaking: it offers a unified physics, cosmology, and geology, including mechanical explanations of everything from magnetism to earthquakes. Planets move as they do because they are embedded in a whirling vortex of subtle matter, according to Descartes, and the interaction among vortices associated with different stars explains various other phenomena, such as comets. This entire system was meant to follow from an austere set of basic principles regarding the nature of bodies and laws governing their motion. Few readers were convinced by Descartes' claim to have deduced these laws of nature from his metaphysical first principles, and even a quick reading of Parts III and IV reveals that the connection between the systematic account of nature offered there and the basic principles in Part II leaves more room for the free play of the imagination than Descartes allows. But it provided a compelling research agenda for the mechanical philosophy. Hypothetical explanations would be judged to be intelligible provided that they invoked only the size, shape, and motion of the fundamental constituents of a system, moving according to fixed laws. In the new mechanical philosophy, the laws of motion, including rules for the collision of bodies, would be the linchpin of theorizing about nature.

Within two decades many leading natural philosophers had grown dissatisfied with Descartes' analysis of collision, which was widely seen as empirically and conceptually inadequate. By adopting Galilean principles, Christiaan Huygens was able to give a satisfactory analysis of collision by the 1660s. (Newton privately had reached the same conclusions.) During that decade a consensus grew around his mathematical treatment when the Royal Society published short pieces by Wallis and Wren leading to similar conclusions (even if they differed in their metaphysical presuppositions). Huygens argued that the quantity preserved in a collision is not mv (as Descartes had supposed) but mv^2, which Leibniz later coined 'vis viva'. Mariotte's *Traitté de la percussion ou chocq des corps* (1673) gave the decisive rejection of Descartes' laws of collision. Yet, the Cartesian mechanical philosophy with its appeal to intelligibility and simplicity of hypothetical explanations (including modified vortex theories and the denial of a vacuum) held on well into the eighteenth century.

While the mechanical philosophy offered the most unified approach to nature, it was not uncontested. In particular, its commitment to the passivity of matter was repudiated by many important natural philosophers. In his influential *De Magnete*, William Gilbert introduced the idea of an "orb of virtue" in describing a body's magnetism. Gilbert's work directly influenced Kepler's discussion of the distant action of the Sun on a planet, ruled unintelligible by strict mechanists. Throughout the century there were numerous proposals that posited action at a distance (including theories developed by Roberval and Hooke), some of these inspired by the publication at mid-century of Gilbert's posthumous *De mundo nostro sublunari philosophia nova*. One author closely studied by Newton, Walter Charleton, who helped revive Epicurean theory in England, advocated a view of matter with innate principles of activity akin to attractive powers. At Cambridge University, the so-called Cambridge Platonists allowed that matter was passive but under strict control of mind-like spiritual substances. Against the Cartesian identification of indefinite space and extension (and its denial of a vacuum), Henry More developed a view of infinite space (and time) as an immaterial entity that emanated from God. More insisted that all entities, material and immaterial, including God, occupied some place in space. This view was also adopted by Newton.

By the 1660s, when Newton began his study of natural philosophy, the most sophisticated natural philosopher was Huygens. Huygens significantly extended Galileo's study of accelerated motion and developed the idea of relativity principles in his derivation of the laws of elastic collision. In the 1650s Huygens advanced the Galilean program by creating a theory-mediated measurement of the acceleration of bodies in the first half second of fall at Paris that was accurate to four significant figures, using different kinds of pendulums. Huygens' approach was aided by his ground-breaking mathematical analysis of the pendulum. Huygens' discovery that the cycloid was an isochronous curve opened up precise time-keeping (useful in astronomy, geography, and mechanics) and fueled the search for a practical solution to finding longitude at sea. It also raised the question of whether gravitation was uniform around the globe. The strength of surface gravity (reflected in the length of a seconds pendulum) varied with latitude, but without apparent systematicity. Huygens's *Horologium Oscillitorium* (1673) represented the state of the art in mechanics before the *Principia*, and Newton greatly admired the book.

A second aspect of Newton's intellectual context was the development of predictive astronomy. Kepler's innovations were mostly neglected by Cartesian philosophers, in part due to his problematic mix of neo-Platonism and ideas that were incompatible with the mechanical philosophy. Cartesian philosophers did not develop quantitatively precise versions of the vortex theory to rival Kepler's account; in the context of Descartes's theory, it was after all not clear whether the planetary orbits exhibit stable regularities or are instead temporary features subject to dramatic change as the vortex evolves. Huygens, despite work in observational astronomy including the discovery of Saturn's moon, Titan, only published a detailed cosmology in response to Newton's *Principia*. Kepler's work did set the agenda for those interested in calculating the planetary tables, and his proposals led to a substantial increase in accuracy. Kepler's *Rudolphine Tables* received a major boost when Gassendi observed the transit of Mercury in 1631 in Paris. The gifted English astronomer, Jeremiah Horrocks observed a transit of Venus in 1639, but his work remained little known during his (brief) life. Despite the success of Kepler's innovations in leading to more accurate planetary tables, his physical account of planetary motion was controversial — Boulliau, for example, dismissed Kepler's physical account as "figments."[3] When Newton began his study of astronomy with Streete's *Astronomia Carolina*, there was an active debate underway regarding the best way of calculating planetary orbits. Kepler had motivated what we call his "area law" on physical grounds, but Boulliau, Streete, and Wing had each proposed alternative methods for calculating planetary positions with comparable levels of accuracy. It is certainly not the case that astronomers prior to the *Principia* took "Kepler's three laws" to reflect the essential properties of planetary orbits that should inform any physical account of their motion. The first to single out

[3] Here we draw on Wilson (1970)'s description of the debates in assimilating Kepler's innovations in astronomy and Newton's responses to it; see also (Smith, 2002). (See p. 107 of Wilson's paper regarding Boulliau's criticism of Kepler.)

"Kepler's laws" was in fact Leibniz (1689), who perhaps intended to elevate Kepler's contributions at Newton's expense.

Newton stands at the convergence of Keplerian astronomy and Galilean-Huygensian mechanics, uniquely able to use the latter to provide firmer physical footing for the former because of his enormous mathematical talent. The third aspect of Newton's intellectual context is the development of mathematics and the central role of mathematics in his new mode of inquiry. Descartes is again the pivotal figure; Newton's early work was guided by his close study of van Schooten's second edition of the *Géométrie*.[4] Among the central problems in mathematics at the time were the determination of the tangent to a given curve and quadrature (finding the area under the curve), for curves more general than the conic sections. Descartes and others had solved these problems for a number of special cases, but from 1664 - 1671 Newton developed a general algorithm for solving these problems and discovered the inverse relation between finding the tangent and performing quadratures — and in that sense he "invented the calculus." His generalization of the binomial theorem and use of infinite series allowed him to handle a much broader class of curves than those treated by Descartes. While this is not the place to review these contributions in more detail (see, in particular, Whiteside's *Mathematical Papers* and Guicciardini 2009), we will briefly describe the distinctive mathematical methods employed in the *Principia* below.

Newton's mathematical talents and new techniques enabled him to tackle quantitatively a much wider range of problems than his contemporaries. But equally important was Newton's view that the judicious use of mathematics could be used to reach a level of certainty in natural philosophy much greater than that admitted by the mechanical philosophers. Descartes and others regarded the mechanical models they offered as intelligible and probably accurate, but Newton claimed to be able to achieve more certainty. Newton formulated this view quite stridently in his *Optical Lectures*:[5]

> Thus although colors may belong to physics, the science of them must nevertheless be considered mathematical, insofar as they are treated by mathematical reasoning. ... I therefore urge geometers to investigate nature more rigorously, and those devoted to natural science to learn geometry first. Hence the former shall not entirely spend their time in speculations of no value to human life, nor shall the latter, while working assiduously with an absurd method, perpetually fail to reach their goal. But truly with the help of philosophical geometers and geometrical philosophers, instead of the conjectures and probabilities that are blazoned about everywhere, we shall finally achieve a science of nature supported by the highest evidence.

Here the level of certainty to be attained contrasts with that of the Cartesian program and also with that of Newton's immediate contemporaries, members of the Royal Society such as Hooke and Boyle. The degree of certainty Newton had achieved with his "New Theory of Light and Colors" (1672) soon became the focus of a contentious debate, drawing in Hooke and Huygens, among others. But Newton continued to advocate the importance of mathematics in his new way of inquiry, and we next turn

[4] This edition, published in 1659 (the original appeared in 1637) contained extensive supplementary material, including correspondence between Descartes and other mathematicians and further work by Van Schooten's Dutch students – including Hudde, Heuraet, and de Witt (later the leader of the Dutch republic) – on problems posed by Descartes.

[5] The lectures were deposited in October 1674 and perhaps delivered in the period 1670-72. The quotation is from *Optical Papers*, Volume 1, pp. 87, 89; for further discussion of Newton's position in relation to his optical work and more broadly, see Guicciardini (2009), Chapter 2; Shapiro (2002, 2004); and Stein (ms).

to a study of the contents of the *Principia* and the essential role of mathematics in enabling his deduction of gravity from the phenomena of celestial motions.

1.3 Overview of the *Principia*

1.3.1 From *De Motu* to the *Principia*

Newton took the first steps toward writing the *Principia* in response to a problem posed by Edmond Halley in the summer of 1684.[6] Christopher Wren had offered Halley and Robert Hooke the reward of a "forty-shilling book" for a proof that elliptical planetary trajectories follow from a force varying as the inverse square of the distance from the sun. The challenge proved too great for Halley and Hooke, and Halley consulted Newton on a visit to Cambridge.[7] Newton replied in a 9 page manuscript that November, bearing the title *De Motu Corporum in Gyrum* (hereafter *De Motu*). Newton's results in this brief paper alone would have secured him not only Wren's reward but a place in the history of mechanics, and we will describe its contribution as a prelude to the *Principia*.[8] The most striking contribution is bringing together the Galilean-Huygensian tradition in mechanics with astronomy, unified via the new conception of centripetal force. But committing these initial insights to paper was only the first step in a line of inquiry that Newton would pursue with incredible focus and insight for the next three years.

Halley, Hooke, and Wren had a plausible physical motivation for considering a force whose intensity decreases with the inverse square of distance from its source. Huygens's treatment of uniform circular motion in terms of centrifugal force combined with Kepler's "third law" implied that the force varies as the inverse square of the distance for an exactly circular trajectory.[9] What they lacked was a conceptualization of force sufficiently clear to allow them to relate this hypothesized variation with distance to a trajectory, and to assess the implications of this idea for an elliptical trajectory. In correspondence in 1679, Hooke had already pushed Newton to take an important step in the right direction, to conceiving of planetary trajectories as resulting from a tendency to move in a straight line combined with a deflection due to an external force.[10] But it was only in the *De Motu* that Newton combined this idea

[6] See (Cohen, 1971) for discussion of the circumstances leading to the publication of the *Principia*, including Halley's visit. Newton reportedly answered Halley's question during the visit, but could not find the paper where he had already performed the calculation. We do not know how this earlier calculation compared to the manuscript he then produced.

[7] It is now customary to distinguish between two related problems: the direct problem — *given* the orbit or trajectory, find a force law sufficient to produce it, and the inverse problem — *given* the force law and initial position and velocity, determine the trajectory. It is unclear precisely what problem Halley posed to Newton, but the *De Motu* addresses the direct problem.

[8] Here we emulate the effective presentation in (de Gandt, 1995), which also includes a discussion of the *Principia*'s mathematical methods and historical context.

[9] Kepler's third law states that $\mathsf{P}^2 \propto \mathsf{a}^3$ for the planets, where P is the period and a is the mean distance from the sun. For a discussion of the understanding of Kepler's "laws" among Newton's contemporaries, see (Wilson, 1970).

[10] See Hooke's correspondence with Newton in 1679-80 (in *Correspondence of Isaac Newton*, Volume 2), and his earlier work cited there. Hooke did not formulate inertial motion as Newton later would, in that he does not say that bodies move *uniformly* in a straight line. However, Hooke certainly deserves

with other insights to establish the connection between a given curvilinear trajectory and the force responsible for it.

The *De Motu*'s beautiful central result, Theorem 3, starts from a generalization of Galileo's treatment of free fall. Galileo established that under uniform acceleration, the distance traveled by a body starting at rest is proportional to the square of the elapsed time. What Newton required was a precise link between a quantitative measure of the trajectory's deviation from a straight line at each point of the orbit and the magnitude of the force producing this deviation, for forces whose magnitude varies from point to point. The initial draft stated a generalization of Galileo's result as an hypothesis: namely, the deviation produced by *any* centripetal force is proportional to the square of the elapsed time, "at the very beginning of its motion". The proof of this result as Lemma 10 in the *Principia* clarifies the importance of the last clause: it is only "ultimately" (or "in the limit" as the elapsed time goes to zero) that the proportionality holds. In effect, Galileo's result holds for *finite* elapsed times in the special case of uniform acceleration, but Newton recognized that it is valid *instantaneously* for arbitrary centripetal forces. Newton's next step established that the elapsed time is represented geometrically by the area swept out by a radius vector from the force center following the trajectory. Theorem 1 of the *De Motu* established this result, now known as Kepler's area law, granted Newton's conception of inertial motion and the restriction to centripetal forces that depend solely on the distance to a force center (i.e., central forces). (Newton would later establish the converse as well, namely that a body sweeping out equal areas in equal times around a given point experiences a net impressed force directed to that point.) Combining these two results leads to an expression relating the magnitude of the force to geometrical properties of the trajectory. In terms of Newton's diagram reproduced as Figure 1.3.1, the deviation produced by the force acting at point P is represented by the segment QR. This displacement is proportional to the product of the force F acting on the body with the square of the time elapsed, $\mathsf{QR} \propto \mathsf{F} \times \mathsf{t}^2$, as shown by the generalization of Galileo's law.[11] From Kepler's area law, $\mathsf{t} \propto \mathsf{SP} \times \mathsf{QT}$, and it follows that $\mathsf{F} \propto \frac{\mathsf{QR}}{\mathsf{SP}^2 \times \mathsf{QT}^2}$.

This result states an entirely general relation between the centripetal force law and the properties of a given trajectory. It holds instantaneously, in the limit as the point Q approaches the point P. But in applying this result, Newton establishes connections between the "evanescent" figure QRPT and finite quantities characterizing the trajectory, such as the radius of a circle or the *latus rectum* of an ellipse. This leads to an expression characterizing the force law that makes no reference to quantities that vanish in the limit as $\mathsf{Q} \to \mathsf{P}$. Consideration of the figure QRPT allows Newton to handle the differential properties of the curve geometrically. Along with other special cases, Newton noted that for motion on an ellipse with a force directed at one focus this result implies that the force varies inversely as the square of the distance from the focus.

Newton achieved a great deal in the few pages of *De Motu.* The central achievement is a unification of Kepler's treatment of planetary motion with the Galilean-Huygensian

more credit than Newton was willing to acknowledge for pushing him to treat curvilinear motion as resulting solely from inertia and a centripetal force.

[11] In terms of the *Principia*'s definitions, F is the accelerative measure of the force.

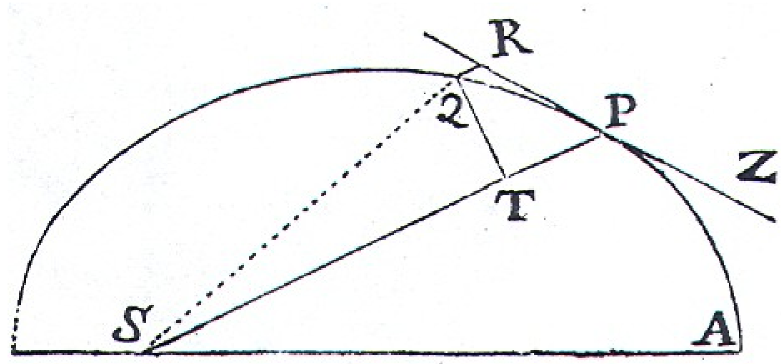

Fig. 1.1 Figure from Proposition 6 in the *Principia* (and in Theorem 3 of the *De Motu*).

theory of uniformly accelerated motion based on Newton's innovative treatment of force. Theorem 3 allows one to determine a force law sufficient for motion along a given trajectory with a given force center. Newton generalizes Huygens's earlier treatment of uniform circular motion to arbitrary curvilinear trajectories, and this opens the way for considering a variety of possible central forces sufficient for motion along different plane curves. In *De Motu* Newton took on two different types of problems: projectile motion in a resisting media, and the motion of celestial bodies under an inverse-square force law.

Kepler's area law is strikingly given special standing as a direct consequence of any central force, rather than just being one among many calculational devices in astronomy. Similarly, Newton's work clarifies the status of Kepler's first and third laws. According to Kepler's first law, the planets follow elliptical trajectories. An inverse square force directed at the focus is sufficient to produce this motion if, in addition, the second law holds with respect to the focus of the ellipse (that is, the radius vector from the focus sweeps out equal areas in equal times).[12] Thus, insofar as Kepler's laws hold exactly for each planet, one can infer an inverse square force between the sun and each of the planets. Kepler's third law is a specific instance of a general result linking periodic times to the exponent in the force law (Theorem 2). Furthermore, the ratio of the radii to the periods ($\mathsf{a}^3/\mathsf{P}^2$) is the same for all of the planets, leading to the conclusion that a *single* inverse-square force directed at the sun suffices for the motions of the planets. These results lead Newton to announce in a scholium that the planets move in their orbits "exactly as Kepler supposed." Although he soon recognized that this conclusion is too hasty, he had persuasively answered what may have been Halley's original query. Halley et al. would have naturally wondered whether the inverse-square force sufficient for perfectly circular orbits would have to be supplemented by a secondary cause to account for elliptical motion. Newton's results show convincingly that a simple inverse-square force alone is sufficient for Keplerian motion. (But these results do not establish that it is *necessary* for Keplerian motion.)

[12] Elliptical orbits are also compatible with a force law that varies directly with the distance if the area law holds with respect to the *center* of the ellipse rather than a *focus*, as Newton showed.

The *Principia* grew out of a number of questions provoked by *De Motu*.[13] Two questions would have been particularly pressing to Halley upon reading the manuscript. First, what do Newton's ideas imply regarding a glaring potential counter-example to the claim that all of the celestial bodies move in accord with Kepler's laws — namely, the moon? The moon's motion is far more complicated than that of the planets, and it was unclear whether Kepler's laws hold for the lunar motion as even a rough approximation. The challenge facing Newton was to see what an inverse-square force implies regarding lunar motion. Second, how do the ideas in *De Motu* apply to comets? The nature of comets was the focus of an active debate among Hooke, Halley, Flamsteed, Newton, and others.[14] A central issue in this debate was whether the regularities observed in planetary motion hold for comets as well. In the early 1680s Newton argued against the idea that various sets of observations could be described as those of a *single* body, with a sharp button-hook trajectory around the sun. In a draft letter to Flamsteed in 1681, he considered the possibility that comets move in this way as a result of an attractive force analogous to magnetism. But the advances of *De Motu* opened up the possibility of a precise quantitative treatment of cometary motion. Newton suggested a procedure for determining cometary orbits that would prove unworkable. A successful account of cometary motion based on an inverse-square force would show that the force effects the motion of bodies other than the planets, over a much wider range of distances from the sun than those explored by planetary orbits.

1.3.2 Definitions and the Laws of Motion

The opening part of the *Principia* extends and refines the ideas of *De Motu* in several significant ways. First, Newton gives a much more precise characterization of force in the three Laws of Motion.[15] A predecessor of the First Law appears in *De Motu* as Hypothesis 2: "Every body by its innate force alone proceeds uniformly to infinity in a straight line, unless it is impeded by something extrinsic." This formulation makes one important advance on earlier formulations of a inertial principle, such as that due to Descartes. Newton explicitly remarks that the motion will be *uniform*, covering equal distances in equal time, as well as rectilinear. The formulation in the *Principia* marks another important step, in that Newton drops the implication — present in all earlier formulations of the law — that an external impediment must be present to deflect a body from inertial motion. The formulation of the *Principia* requires only that an impressed force acts on the body, but this force is treated abstractly without commitments regarding its mode of operation or source.

The Second Law is implicit in *De Motu*'s quantitative treatment of forces as measured by the deflection from an inertial trajectory.[16] But it is not stated explicitly as it is in the *Principia*:

[13] Although we will not trace the details here, the original "De Motu" manuscript was extended step by step in a series of revisions leading up to the *Principia*; see (Herivel, 1965) and Volume 4 of the *Mathematical Papers*.

[14] See (Wilson, 1970, pp. 151-160) for a brief overview of the debate regarding comets.

[15] We are reversing the order of exposition in the *Principia*, where the definitions and the scholium on space and time, discussed below, precede the Laws.

[16] Actually, only part of Second Law is used: what is usually called the parallelogram law for the addition of forces, and the generalization of Galileo's treatment of acceleration discussed above.

Law II: A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.

The "change in motion" is measured by the deflection QR in the diagram above; that is, the distance between the point the body would have reached moving inertially and the point actually reached due to the action of an impressed force. "Motive force" is introduced in the definitions as one of the three measures of a centripetal force, given by the motion generated in a unit time. Earlier definitions stipulate that quantity of motion is given by the mass times the velocity, and hence the motive force measures the impressed force by the product of the mass of the body and the resulting acceleration. Newton applies the law to cases involving discrete impressed forces, such as impacts, as well as continuously acting forces.

In addition to the motive measure of force, Newton also introduces the accelerative and absolute measures. These three distinct measures quantify different aspects of a force: the absolute measure of force characterizes the overall strength of the force — in the case of gravity, this would correspond to the mass of the body producing the force. The accelerative measure characterizes the intensity of the force as a function of radial distance, as revealed in the acceleration it produces on *any* body at that distance.[17] Finally, the motive measure characterizes the force impressed upon a body of a given mass, and is given by the product of the mass of the body with the accelerative measure.

Newton claimed no originality for the first two Laws, and there are antecedents in earlier work, particularly Huygens's *Horologium Oscillatorium*.[18] But there are nonetheless two innovations in Newton's formulation that deserve emphasis. First, Newton treats force as an abstract mathematical quantity independent of any commitments regarding its physical sources. What this allows is the decomposition of a force into arbitrary oblique components, explicitly stated as Corollary 2 of the Laws. The importance of this move is easily overlooked from a modern perspective, but this Corollary is crucial to many of the results in Book 1. For example, Prop. 1.40 establishes that the velocity acquired falling through a given radial distance from a force center is independent of the path taken as a direct consequence of Corollary 2; Huygens, lacking a similarly abstract conception of force, had to give a more laborious proof of the same result (Proposition VIII of the *Horologium*). The second innovation is the introduction of a concept of "mass" distinct from the "weight" or "bulk" appealed to in earlier work.[19] Mass does not even appear in the initial draft of the *De Motu*; it only emerged in the course of clarifying the laws.[20] The definitions in

[17]Stein (1967) argues that for gravity this can be treated as an "acceleration field", but for other types of interaction this need not be the case.

[18]For more detailed discussion of Newton's formulation of the laws and his conception of force, see, in particular, (McGuire, 1968; McGuire, 1994; Gabbey, 1980), and Chapter 3 of (Janiak, 2008).

[19]Huygens introduced a concept close to Newtonian "mass" in formulating a law for momentum conservation in elastic impacts in 1669, in response to the Royal Society's competition regarding collision laws. Although Huygens's concept of mass agrees with Newton's in particular cases, Huygens did not formulate the concept at the same level of generality as Newton did in the *Principia*. See (McMullin, 1978) and Chapter 4 of (Janiak, 2008) for entry points into discussion of Newton's innovative concept of mass.

[20]It first appears in *De Motu Corporum in Mediis Regulariter Cedentibus*; see (Herivel, 1965).

conjunction with the first two laws make it clear that mass or quantity of matter is to be measured by a body's response to an impressed force, its resistance to acceleration. Newton then establishes experimentally that mass can be measured by weight.

The Third Law is entirely absent from the *De Motu*, although Newton had used it as an illustrative example in lectures on algebra much earlier.[21] There the law is formulated as the principle of conservation of linear momentum, but in work leading to the *Principia* Newton considered several different equivalent statements of the Third Law before deciding on the following:

> *Law III*: That to an action there is always a contrary and equal reaction; or, that the mutual action of two bodies upon each other are always equal and directed to contrary parts.

This formulation emphasizes that forces should be understood as interactions between bodies; to speak of *separate* impressed forces acting on two bodies, which happen to come in an action-reaction pair, is misleading. The "force" corresponds to a mutual interaction between bodies that is *not* broken down into separate "actions" and "reactions," except in our descriptions of it. Given its novelty, Newton's discussion in the scholium following the laws and their corollaries focuses on defending the Third Law. The examples used there illustrate the connection between the Third and First Laws. In order for the First Law to hold for the center of mass of a closed system of interacting bodies, the Third Law must hold for the interactions among the bodies.[22]

In *De Motu* the centripetal forces are treated just as accelerative tendencies towards a fixed center, without regard to whether they are produced by a body. The Third Law did not figure in the discussion, and indeed only the accelerative measure of the force is relevant. But in the *Principia* the Third Law supplements the first two laws by allowing one to distinguish apparent from real forces. Given a body in motion, the first two laws allow one to infer the existence of a force producing the motion that may be well defined quantitatively (given a definite magnitude and direction). But the Third Law further requires that the force results from a mutual interaction. The Coriolis force—the apparent force revealed, for example, in the deflection of a ball rolling along the floor of a carousel—illustrates this distinction: the force is well defined quantitatively and could be inferred from observing a body in motion using the first two laws of motion and results from Book 1, but there is no "interacting body" to be found as the source of the force. The first two laws figure primarily in treating forces from a mathematical point of view whereas the introduction of the Third Law marks an important physical constraint. Although Newton famously abstained from requiring a full account of the "physical cause or reason" of a force as a precondition for establishing its existence, any further account of the physical nature of the force would have to satisfy the constraint imposed by the Third Law.

Newton came to consider the implications of the Third Law for planetary motion soon after completing the first version of the *De Motu*. His initial results suggested that the motions of the planets resulted from an inverse-square centripetal force produced by the Sun. Similarly, the motion of the satellites of Jupiter suggested the existence of an independent centripetal force produced by Jupiter. How do these two forces relate

[21] See *Mathematical Papers*, Volume 5, pp. 148-149, f.n. 15.

[22] Newton's illustrative examples involve contiguous bodies, but he does not hesitate to extend the third law to cover interactions between non-contiguous bodies as well.

to one another? How far does their influence extend? How do they effect the motion of a comet? Newton clearly considered such questions before adding what Curtis Wilson has aptly called the "Copernican Scholium" to a later version of the *De Motu*. In this passage (quoted more fully below), Newton argues that the "common centre of gravity ... ought to be considered the immobile center of the whole planetary system," and adds that this proves the Copernican system *a priori*. In order to define the common center of mass of the solar system, Newton applied the Third Law to combine the distinct accelerative tendencies produced by the planets and the sun. Thus by this point Newton had taken one step towards universal gravity, by apparently treating the forces responsible for celestial motions as a mutual interaction between the sun and the planets.[23] But, as we will see shortly, with this Newton further recognized that the problem of inferring the force responsible for celestial motions from observations was far more challenging than the problem posed by Halley. The need to face this challenge squarely led to the more elaborate theory of generic centripetal forces developed over the course of Book 1 and dictated the form of the argument for universal gravitation in Book 3.

Before turning to that issue, we need to address the part of the *Principia* that has received by far the most attention from philosophers: the scholium to the definitions, regarding space and time. Unfortunately much of this philosophical commentary has been based on misreading Newton's aim as that of proving the existence of absolute space and time. By contrast, we follow Stein (1967) and other recent commentators in reading Newton as clarifying the assumptions regarding space and time implicit in the Laws of Motion.[24] The laws draw a fundamental distinction between inertial and non-inertial motion. What conception of space and time is needed for this distinction to be well-founded?

Newton argued that the laws presuppose a sense of "absolute motion" that cannot be adequately analyzed in terms of the relative positions among bodies. Although he does not identify a target of criticism in the *Principia*, in an unpublished manuscript "De Gravitatione ...," Newton trenchantly dissects Descartes' relational definition of motion and reveals its inadequacy as a basis for physical reasoning (including Descartes' own).[25] The dispute does not turn on whether there is a distinction be-

[23] Exactly how to characterize this stage in the development of Newton's thought is not entirely clear. As George Smith has emphasized, Newton can infer from Law 4 in *De Motu* that the distances of the Sun and Jupiter to the center of mass must hold in fixed proportion, with the proportion determined by a^3/P^2 as a measure of the absolute strength of the centripetal force. Thus the step to the two-body solution could be driven by the idea that the center of gravity should remain stationary, rather than by conceiving of gravity as a mutual interaction subject to Law 3 of the *Principia*.

[24] See (Rynasiewicz, 1995*a*; Rynasiewicz, 1995*b*; DiSalle, 2006) for more detailed readings of Newton's arguments in the scholium that are in broad agreement with the line we sketch here. See (McGuire, 1978) for an influential assessment of Newton's debt to Gassendi, More, and others, and the relationship between the views in the Scholium and Newton's theological views.

[25] This manuscript was first published (and translated) in (Hall and Hall, 1962). The date of composition has been the subject of some dispute: it was initially regarded as an early manuscript, composed during Newton's days as an undergraduate, but more recent commentators have argued for a later date given the maturity of the views expressed and the connection between the positions adopted in the "De Gravitatione" and the *Principia*. A great deal of attention has been devoted to this manuscript because Newton is more forthright in addressing metaphysical issues in it than in his published work.

tween two different senses of motion: motion as defined to be "merely relative" to an arbitrarily chosen set of reference bodies, versus a body's unique "true" or "proper" motion. Rather, the question is whether an adequate definition of the absolute motion can be given solely in terms of relative quantities — spatial and temporal relations with respect to other bodies. Newton abrogates Descartes' proposed definition of true motion and further argues that any adequate definition requires reference to the absolute structural properties of space and time.

The required structures are intervals of spatial distance and temporal duration, along with a unique identification of locations over time.[26] Newton characterizes these structures as "absolute" in several different senses: they are "immutable" and do not change in different regions or in response to the presence of bodies; they are intrinsic rather than conventional; and they are not defined in terms of the relations between material bodies. Based on these structures, Newton defines absolute motion as "the translation of a body from one absolute place into another." By contrast, relative motion is defined with respect to relative spaces, delimited by some particular bodies regarded as immovable. Newton argues that the two are distinguished "by their properties, causes, and effects." As an illustration, in the argument that absolute and relative motions differ in their effects Newton invokes the famous example of a bucket of water hanging from a rope, wound up and then released. When the water in the bucket is rotating, it moves away from the axis of rotation leading to a curved surface. But this curved surface is neither a necessary nor a sufficient condition for *relative* rotation between the water and the bucket: not necessary because initially the surface of the water remains flat despite the relative rotation of the water and the bucket, and not sufficient because after the water has caught up with the rotating bucket, the surface of the water is curved despite the lack of relative rotation of the water and the bucket. As with the arguments from properties and effects, Newton concludes that absolute motion (illustrated here by the dynamical effects of rotation) cannot be analyzed in terms of relative motion.[27]

Yet despite this clear distinction between absolute and relative motion, Newton acknowledges that we lack direct access to absolute motion. Our observations are always of relative motions, given some stipulated relative space that we may provisionally take to be absolute space. How then is it possible to determine the true, absolute

[26] In more modern mathematical treatments, Newtonian space-time is described as a differentiable manifold that is topologically $\Sigma \times \mathfrak{R}$. Σ is a three-dimensional space representing "all of space at a single instant of time," and time is represented by $\mathfrak{R}$ (the real numbers). (To be more precise, these structures are represented by *affine* spaces, in which there is no preferred "origin".) Each of these spaces is endowed with separate metrics, such that the following quantities relating events, points in space-time, are well-defined: (1) the spatial distance between two events at a single instant, and (2) the time elapsed between any two events. In order to distinguish between motion along a "straight" vs. "curved" line further structure is needed: the "kinematical connection" relating locations at different times, which is not fixed by the structures assigned separately to space and time. The kinematic connection can be characterized by its symmetry properties. In what is now called "full Newtonian space-time," these symmetries include time-independent translations, rotations of the spatial coordinates, and translation of the time coordinates. See Stein (1967) and Earman (1989) for discussions of this approach.

[27] Newton directed this argument against a *Cartesian* account of relative motion, in which motion is defined relative to immediately contiguous bodies, and he apparently did not consider the possibility of a Machian alternative that drops the restriction to contiguous bodies.

motions? Newton remarks that the entire *Principia* was composed to elucidate this problem, and "the situation is not entirely desperate." Given the connection between forces and absolute acceleration, it is possible to determine whether motions described with respect to a relative space could be taken as absolute motions. Only for the case of absolute motions will the impressed forces exactly match with accelerations, for relative motions there will in principle always be some discrepancy — for example, the Coriolis forces that arise if one choses a relative space that is in fact rotating. This force is not due to a mutual interaction among bodies; instead, it indicates that the proposed relative space cannot be taken as absolute space. Thus Newton's comment that the entire *Principia* is needed to differentiate absolute from relative motions should be taken seriously, for this can be accomplished only by accurately identifying the relevant forces acting on bodies. In the empirical study of motion, absolute motion does not enter directly into the analysis of motions because it is inaccessible. But the distinction between absolute and relative motions plays a fundamental role in the analysis, given Newton's argument that merely relative ideas are not sufficient to capture the distinction between accelerated and non-accelerated motions.

A final point of clarification concerns the identification of different locations over time. Newton's discussions in the scholium and "De Gravitatione" suggest a *unique* way of identifying a given position in absolute space over time, which would imply that absolute position and velocity are well-defined quantities. But this is more than is required to fulfill the project of the *Principia* of determining the forces responsible for motions, as indicated by absolute accelerations. Newton states a version of Galilean relativity as Corollary 5 to the Laws: the relative motions of a closed system of bodies are not affected if the entire system moves uniformly without rotation. This makes it clear that absolute positions and velocities are irrelevant to determining the absolute accelerations. In modern terms, insuring that absolute acceleration is well-defined requires a weaker structure than a unique identification of locations over time. This structure, called an *affine connection*, allows one to define the amount of curvature of a space-time trajectory *without* introducing absolute position and velocity. Inertial trajectories correspond to straight lines through space-time and curvature of a path represents acceleration. The next corollary states a further sense in which the structures elucidated in the scholium go beyond what is required:

Corollary 6: If bodies are moving in any way whatsoever with respect to one another and are urged by equal accelerative forces along parallel lines, they will all continue to move with respect to one another in the same way as they would if they were not acted on by those forces.

Thus Newton recognized a sense in which acceleration is also relative: in this case, the accelerations of a system of bodies are judged relative to a common, shared acceleration, which is itself irrelevant to the internal relative motions. This poses a deep challenge to the framework of the *Principia*, as it suggests that in the case of gravity the distinction between inertial and non-inertial motion is not well founded empirically. Einstein clearly recognized the importance of this problem 220 years later, and his efforts to solve it guided his path towards a new theory of gravity.

While Huygens developed a very insightful relational response to Newton in his notebooks, public disputes regarding Newton's views on space and time began within Newton's lifetime in the famous Leibniz-Clarke correspondence. Leibniz initiated a tradition of criticizing Newton from a relationalist perspective. Some of the criticisms in the resulting literature misinterpret Newton as offering something other than a clarification of the assumptions regarding space and time presupposed by the empirical project of the *Principia*. (This is not surprising: Newton does not address many basic questions about the nature of space and time in the scholium, although he did address some of them in the unpublished "De Gravitatione.") Even regarded as such, Newton's account can be criticized as introducing excessive structure, as indicated by Corollaries 5 and 6. These issues were significantly clarified only in the nineteenth century with the emergence of the concept of an inertial frame, and further, in the twentieth century, with the re-formulation of Newtonian theory based on Cartan and Weyl's idea of an affine connection following Einstein's discovery of general relativity.

The advent of relativity theory is often taken to vindicate the relationalist critiques of Newton. However, although the transition to general relativity leads to significant conceptual differences, space-time geometry still underwrites a fundamental empirical distinction between different types of motion. Newton's distinction between inertial and non-inertial motion is replaced by a distinction between freely-falling and non-freely-falling motion in general relativity. In Newtonian terms, an object is "freely-falling" if the net non-gravitational force acting on the object is zero, and the resulting motion is described as a consequence of gravity and inertia. General relativity treats inertia and gravitation as manifestations of a single underlying "inertio-gravitational field," and freely-falling motion is represented by geodesic curves in a curved space-time geometry. At least in this sense, that there is a physically distinguished type of motion closely tied to space-time geometry, Newton's seminal analysis remains valid in contemporary theories.

1.3.3 Book 1

The mathematical theory developed in the *De Motu* gave Newton a variety of results that would allow him to infer the centripetal force given an exact trajectory and center of force. But he soon realized that these results were not sufficient for determining the force responsible for the planetary trajectories, as he noted in the "Copernican Scholium" (Hall and Hall, 1962, p. 280):

> By reason of the deviation of the Sun from the center of gravity, the centripetal force does not always tend to that immobile center, and hence the planets neither move exactly in ellipses nor revolve twice in the same orbit. There are as many orbits of a planet as it has revolutions, as in the motion of the Moon, and the orbit of any one planet depends on the combined motion of all the planets, not to mention the action of all these on each other. But to consider simultaneously all these causes of motion and to define these motions by exact laws admitting of easy calculation exceeds, if I am not mistaken, the force of any human mind.

How should one make inferences from observations given this daunting complexity? Newton recognized the multifaceted challenge to reasoning from the phenomena to claims regarding forces, and Book 1 lays the groundwork for Newton's response. Below we will highlight aspects of Book 1 that reveal Newton's novel and sophisticated response to this challenge. But this is only half of the task — discovering the forces from the phenomena of motions — Newton identified as the basic problem of philosophy in the Preface. We will also note how the results of Book 1 contribute to the second task, that of demonstrating new phenomena from these forces.

Up to the end of section 7, Book 1 buttresses the results stated in the *De Motu* without extending them into entirely new domains.[28] Section 1 provides a mathematical prologue, a series of lemmas regarding the use of limits that Newton takes to justify his innovative geometrical techniques. The next 9 sections treat the motion of bodies in response to forces, treated as accelerative tendencies directed towards a fixed center. That is, Newton does not consider the physical origin of the forces, and only the accelerative measure of the force is relevant to these results. Section 2 states a number of results licensing inferences from a trajectory to a force law (*De Motu*'s Theorem 3 appears here as Proposition 6). The next section shows that the trajectories will be conic sections if and only if the force varies inversely with the square of the distance from the focus.[29] Sections 4 and 5 collect a number of geometrical results Newton intended to utilize in determining cometary trajectories. He later devised a simpler method (Proposition 3.41), so these results play no role in the *Principia* itself.[30] Section 6 augments an earlier result in Section 3 that gives a solution for the velocity along the trajectory, and addresses the general problem of determining a body's location along a conic section trajectory given an initial position and an elapsed time. Section 7 adds to the *De Motu*'s treatment of rectilinear ascent and descent, establishing how to recover results of Galileo and Huygens relating time, distance, and velocity in this case by treating the rectilinear ascent as the limiting case of motion along a conic section treated earlier. Section 7 starts by considering specific force laws — $f \propto r^{-2}$ and $f \propto r$ — but the final result solves the problem of rectilinear motion under *arbitrary* centripetal forces, up to quadrature.[31]

Section 8 builds on this generalization of Galileo's treatment of free fall and carries Newton beyond what he had achieved in the *De Motu*. Propositions 41 and 42 give a general solution for projectile motion under arbitrary central forces, again up to quadrature, derived with some of the most sophisticated mathematics in the entire *Principia*. They are based on a fundamental physical insight, namely that Galileo's principle that the velocity acquired in falling through a fixed height is independent of

[28] In the second edition, Newton added important material based on his way of measuring the "crookedness" of a curve using an osculating circle; see (Brackenridge, 1995) for discussions of these changes and the importance of this different approach.

[29] Newton stated this result cited in the text as Corollary 1 to Prop. 13. Few of his contemporaries were convinced by his terse argument in its favor, although arguably his subsequent results in Section 8 do justify this claim (see Guicciardini 1999, pp. 54-56, 217-223, for discussion). The other propositions describe a variety of properties of motion along conic sections. For example, for the case of a circular trajectory, the force directed at the *center* varies either as r^{-2} or r. Later, in Section 7, Newton treats direct fall along a rectilinear trajectory as a limiting case of motion along a conic section.

[30] A conclusion to Lemma 19 makes it clear that Newton's mathematical aim was to provide "a geometrical synthesis, such as the ancients required, of the classical problem of four lines" (Newton, 1726, p. 485). The problem in question was the Pappus problem, which requires the construction of a plane curve bearing a specified relationship to n lines. Descartes claimed that his ability to solve the Pappus problem for n lines was one of the main advantages of his algebraic analysis, and Newton clearly intended to counter this argument; see (Bos, 2001) and (Guicciardini, 2009), Chapter 5. We thank Niccoló Guicciardini for pointing out to us that Newton's intented rebuke to Descartes misses the mark. Descartes's case for the superiority of his method in fact rested on the claim that in principle he could solve the Pappus problem for $n = 8, 10, 12, ...$ lines, but Newton's solution does not generalize to $n > 4$.

[31] "Up to quadrature" means that there is still an integral to be performed to obtain the full solution, but *in principle* this can be done once the force law is specified.

the path traversed also holds, suitably generalized, for motion under arbitrary central forces. Although Newton did not use Leibniz's terminology, this is equivalent to conservation of "vis viva." Results from earlier sections generally draw inferences from a given trajectory to a claim regarding the force law. But the results in Section 8 allow for the study of the effects of a force given only an initial position and velocity (rather than an entire trajectory), and thus Newton has, in a sense, fully solved the "inverse problem" with these propositions. (We discuss the limitations of these results in §1.3.7 below.)

The central result of Section 9 is that apsidal precession of an orbit provides a sensitive measure of the exponent of the force law producing that motion. (The apsides are the points of maximum and minimum distance from one focus of an elliptical orbit; in the case of apsidal precession, the body does not form a closed orbit and the apsides shift slightly with each revolution, by an amount given by the apsidal angle.) Proposition 1.45 considers the relationship between apsidal motion and the force law producing it; the first corollary states that for nearly circular orbits if the apsidal angle θ is given by $n = (\frac{\theta}{\pi})^2$, then the force is given by $f \propto r^{n-3}$. This result is robust in the sense that if the apsidal angle is *approximately* θ, the force law is *approximately* $f \propto r^{n-3}$.[32]

Obtaining this striking result required a great deal of mathematical ingenuity, but our main interest is in how it allows Newton to more reliably draw inferences from the phenomena. For the sake of contrast, consider an argument employing Theorem 3 of the *De Motu* (Prop. 1.11 of the *Principia*) to infer an inverse-square force from the phenomena of motion — an argument often mistakenly attributed to the *Principia*'s Book 3. What does the theorem imply for an *approximately*, rather than *exactly*, elliptical orbit? How does one establish observationally that the force is directed at the focus rather than the center?[33] As Smith (2002) argues convincingly, throughout the *Principia* Newton relies on inferences from the phenomena that are not fragile in the sense of being valid only if the antecedent holds exactly. Instead, propositions like 1.45 establish that an observable quantity (the apsidal angle) can be taken as measuring a theoretical quantity (the exponent of the force law), within some clearly delimited domain, due to the law-like relationship that holds between the two. Inferences based on these results are robust given that the law-like relation holds over some range of phenomena, *not* merely for special cases of exact values for the observable quantities.

Constrained motion, such as motion along an inclined plane or of a pendulum bob, was a central topic in the Galilean-Huygensian tradition. In Section 10 Newton shows how to recover essentially all of the earlier results within his more general framework. A crucial motivation for this section was to assess the validity of Huygens's experimental technique for measuring surface gravity using a pendulum. In *Horologium Oscillatorium*, Huygens had established that a cycloidal pendulum is isochronous — that is, the oscillations have the same period regardless of the starting point of the

[32]The result is also robust in a second sense, namely that the relationship between precession and the force law holds for orbits of increasing eccentricity; strikingly, although Newton did not show this, the relationship holds *more precisely* as eccentricity increases (Valluri, Wilson and Harper, 1997).

[33]Newton derives two distinct force laws for an object moving along an elliptical orbit: for a force directed at a focus, $f \propto r^{-2}$, and a the center, $f \propto r$. For an ellipse with eccentricity very close to 1 (such as the planetary orbits), the foci nearly overlap with the center.

bob — and placed bounds on how closely a small-arc circular pendulum approximates the isochrone. But Huygens treated gravity as constant (not varying with height) and directed along parallel lines, and it is natural to ask whether his reasoning remains valid for truly centripetal forces that vary with distance and are directed towards a center of force. The question is pressing because a crucial step in the argument for universal gravitation, the "moon test," depends on Huygens's result. In an impressive mathematical display Newton introduces curves more general than the plane cycloid studied by Huygens and shows that the true isochrone for a force law varying directly with distance is given by one of these curves, the hypocycloid. The most striking feature of this section is the attention Newton paid to the assumptions involved in Huygens's measurement.

Newton's results in Sections 2 through 10 establish that motion under centripetal forces is a theoretically important *category* of motion, and that it subsumes the Galilean-Huygensian study of uniformly accelerated motion as a special case. But it is also general in the sense that Newton establishes many results that hold for *arbitrary* central forces.[34] This level of generality is crucial to inferring forces from the phenomena, as the force law itself is the unknown quantity to be inferred from observations. The ability to consider arbitrary central forces leads to a much stronger inference: it opens up the space of competing proposals, rather than restricting consideration to checking the consequences of a single postulated force law. Newton's approach allows him to characterize the physical consequences of a variety of different proposals, providing a rich set of contrasts to compare with observations. Furthermore, it allows for the possibility that specific phenomena can be taken as measuring the parameters appearing in the force law, in the sense that there are law-like connections between the phenomena and parameters within some delimited domain.[35] Establishing such connections requires quantifying over a range of different force laws, which would not be possible without Newton's level of generality.

The generality of the treatment of forces in Book 1 is significant in a different sense: in principle it allows Newton to consider empirical contrasts among the phenomena deduced from distinct force laws.[36] Several results in Book 1 isolate striking features of motion under specific force laws. These results contribute not to the initial inference of a force law but to further assessment of the force law by contrast with other alternatives.

Newton begins Section 11 by acknowledging that the preceding results are all based on an unphysical assumption: they concern "bodies attracted toward an immovable center, such as, however, hardly exists in the natural world." For conceiving of forces physically as arising from mutual interactions, the Third Law implies that interacting bodies will move about a common center of gravity. Newton shows that a systematic treatment of motion is still possible without the simplifying assumption of the previous

[34] Most of Book 1 considers only $f \propto r^n$ for integer values of n, but several important theorems allow for rational values of n (e.g., 1.45).

[35] One of the main themes of Harper (2011) is that Newton uses the phenomena to measure theoretical parameters in roughly this sense, and that Newton judged a theory to be empirically successful if diverse phenomena give agreeing measurements of the same parameters.

[36] In practice, however, Newton's mathematical methods were sufficient to handle only a handful of force laws.

sections, and obtains as before solutions for the trajectories of bodies given a force law and initial positions and velocities. In fact the results of the previous section generalize directly to the case of two interacting bodies: Proposition 58 shows that for a given force law a body follows the same trajectory around the second interacting body, also in motion, *and* around their common center of gravity, as it would around an immobile force center for the same force law. Kepler's area law also holds for either trajectory described with respect to the common center of gravity *and* also for the trajectory of one body described with respect to the other. Applied to a two-body system with an inverse-square force law, for example, this result shows that each body describes an ellipse with the other body at one focus (Corollary 2). Newton further shows in Proposition 59 that Kepler's harmonic law has an important correction in the two-body case. Proposition 61 solves for the force law from the common center of gravity that would be needed to treat a two-body problem as an equivalent one-body problem, allowing Newton to apply earlier results (in Propositions 62-63) to find trajectories for arbitrary initial velocities.

Relaxing the simplifying assumption even further to solve for the trajectories of 3 or more interacting bodies proves much more challenging. In the two body case, the center of gravity is co-linear with the two bodies, and the distances of each body to the center of gravity holds in fixed proportion. This consequence of the Third Law is what makes the two-body case fairly straightforward. But it is only in the case of a force law that varies as $f(r) \propto r$ that the general n-body problem is also straightforward due to the Third Law; Proposition 64 shows that in this case the trajectories will be ellipses with equal periodic times with the common center of gravity at the center of the ellipsis.

The three-body problem for $f(r) \propto r^{-2}$ is not nearly so simple. Newton strikingly drops the mathematical style of Book 1 and describes in prose the effects of gravitational perturbations to Keplerian motion in a series of 22 corollaries to Prop. 1.66, for a situation like that of the Earth-Moon-Sun system. Although Newton characterized his efforts on this problem as "imperfect" (in the Preface), his results provided an initial step towards lunar theory, treatment of the tides, and an account of precession of the equinoxes (all taken up in Book 3). These results provide some indication of the empirical contrasts between a two-body description of motions and a more realistic account incorporating the effects of multiple bodies.

Sections 12 and 13 relax a further simplifying assumption, namely that the finite extension of bodies could be entirely neglected, in the sense that there is a single trajectory characterizing the body's motion and that the accelerative tendencies directed toward a body are directed towards a point. These sections reveal a further aspect of Newton's conception of force: he treats the force acting on or produced by a macroscopic body compositionally, as a sum of the forces acting on constituent parts. How does acknowledging the finite extent of bodies, and treating the forces compositionally, affect the results already obtained? The case of highly symmetric bodies is the most tractable mathematically, and Section 12 proves a series of stunning results regarding the net force due to spherical shells on bodies inside or outside the shell. This approach culminates in Propositions 75-76: two spherical bodies interacting via an inverse-square attraction to all parts of both bodies can be described as interacting via an inverse-square attraction directed to their respective centers, with the absolute measure of the force given by their respective masses. Proposition 75 establishes this result for spheres of uniform density, and 76 shows that it holds as well for spheres whose density varies only as a function of the radial distance. Newton further considers the same question for an $f(r) \propto r$ force law (Props. 77-78) and generalizes to an arbitrary force law (Props. 79-86). But the inverse-square force law is quite distinctive in admitting of the remarkable simplification established by Propositions 75-76, and Newton later emphasized this striking feature of the inverse-square law (in Prop. 3.8). Section 13 goes on to consider how to treat bodies of arbitrary shape. The $f(r) \propto r$ force law is tractable: bodies of any shape can be treated as if their mass were concentrated at their center of gravity (Props. 88-89). For the case of an inverse-square force, the general case is mathematically intractable but Newton obtains several results for special cases, such as the force felt by a body along the axis of rotation of a spheroid (Props. 90-91).

These three sections reveal a quite sophisticated approach to handling idealizations. Rather than acknowledging the unphysical assumptions built into the account of motion developed in Sections 2-10, and then arguing that the mathematical theory nonetheless was a good approximation for some phenomena, as his predecessors had done in similar situations, Newton developed the mathematics to assess the effects of removing the idealizations. Although his results were by no means complete, they could be used to characterize qualitatively the departures from the initial idealized treatment due to many-body interactions and the finite extent of real bodies. And indeed many of the effects identified in these sections were already relevant to assessing the application of the theory to the solar system, and Newton returns to these issues in Book 3.

This sequence of results illustrates Newton's approach to the complexity of observed motions through a series of approximations, which Cohen (1980) called the "Newtonian style."[37] The simplest idealized description of motion obtained in the earlier sections does not exactly describe observed motions due to its false simplifying assumptions. Even so, the idealized models can be used in making inferences concerning values of physical quantities as long as these inferences are robust in the sense described above. The further results put Newton in a position to assess whether particular systematic deviations from the idealized models can be eliminated by dropping specific assumptions and developing a more complicated model. If the empirical deviations from the ideal case are of this sort, then the research program can proceed by relaxing the simplifying assumptions. But it is also possible to identify systematic deviations that cannot be traced to a simplifying assumption, which may instead reveal deeper problems with the entire framework of Book 1. Thus Newton's treatment of idealizations allows for empirical results to continue to guide research, even though the identification of the deviations in question presupposes that the simplest idealized models are approximately correct, or, as he says in the fourth rule of reasoning, "nearly true."

[37] In this brief discussion, we draw on more recent studies of the Newtonian style, in particular (Harper and Smith, 1995; Smith, 2001; Smith, 2002; Smith, 2009; Harper, 2011).

1.3.4 Book 2

The final two propositions of the *De Motu* concern the motion of projectiles in a resisting medium. Galileo had been able to treat projectile motion only in the idealized case of a projectile that does not encounter resistance, and Newton considered the consequences of including a resistance force that depends on the velocity ν of the projectile relative to the medium. This initial discussion of the effects of resistance grew into a treatise on a variety of issues in fluid mechanics, incorporating some of the most challenging mathematics of the *Principia* and its most detailed experimental results.

The resulting collection of results has a less coherent structure than Books 1 and 3, although most of the propositions in Book 2 are directed towards three main goals. The first of these was to extend the Galilean-Huygensian study of motion to cover motion in resisting media. A central problem in this area was that of determining the trajectory of a projectile in the air and its dependence on various parameters, and Newton's results constituted substantial progress, albeit not a full solution. A second goal was to provide a framework for understanding resistance forces quantitatively that would allow one to infer properties of resistance forces in actual cases experimentally, in a style similar to the treatment of gravity in Book 3. Finally, Newton gave an empirical argument against the influential vortex theory of planetary motion proposed by Descartes. Unlike Descartes, Newton assessed in quantitative detail the properties of motion exhibited by a body immersed in a fluid vortex, and argued that it was incompatible with the approximately Keplerian motion of the planets.

The first three sections of Book 2 consider the effects of resistance forces that vary with different powers of the relative speed of the object and medium. These sections consider the motion of a body under Galilean gravity (uniform gravity directed along parallel lines) through a medium with resistance proportional to ν, ν^2, or a linear combination of both terms. The challenge was to recover results like those of Book 1, section 8, where Newton reduced to quadrature the problem of determining the trajectory of a body moving under a specific central force, given the initial position and velocity. Newton made several important steps toward a fully general result of this kind. Proposition 2.10, for example, shows how to find the density of a medium that will produce a given trajectory, assuming Galilean gravity and a resistance force proportional to $\rho\nu^2$ (where ρ is the density of the medium). This is not a solution of the inverse ballistic problem — that is, a solution for the trajectory given initial position, velocity and the forces. Instead Newton "adapts the problem to his mathematical competence" (to borrow a phrase from Guicciardini), and the result is fairly limited. Newton showed how to utilize this result to obtain a variety of approximate solutions in the scholium following 2.10.

The historical impact of these sections, and Proposition 2.10, is partially due to the mathematics Newton employed in them. Here Newton explicitly introduces analytical techniques, in the form of a brief introduction to the method of "moments" he had developed in the 1670s (Lemma 2), and the proof of 2.10 relies on infinite power series expansions. It is not surprising that these sections drew the attention of the best continental mathematicians. Johann Bernoulli famously found a mistake in the first-edition proof of 2.10 that his nephew Niklaus communicated to Newton after that part of the second edition had already been printed, forcing Newton to insert a correction.[38] Once the priority dispute regarding the invention of the calculus had arisen, this mistake was taken, erroneously, as evidence that Newton had not mastered the calculus by the time the *Principia* was written.

Later in Book 2 Newton reported a series of ingenious pendulum experiments designed to determine features of the resistance forces. In the *De Motu* he proposed using projectile motion for this purpose, but this was not possible without a solution for the inverse ballistic problem. But in the case of a pendulum the trajectory is already given and the effect of resistance is merely to damp the oscillations. Propositions 2.30-31 establish systematic relationships between the amount of arc length lost due to resistance with each swing and the parameters appearing in the hypothesized total resistance force. Newton treated the total resistance as a sum of factors depending on powers of the relative velocity — that is, $f_r = c_0 + c_1 v + c_2 v^2$. The beauty of this approach was that Newton could, in principle, determine the relative contribution of these different factors (the coefficients c_0, c_1, c_2) merely by varying the starting point of the pendulum. However, Newton was only able to reach conclusions about the v^2 contribution to resistance acting on spheres: he argued this term was the dominant contribution to resistance (when v is large), and that it depended on the density of the medium ρ and diameter of the sphere d as ρd^2. The pendulum experiments did not measure the other contributions to resistance, and there were a number of discrepancies in the experimental results that Newton could not account for.[39]

Newton's dissatisfaction led him to substantially revise Book 2 for the second edition and to perform a completely different set of experiments. These experiments involved timing the free fall of globes dropped in water (in a 9 foot trough) and in air (from the top of St. Paul's Cathedral). The purpose of these experiments was similar, namely that of determining the properties of resistance forces. But they were useful in this regard only given a theoretical framework different than that in the first edition. Newton had classified different types of resistance in Section 7 as arising from different properties of a fluid. In the second edition, he further gave a theoretical derivation of what he took to be the dominant contribution to resistance, namely "inertial resistance" due to the impacts between the particles of the fluid and the body. This result allowed him to predict a time for the freely falling globes, and differences from the predicted value would then potentially reveal the other contributions to resistance, due to the "elasticity, tenacity, and friction" of the parts of the fluid.

The general approach here is similar to that in Book 3: Newton hoped to use observed phenomena to measure parameters appearing in a general expression for the force of resistance.[40] Doing so would directly respond to long-standing skepticism, expressed forcefully by Galileo, about the possibility of a real science of resistance forces. But despite the similar methodology, the outcome of the two lines of work

[38] See, in particular, Whiteside's exhaustive discussion in *Mathematical Papers*, Volume 8.

[39] Here we draw on (Smith, 2001), who gives a more detailed analysis of the experiments and their results, an explanation of why the approach is so promising, and an assessment of what may have gone wrong from a modern point of view.

[40] Here we emphasize some of the conclusions reached in (Smith, 2001); see also (Truesdell, 1968) for a critical assessment of Book 2 and discussions of the historical development of hydrodynamics in the 18^{th} century.

contrasts sharply. Newton's starting point was fatally flawed, given that there is in fact no separation between an inertial contribution to resistance and other factors such as viscosity. This is vividly illustrated by D'Alembert's "paradox": in 1752, D'Alembert showed that a body of *any* shape encounters zero resistance to motion through a fluid with zero viscosity. In addition, experiments like the ones initiated in the *Principia* would not have directly revealed the limitations of this starting assumption. Unlike the case of celestial mechanics, it was not possible to develop a full account of the nature of resistance forces from Newton's starting point via a series of successive approximations guided by further experimental results.

The closing section of Book 2 argues that a Cartesian vortex theory cannot satisfy both Kepler's second and third laws, and is therefore incompatible with observed celestial motions. Newton formulates this argument as conditional on an explicit hypothesis that fluid friction leads to a resistance force proportional to the relative velocity. From Newton's point of view, the resulting argument has the weakness of depending on this hypothesis, which he presumably had hoped to establish via the earlier experiments on resistance.[41] One notable feature of Newton's argument in this section is that on behalf of the vortex theorist Newton develops a quantitative account of vortex motion (probably the first of its kind) that is not obviously contrived to fail. However, Newton's approach is deeply flawed, as Johann Bernoulli and Stokes would later emphasize, due to its erroneous treatment of torque. Even so, Newton had shifted the burden of proof: vortex theorists would need to show that a vortex is compatible with the observed regularities of celestial motions. In addition to planetary motions, the motion of comets was particularly difficult to account for in a vortex theory; comets cut across the planetary orbits, in some cases (such as the comet of 1682 Newton had observed) moving in the opposite direction as the proposed planetary vortex. In the second and third editions, it was also clear that the vortex theorists would have to further defend the impossibility of a vacuum. Roger Cotes's preface and Newton's General Scholium (both added in the second edition) both appealed to Boyle's air pump experiments to bolster the claim that space could be freed from (most) matter.

1.3.5 Book 3

In the opening sequence of propositions in Book 3 Newton presents an argument for the law of universal gravitation. This line of reasoning is the centerpiece of the *Principia*.[42] It leads to the striking conclusion that every body in the universe attracts every other body with a force that varies as $f \propto \frac{m_1 m_2}{r^2}$, for bodies with masses m_1, m_2 at a distance r. We cannot overstate how shocking this claim was at the time; it far exceeded what many, even those well-versed in the mechanical tradition such as Huygens, expected could be established in natural philosophy. In what sense is the conclusion "deduced from phenomena," as Newton described it, and what is the character of this line of argument? In particular, how does it differ from a hypothetical approach, as Newton

[41] In the scholium to Proposition 2.52, Newton argues that this line of argument is robust in the sense that even if the hypothesis fails to hold exactly the vortex theory still fails to account for Kepler's third law.

[42] See also (Stein, 1991) for a clear discussion of the structure of Newton's argument in Book 3.

insisted that it did? Finally, how does this opening sequence relate to the remainder of Book 3?

The first three propositions establish that the planets and their satellites, including the moon, are held in their orbits by inverse-square centripetal forces directed towards their respective central bodies. The premises of these arguments are provided by phenomena stated at the outset of Book 3. "Phenomena" do not refer to individual observations; rather, Newton uses the term for law-like regularities inferred from the data. The phenomena included the claims that the satellites and the planets satisfy Kepler's harmonic law and area law.[43] Newton then invokes propositions from the opening of Book 1 to determine the explicit form of the force law. But in addition to these phenomena, Newton notes that the conclusion can be established "with the greatest exactness" based on Proposition 1.45 and the observation that the aphelia are at rest. The argument does not rely on Kepler's first law as the textbook tradition would have it; as we noted above, the argument Newton in fact gives is not fragile in the sense of requiring the phenomena to hold exactly, as an argument based on the first law would. This first step of the argument was already clearly presented in the *De Motu*, and it was uncontroversial among Newton's contemporaries — Huygens immediately accepted it. But it is only a first step towards universal gravitation.

The next step was far more striking for Newton's contemporaries. Proposition 4 identifies the centripetal force maintaining the moon in its orbit with terrestrial gravity. Newton considers what the acceleration of the moon would be were it brought to near the earth's surface, as calculated from orbital acceleration in conjunction with the inverse-square variation in the centripetal force. But this turns out to be very nearly the acceleration of terrestrial bodies due to gravity, as measured by Huygens. Newton invokes the first and second "Rules of Reasoning" to conclude that the two forces should be identified as a single force. A different way of considering the argument is to ask how Newton individuates different fundamental forces. The moon test establishes that the centripetal force holding the moon in its orbit and terrestrial gravity agree on all measures (absolute, accelerative, and motive). If we take forces to be individuated by having different measures, or different laws governing their variation, there is then no ground for claiming that there are two distinct forces rather than one. Proposition 5 extends this reasoning to the other planets and the sun, concluding the centripetal force responsible for the celestial motions is gravity.

How gravity towards one of the planets varies with the mass of the attracted body is taken up in Proposition 6. Newton first argues that pendulum experiments have established that weight varies with mass for bodies at a fixed distance from the earth's center. The corollaries emphasize the contrast here with other forces such as magnetism, in that *all* bodies are affected in the same way at a given distance. He then argues that the satellites of Jupiter have the same acceleration towards the sun as Jupiter itself, directly from Proposition 1.65 and also based on the absence of observable eccentricity of the orbits that would be the consequence of a difference in acceleration. (That is, the sun's gravitation does not appreciably displace or distort

[43] Kepler's first law – that the planets move in elliptical orbits with the sun at one focus – is *not* stated as a phenomena.

the motion of the satellites.)[44] Finally, the weights of each part of a planet varies with the mass of that part, or else the response of the planet to the Sun's gravitational field would depend upon its composition (with parts allowed to have different ratios of mass to weight) rather than just its mass. This step of the argument must have also been quite striking for Newton's contemporaries. Many of them would have considered the possibility that gravity is analogous to magnetism, but Newton turned this speculative question into a precise empirical contrast between gravity and other forces.

The final step in Proposition 7 is to the claim that all bodies produce an attractive force directed towards them whose absolute strength is proportional to their masses. The conclusion follows directly from Proposition 1.69. This earlier result establishes that for a central force whose accelerative forces depend only on distance (and *not* on any other properties of the interacting bodies), the absolute measure of force is given by mass.[45] This conclusion follows from the Third Law and the definitions of the accelerative and motive measures of force. For given any pair of interacting bodies A and B, the Third Law implies equality of the motive forces acting on the two bodies. But given that the motive force is the mass of the body times the accelerative force, it follows that the ratio of accelerative forces acting on the two bodies is given by the ratio of their masses. Finally, for a central force independent of any other properties of bodies, at a fixed distance the acceleration towards A or B is the same for all bodies. Hence, for *any* distance, the ratio of the accelerative forces to A and B is given by the ratio of their masses. The mass thus measures the strength of the accelerative tendency towards each body, i.e. it is proportional to the absolute measure of the force.

With this Newton has completed the initial argument for universal gravitation. However, Newton clearly endorses a further claim, based on his comments in the Preface but not explicitly argued for in Book 3, that gravity should be taken as one of a few natural powers or general principles of motion. Thus in addition to establishing how gravity depends on masses and distance, Newton treats it as something akin to a "fundamental force" in roughly the same sense as that term is used in contemporary physics.[46]

Even a reader as sympathetic as Roger Cotes, the insightful editor of the second edition, balked at the application of the Third Law in Proposition 7 (and earlier in corollaries to Proposition 5). Cotes objected that the Third Law itself does not license the conclusion that the force equal and opposite to the one holding a planet in its orbit is a force impressed on the *Sun*. If there were an invisible hand holding a planet in its orbit, to use Cotes's vivid illustration, then the Third Law implies the existence of a force acting on the *hand* rather than on the *Sun*. (This suggestion is not mere fancy on Cotes's part; the then dominant vortex theories similarly involve pressure

[44] There would be *no* "distortion" of the motion of one of Jupiter's satellites in the idealized case where $d/r \to \infty$, where d is the Jupiter-Sun distance and r is the radius of the orbit of the satellite. In the real case, Newton's theory predicts tidal effects, but these would be quite small for the case of Jupiter.

[45] Newton states this as Corollary 2 to Proposition 1.69; the proposition itself is formulated in terms of inverse-square forces. For a central force whose accelerative measure does not depend upon properties of the interacting bodies, then the absolute measure must be a scalar quantity assigned to each body.

[46] We follow (Stein, 2002) in emphasizing this point.

exerted on the planet by neighboring bodies.) Newton's argument thus apparently required a hypothesis regarding the nature of gravitation, contrary to the methodological pronouncements added to the second edition's General Scholium. Explicitly, the hypothesis is that the impressed force on the orbiting body is due to a mutual interaction with the Sun. While this is certainly a plausible claim, given that the impressed force is directed towards the Sun and varies with distance from it, Newton is not in a position to offer an argument from the phenomena to this effect as he had the earlier propositions. Furthermore, the evidence offered in defense of the Third Law itself involved only cases of interaction among contiguous bodies, and here Newton has significantly extended its application to cases of non-contiguous interacting bodies.

In what sense, then, can we say that the law of gravity has been "deduced" from the phenomena, and to what extent is Newton's conclusion warranted? There has been controversy regarding these questions from Newton's time to the present, but one thing is abundantly clear from the text of the *Principia*: Newton did not treat the empirical case for universal gravity as completed after the opening seven propositions. Instead, the opening sequence captures the initial inference to the force law, but the empirical case in favor of universal gravitation *includes* the assessment of its further consequences. From Proposition 8 onwards Newton derives results taking the law of gravity as given in conjunction with a number of further empirical claims. Propositions 8 and 9 immediately note two important aspects of universal gravity (based on the results from Section 12-13 in Book 1): that outside a spherical mass with spherically symmetric density, $\mathtt{f}(\mathtt{r}) \propto \mathtt{r}^{-2}$, and inside a spherical mass of uniform density, $\mathtt{f}(\mathtt{r}) \propto \mathtt{r}$. In the Corollaries to Proposition 8, Newton calculated the masses and densities of other planets and the sun — and Newton's contemporaries, such as Huygens (in Huygens, 1690), singled this out as a striking consequence of universal gravity.

The remainder of Book 3 exploits the framework provided by Book 1 to establish a number of distinctive consequences of universal gravitation. Propositions 10-17 elaborate and defend a Copernican-Keplerian account of orbital motions. Newton next considered the shape of the planets and the possibility of determining the Earth's shape by measuring local variations in surface gravity. Newton added Propositions 18-20 after Halley informed him in 1686 of the discovery that the length of a seconds pendulum varies with latitude. Huygens was already aware of the importance of the pendulum measurements, and immediately recognized the importance of these propositions. Newton's prediction of the shape of the earth provided a sharp empirical contrast between universal gravity and other gravitational theories, and the contrast was accessible to contemporary pendulum measurements (see Schliesser and Smith, ms).

The most difficult parts of Book 3 concern the question that Halley would have probably posed in response to *De Motu*: what does the inverse-square law imply for the moon and for comets? The study of the Earth-Moon-Sun system takes up the bulk of Book 3. Newton's main aim was to give an account of the various inequalities in the moon's motion based on his earlier qualitative treatment of the three-body problem in Prop. 1.66 and its corollaries. But he also argues for two other striking conclusions: first, that the tides result from the gravitational attraction of the Moon, and second, that precession of the equinoxes is due to the Moon's gravitational pull on the Earth's equatorial bulge. The final three propositions (40-42) turn to the problem of comets. Newton had delayed the publication of the *Principia* as he struggled to find a way to use observations of comets to determine their trajectory. He did ultimately find a method and used it to determine the orbit of the comet of 1680/81 as an illustration of the technique.[47]

Many of the results in Book 3 were startling breakthroughs. Yet Newton's success in providing a qualitative dynamical understanding of these phenomena based on gravity was achieved in spite of a variety of errors and illicit assumptions. This is in part due to the difficulty of the problems Newton tackled. His most astute eighteenth century readers would soon discover deep flaws in most of the arguments in Book 3, and correcting these flaws would require significant advances in mathematical physics. But, as we will discuss briefly below, a century of further work on these problems ended up strengthening the case for Newton's claim that these phenomena can all be understood as consequences of gravity.

Taken as a whole, these results transformed the study of celestial mechanics and set the agenda for the eighteenth century.[48] One aspect of this transformation was to treat celestial motions as problems within gravitational physics. Prior to Newton, predictive astronomy focused on finding calculational schemes that would allow one to compute celestial positions accurately. By contrast, Newton treated all these motions as consequences of the gravitational interactions among celestial bodies. An equally important second part of this transformation was the level of quantitative detail Newton demanded in a satisfactory treatment of celestial motions. Even in the face of the daunting complexity of observed celestial motions, Newton emphasized the importance of control over the theoretical description of the system. Rather than taking a rough qualitative agreement between theory and evidence as the appropriate end point of inquiry, Newton insisted on seeking precise quantitative agreement. Demanding such agreement would make it possible to recognize residual discrepancies between theoretical descriptions and observed motions, discrepancies that might reveal physically important features of the real world that had been initially excluded. This would make it possible to develop ever more accurate models capturing more of the complexity of real phenomena.

The lunar theory provides the clearest illustration of this aspect of Newton's approach.[49] The moon's motion is enormously complicated compared to that of the planets, and seventeenth century astronomers struggled to give a descriptive account of lunar motion with comparable accuracy. Newton thus faced two problems: first, to develop a descriptively accurate account of the moon's motion, and second, to assess whether all of the details of that account could be derived as a consequence of gravity. What was so distinctive and transformative about Newton's approach was that he responded to both problems at once; as he puts it, "I wished to show ... that the lunar

[47] We do not have the space to discuss these results; see (Kriloff, 1925) for a detailed reconstruction and defense of Newton's techniques for determining the comet's trajectory, and (Hughes, 1988) for a more schematic overview of Newton's work on comets.

[48] See (Wilson, 1989) for a more detailed account of Newton's contribution to celestial mechanics.

[49] We thank George Smith for emphasizing to us the importance of Newton's lunar theory as exemplifying the "Newtonian style".

motions can be computed from their causes by the theory of gravity" (Scholium to Proposition 3.35, added in the second edition). In rough outline, Newton's aim was to account for the various known inequalities in the lunar orbit as a consequence of the perturbing effect of the Sun's gravity and other features of the Earth-Moon-Sun system.[50] If the Earth and the Moon were spheres of uniform density and the Sun had no perturbing effects, then the Earth and Moon would each follow elliptical orbits with their common center of mass at one focus, without apsidal precession. The departures of the moon's motion from this simple idealized case can reveal what physical features are needed for a more accurate description. The computation of lunar motions from first principles is crucial to this process, as one can seek to identify more subtle effects based on the character of the residual discrepancies remaining once other contributions have been taken into account.

The idea of approaching lunar theory on these terms was a more lasting contribution than the details of Newton's own account. Newton and his advocates often overstated the success of his lunar theory. In fact, he was not able to make substantial progress with regard to accuracy over the ideas of Jeremiah Horrocks. One glaring problem with the lunar theory was Newton's failure to derive the amount of apsidal precession due to the Sun's perturbing effects. Newton required a number of unmotivated steps to conclude that the rate of apsidal precession was compatible with observations. Newton also faced a more general obstacle: within his geometric approach it was not possible to enumerate all of the perturbations at a given level of approximation, as one could later enumerate all of the terms at a given order in an analytic expansion. It was only with a more sophisticated mathematics that astronomers could fully realize the advantages of approaching the complexities of the moon's motion via a series of approximations.

1.3.6 The General Scholium and the Rules of Reasoning

At several points above we have indicated that Newton made many refinements to the *Principia*. In this section we discuss three significant, interconnected changes that fundamentally influenced the content and reception of the *Principia*. First, the re-labeling and rewording of nine "hypotheses" (into "phenomena" and "rules of reasoning") at the start of Book 3; second, the addition of the General Scholium, which provided a completely new ending to the *Principia*; third, changes to matter theory of the *Principia* that removed vestiges of atomism. The first two changes are linked because together with Roger Cotes's new preface, they explicitly deal with methodological issues and they also connect the *Principia* to wider metaphysical and theological concerns. In the General Scholium, Newton provides a list of discoveries as exemplars of the fruits of this method: "the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation." These overlap significantly with his treatment in the third rule of reasoning. There is, thus, no doubt that the methodological claims of the General Scholium are meant to be read alongside the rules of reasoning.

[50] Wilson (1989) gives a concise introduction to Newton's lunar theory.

The three changes are linked in a complex fashion. Atomism is one of the hypotheses that gets dropped in the second edition as we will show below. This change is connected to a little noticed oddity: the first edition of the *Principia* starts with Halley's laudatory poem (which is clearly modeled on Lucretius' ode to Epicurus) and ends rather abruptly after a technical discussion of how to compute the trajectory of comets and some hints on how these might play a role in the circulation of cosmic materials. We can infer from Newton's reaction to now lost letters from Bentley that Bentley (first and foremost a classicist) imputed Epicurean doctrines to Newton. Bentley heavily edited Halley's poem for the second edition in order to remove any sign of impiety or Epicureanism.[51] The carefully crafted General Scholium tacitly addresses various unnamed critics. In particular, it disassociates Newton from any Epicurean reading. It aligns Newton with — and also created crucial intellectual authority for — the newly influential efforts at creating a Newtonian physical theology (or natural religion) in Great Britain and the Netherlands. So, even though the General Scholium is under 2500 words it dramatically changes the framing of the whole *Principia*. Below, we discuss it in more detail, but first we the discuss the changes to Newton's matter theory and analyze the rules of reasoning.

From a methodological approach the most significant of the Newton's changes to the second edition was the re-labeling and rewording of nine "hypotheses" at the start of Book 3 of the *Principia*. Five of these became empirical "phenomena" that Newton lists just before his argument for the existence of the inverse-square law. (Newton added an original, "phenomenon 2," to the second edition.) The first two hypotheses were renamed the first two "rules for the study of natural philosophy." The "hypothesis" that became the third rule of reasoning was replaced in the second edition with the third rule. The fourth rule was added only in the third edition of the *Principia*. Only one hypothesis (the fourth) survived as a "hypothesis" in the second and third editions. It is used in a reductio (in proposition 3.11) and, thus, is not a counterexample against the General Scholium's famous injunction against hypotheses.

One hypothesis, the original hypothesis 3, was dropped entirely; it reads "Every body can be transformed into a body of any other kind and successively take on all the intermediate degrees of qualities" (*Principia*, p. 198).[52] This transformation thesis is a very broad assertion of the homogeneity of matter. Something like this claim was a staple of the mechanical philosophy and may have also motivated alchemical search to turn lead into gold. Newton appealed to it only once in the *Principia* (Book 3, proposition 6, corollary 2; he dropped the corollary and reworded the proposition in subsequent editions.) Mass as a quantity or measure also presupposes homogeneity of matter in a weaker sense, but does not require the transformation thesis. In *An Account of Sir Isaac Newton's Philosophical Discoveries* (posthumously published in 1748) the leading and most sophisticated Scottish Newtonian, Colin MacLaurin, even goes so far as to suggest that different kinds of matter that have different kinds of resistance to change might well exist (MacLaurin, 1748, p. 100). Something of the spirit behind this dropped hypothesis reappeared in Query 30 of the *Opticks*: "Are

[51] Much of Halley's original poem got restored in the third edition (see Albury, 1978).

[52] See, in particular, (McGuire, 1970) for a discussion of the philosophical significance of this hypothesis and its role in Newton's matter theory.

not gross Bodies and Light convertible into one another, and may not Bodies receive much of their Activity from the Particles of Light which enter into their Composition? The changing of Bodies into Light, and Light into Bodies, is very conformable to the Course of Nature, which seems delighted with Transmutations" (Newton, 1730, p. 374).

The dropped hypothesis reflects a more important change from the first edition of the *Principia*, where Newton seems committed to atomism. This is not only reflected by the hypothesis, but more clearly in Proposition 3.6, Corollary 3. There Newton relies on the counterfactual assumption that if matter is fully compressed to eliminate all interstitial void spaces, it would be of the same density. In the second edition this was reworded so as to remove the commitment to atomism.[53] Even so, Newton may well have remained committed to atomism throughout his life. For example, in Query 31 of the *Opticks*, Newton freely speculates about "the small Particles of Bodies" that have "certain Powers, Virtues, or Forces, by which they act at a distance, not only upon the Rays of Light for reflecting, refracting and inflecting them, but also upon one another for producing a great Part of the Phenomena of Nature." We should be careful not to conflate Newton's undoubted corpuscularianism with his atomism. Nevertheless, at several places in the *Opticks*, Newton speculates about perfectly "hard bodies"out of which other bodies are composed as possible and likely (Newton, 1730, pp. 364ff; 370; 375-8).

Newton's four rules of reasoning also change the framing of the argument in Book 3 in response to criticisms of the first edition. In particular, all the rules are meant to underwrite steps in the argument for universal gravity. Newton wanted to elucidate and justify the inference and also respond to the common criticism that the entire line of argument was based on the "hypothesis of attraction". The four rules can be read as providing norms for causal ascription. We now turn to a careful analysis of the four rules of reasoning. The first two rules are as follows (*Principia*, pp. 794-95):

Rule 1: No more causes of natural things should be admitted than are both true and sufficient to explain their phenomena.

As the philosophers say: Nature does nothing in vain, and more causes are in vain when fewer suffice. For nature is simple and does not indulge in the luxury of superfluous causes.

Rule 2: Therefore, the causes assigned to natural effects of the same kind must be, so far as possible, the same.

Examples are the cause of respiration in man and beast, or of the falling of stones in Europe and America, or of the light of a kitchen fire and the sun, or of the reflection of light on our earth and the planets.

We treat these two rules together because their wording ("therefore" in the second rule) encourages understanding the second rule as a consequence of the first. In eighteenth century discussions they are also often discussed jointly. The first thing to note about these two rules is their focus on how to match causes and effects; Newton's science is causal. Both rules promote causal parsimony and simplicity. Newton advocates a form of reductionism – universal gravity is a single underlying cause for disparate phenomena – without insisting that this cause be reduced to microscopic or physical

[53] See (Biener and Smeenk, 2011) for a discussion of a problem with Newton's treatment of matter noted by his editor Roger Cotes. Cotes' incisive criticism of this line of reasoning led Newton to back down in the second edition, and acknowledge the hypothetical character of the earlier claim that quantity of matter holds in fixed proportion to quantity of extension.

qualities.

We quote the third rule, but not Newton's lengthy commentary, before discussing them further (*Principia*, p. 795):

Rule 3: Those qualities of bodies that cannot be intended and remitted and that belong to all bodies on which experiments can be made should be taken as qualities of all bodies universally.

While the first two rules promote causal austerity, the third rule promotes a kind of inductive boldness. In particular, it licenses induction for the imputation of properties to very distant and to very small objects. The latter is often called "transduction" in the literature. As will become fully clear in our analysis of the fourth rule, Newton recognized the limits and dangers of induction (see also (Newton, 1730, Query 31, pp. 403-04)). Nevertheless, within the confines of a research program he advocated bold generalization from the empirically available domain to domains beyond our experimental grasp. David Hume recognized something of Newton's boldness; in *The History of England* he wrote that Newton is "cautious in admitting no principles but such as were founded on experiment; but resolute to adopt every such principle, however new or unusual" (Hume, 1985, Volume 6, p. 542).

The third rule clearly presupposes rather strong assumptions about the scale invariance of nature. Now to be clear, Newton had put a lot of experimental and theoretical work into showing that he was allowed to sum "motions of the individual parts" into "the motion of a whole" (*Principia*, Definition 2; p. 404) and, in particular, that the mass of a body could be summed from its parts. By analogy in the explication of the third rule Newton now asserts that "extension, hardness, impenetrability, mobility, and force of inertia of the whole [body], arise from the extension, hardness, impenetrability, mobility, and force of inertia of each of the [body's] parts." Newton underscores the importance of this compositionality by adding that "this is the foundation of all philosophy." The rule, thus, licenses quantitative inferences from empirical evidence to parts of nature beyond the reach of our evidence (see, in particular, McGuire 1970).

We note three things on Newton's discussion of the third rule. First, Newton now takes an agnostic stance on atomism. This fits well with his removal of the original third hypothesis. Second, the rule contains a not-so-subtle dig at Christiaan Huygens's Cartesian skepticism about the very possibility of universal gravitation of all bodies. For, Newton points out that the empirical argument for the "principle of mutual gravitation" is far stronger than the argument for the "impenetrability" of matter (which is presupposed by Cartesians following section 43 of Descartes' *Principia*). The third rule appeals to "sensation" and "experiments" and rejects "reason" as the grounds for asserting impenetrability (and presumably other properties of bodies). Newton clearly intends this as a contrast with the rational insight into the nature of body appealed to by Descartes and his followers.

Third, the rule itself deploys the plural "bodies." The plural is used throughout Newton's discussion. This modifies a bit the nature of the inductive leap that Newton advocates. Newton is offering an account in terms of the behavior of systems of bodies, not an account that has its source in the nature of body. In fact, in the *Principia* Newton never defines the nature of body (McGuire 1966). It is surely tempting to see Newton as effecting a conceptual reversal, by explicitly formulating the laws of motion and implicitly defining bodies as entities that satisfy the laws.[54] In the gloss to the third rule, Newton is careful to distinguish between essential and universal qualities of bodies presumably to block the implication that if a quality is universal it must also be essential. By "essential" Newton probably means what we would call "intrinsic" qualities of bodies, that is, qualities that are presupposed in the very conception or nature of body. In part, the terminology of "universal qualities" marks Newton's contrast with the Cartesians. But he also hoped to avoid the charge of attributing gravity to bodies as an essential property by calling it universal instead, a claim he made explicit in the third edition.[55]

Finally, Rule IV, which was added to the third edition, and its brief commentary read (*Principia*, p. 796) as follows:

> Rule 4: In experimental philosophy, propositions gathered from phenomena by induction should be considered either exactly or very nearly true notwithstanding any contrary hypotheses, until yet other phenomena make such propositions either more exact or liable to exceptions.
>
> This rule should be followed so that arguments based on induction may not be nullified by hypotheses.

This rule has been the subject of considerable recent scholarly attention. Yet, we have been unable to locate a single explicit discussion of it during the eighteenth century! By "hypotheses" Newton means the kind of proposals offered by Mechanical philosophers. The main purpose of this rule is to settle one's attitude toward ongoing research. It encourages one to accept one's going theory as true (or "very nearly" so). Also, the rule disallows a vantage point outside of ongoing research as providing legitimate sources of principles that could motivate theoretical reinterpretations of one's empirical results (of the sort that mechanical philosophers would promulgate).

But the rule has two further important implications. First, notwithstanding the kind of bold inductive leap that the third rule encourages, this rule is a clear expression of Newton's fallibilism. He knows he could be wrong. This echoes his own "Author's preface" to the *Principia*: "I hope that the principles set down here will shed light on either this mode of philosophizing or some truer one" (p. 383). (This was already present in the first edition.) Second, the rule encourages the search for systematic deviations from known regularities. Discrepancies need to be turned into "phenomena." As noted above, it is a major methodological innovation and achievement of Newton's *Principia* that systematic discrepancies are both possible sources of much more subtle evidence than previously imagined, as well as sources of refinements to the theory.

We now turn to a more detailed treatment of the General Scholium. While we discuss it roughly in order of Newton's presentation, we will emphasize four aspects: (1) Newton's reiteration, even expansion of his case against the vortex hypothesis; (2) Newton's embrace of a design argument; (3) Newton's treatment of the role of natural philosophy in knowledge of God; (4) Newton's elaboration of his methodology.

The General Scholium opens with three main arguments against the "hypotheses of Vortices." Newton argues, first, (echoing the end of Book 2) that the observed Keplerian motion of the planets is incompatible with vortices. Second, he argues that

[54] Here we draw on (Brading, 2011), which articulates and defends what she calls the "law-constitutive" approach to the problem of defining and individuating bodies.

[55] For further discussion of the Third Rule, see (McGuire, 1970; McMullin, 1978).

the trajectories of comets, which "observe the same laws" as planetary motions, are incompatible with vortices. Third, he offers an analogical argument that, in accord with all seventeenth century New Philosophy, relies on the fundamental unity between terrestrial and celestial phenomena; it appeals to evidence from Boyle's vacuum experiments to claim that in the absence of air-resistance all celestial bodies will keep their motion in accord with Newton's laws of motion. We can understand these three arguments as successful burden-shifting.

The General Scholium then turns to arguments for the existence and the nature of our knowledge of God. Newton first argues that while the orbits of celestial bodies are law-governed, neither these laws nor the "mechanical causes" of his philosophic opponents can be the cause of the orbits themselves. Newton barely gives an argument for his claim that the laws of motion cannot be the cause of the orbits.

We use "barely" because Newton does claim that it is "inconceivable" that the laws of nature could account for such "regular" orbits. But in what follows Newton packs quite a bit into this claim. In particular, it turns out that for Newton the regularity consists not merely in their being law-governed, but also that the trajectories and mutual attractions of the planets and comets hinder each other least. This culminates in Newton's conclusion that "This most elegant system of the sun, planets, and comets, could not have arisen without the design and dominion of an intelligent and powerful being" (p. 940). Without further argument Newton rules out the possibility that these particular three features (that is, (i) law-governed orbits that (ii) hinder each other minimally and that (iii) are jointly beautiful) could be caused by other causes than God. Newton then offers the "immense distances" among the planetary systems, which thus avoid the possibility of gravitationally-induced mutual collapse, as another, empirical phenomena that supports his argument from inconceivability. Moreover, given that Newton could put almost no constraint on the mass of comets, he must have also found it striking that these do not disrupt the motions of the solar system through which they pass. Comets provide a further hint of providential design, in that at aphelia they are sufficiently far apart so as not to disturb each other's motion.

Before we turn to analyzing Newton's argument from (beautiful) design and his views of God, it is worth noting that Newton's position rules out two contrasting, alternative approaches, both discussed later in the General Scholium: i) that God is constantly arranging things in nature. As he writes, "In him are all things contained and moved; but he does not act on them nor they on him" (p. 941). No further argument is offered against a hyper-active God. ii) That everything is the product of "blind metaphysical necessity." This second view is associated with the neo-Epicurean systems of Hobbes and Spinoza. Newton offers an independent argument against that approach, namely that given that necessity is uniform it seems it cannot account for observed variety. Now, against Hobbes this is a powerful point, but it is only a limited objection against Spinozism. For Spinoza is committed to there being sufficient reason for infinite variety in the modes (E1p16 and E1p28).[56]

[56] We cite Spinoza's *Ethics*, as is customary, by part and proposition; the standard English translation is (Spinoza, 1985).

At best Newton has shifted the burden of proof. Because Newton has the better physics, he can claim to have constrained any possible explanation that will account for the observed variety. But it is not insurmountable: all a necessitarian needs to show is how the laws and the "regular" orbits are possible given some prior situation. Moreover, in the absence of a discussion of initial conditions of the universe and a developed cosmogony, Newton's claim begs the question. One can understand Kant's *Universal Natural History* as taking up the challenge of accounting for the origin of the universe in light of Newton's laws of motion.

Of course, Newton is not merely pressing the existence of variety against the necessitarian; he is also calling attention to the significance of that particular "diversity of created things, each in its time and place" (p. 942). As we have seen Newton argues from (i) law-governed orbits that (ii) hinder each other minimally and that (iii) are jointly beautiful, to the conclusion that an all powerful and intelligent God must have been their cause. Moreover, he uses the distance among planetary systems as a further argument to insist that God must be "wise." Regardless of how plausible one finds such an argument to a providential designer, Newton's version is not an anthropocentric argument. In particular, the beauty of our solar system is mimicked by countless other solar systems, too far apart to be of interest to us. Moreover, natural diversity is suited to times and places regardless of human interest.

Such a non-anthropocentric design argument is also available to careful readers of the first edition of the *Principia*, but it had not been highlighted in it. Newton writes, "Therefore God placed the planets at different distances from the sun so that each one might, according to the degree of its density, enjoy a greater or smaller amount of heat from the sun" (Proposition 3.8, corollary 5). It is the only explicit mention of God in the first edition. Newton suggests that the variation in temperature is suitable given the variation in surface gravity and density of the planets. As Cohen notes, Huygens was astounded that Newton was able to calculate the strength of gravity "the inhabitants of Jupiter and Saturn would feel" (Cohen, 1999, quoting Huygens, p. 219).

Much of core of the General Scholium is then given over to articulating the nature of God and the proper way of talking about him. Here we do not do justice to all the complex theological and metaphysical issues that this material poses.[57] We wish to note six points about this material. First, Newton unabashedly argues that natural philosophy includes empirical research into God ("to treat of God from phenomena is certainly a part of natural philosophy," p. 943). (See McGuire 1978 and Janiak 2008.) Second, our knowledge of the manner of God's existence is strictly empirical and based exclusively on the argument from beneficial/providential design ("we know him only by his properties and attributes and by the wisest and best construction of things and their final causes," p. 942). Of course, the existence of God may be secured by other arguments and sources, including Scripture. Third, if one denies that God is a beneficial designer, talk of "God" refers to fate. Fourth, Newton denies that we can have knowledge of the substance of God. This last point is a consequence of the more general claim that knowledge of substances is not available to human inquirers.

[57] On theological matters see, for example, Snobelen (2001), Ducheyne (2007); for useful treatment on Newton's sources see McGuire and Rattansi (1966) and Smet and Verelst (2001).

(Newton formulates this and related discussions about personhood in very Lockean terms, and this surely encouraged Enlightenment philosophers to link their approaches into a connected program. More recent scholarship tends to disconnect Newton and Locke.)[58]

The first and the fourth of these points fit with a fifth: in three distinct ways knowledge of God's manner of existing is less secure than the empirical knowledge from which it is derived. (This is emphasized by Stein 2002.) I. Newton insists that our knowledge of God's substance is even less available to us than knowledge of the substance of bodies. II. Newton insists that we have no knowledge of the manner of God's (immaterial) activity (his manner of understanding, perception, etc). III. Newton also recognizes that we have a tendency to anthropomorphize God and — strikingly — while he stresses the limitations of this, he seems to think it is the only route available to us when speaking about God. Newton also makes a number of definitive claims about God's attributes that raise complicated and controversial (theological) questions over the exact relationship between God and space/time; many of these questions were raised and pressed by Leibniz in his famous correspondence with Clarke.

Newton concludes the General Scholium with his discussion of the "power of gravity" and a speculation on a subtle spirit. On the "power of gravity" Newton is explicit about the contours of his knowledge and ignorance of it. His reflections on it provide further insight into his mature methodological views. We offer two main observations: first, he explicitly asserts that celestial phenomena and phenomena of the sea are explained by gravity. Gravity is a single cause for Newton. According to Newton we do not know what causes it, but we are in the position to make some explicit claims about the structure of this unknown cause (p. 943):

> ... this force arises from some cause that penetrates as far as the centers of the sun and planets without any diminution of its power to act, and that acts not in proportion to the quantity of the *surfaces* of the particles on which it acts (as mechanical causes are wont to do) but in proportion to the quantity of *solid* matter, and whose action is extended everywhere to immense distances, always decreasing as the squares of the distances. Gravity toward the sun is compounded of the gravities towards the individual particles of the sun, and at increasing distances from the sun decreases exactly as the squares of the distances as far out as the orbit of Saturn ...

Despite Newton's inability to infer the underlying cause of gravity from the phenomena, he does have confidence in the constraints that any to-be-identified cause must obey.

Second, at this juncture Newton resists the temptation to offer his own hypothesis, with his famous claim that "I do not feign hypotheses." He adds that "For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy" (p. 943). In historical context, Newton is clearly ruling out the demand familiar from the mechanical philosophy for causal explanations in terms of the shape, size, and motion of colliding bodies. But the way he phrases it is far broader. Now, Newton does not explain in context what he means by "phenomena." But as our treatment above suggested phenomena are law-like regularities inferred from a given body of data. So, "deducing from phenomena" means something like rigorous inference

[58] For further discussion of Locke and Newton, see (Stein, 1990; Downing, 1997; Domski, 2011).

from well-established, non-trivial empirical regularities. In modern terminology this is something like induction. Confusingly to the modern reader, Newton identifies a second phase of research with induction (i.e., "made general by induction"). But such generalization and systematizing we tend to associate with deduction. So, at the risk of over-simplification: in his methodological statements of the General Scholium, Newton takes for granted the existence of well established empirical regularities that form the basis of inference to a set of general claims that, in turn, form the basis for deduction. In the *Opticks* he identifies these two stages with the so-called Analytic and Synthetic methods.

Strikingly and confusingly, just after denying any interest in hypotheses, Newton concludes the *Principia* with a bit of speculation about an "electric and elastic" ether that might account for all kinds of natural forces. He immediately admits that he lacks the experimental evidence to account for its laws, and on that note of ignorance the book closes.

1.3.7 Mathematical Methods

Newton's sheer mathematical genius is evident throughout the *Principia*, as he tackles an astounding range of different problems using a variety of innovative techniques. Yet Newton's mathematical style is immediately jarring for a modern reader, and it would have been nearly as jarring for many readers within a generation of the publication of the first edition. For a variety of reasons, Newton chose to adopt a geometrical style of reasoning in the *Principia* and to suppress many of his most novel mathematical techniques. Although we lack the space to discuss these issues in depth, we will briefly comment on Newton's mathematical prologue, regarding the method of ultimate ratios (Section 1, Book 1), and discuss two examples illustrating the *Principia*'s mathematical style.

In the priority dispute with Leibniz regarding the invention of the calculus, Newton and his proponents claimed that the *Principia* was initially formulated using the methods of fluxional analysis Newton had discovered in the 1670s, but publicly presented in the superior geometrical style exemplified by Huygens's *Horologium Oscillatorium.* This claim is certainly an exaggeration: the *De Motu* drafts and other manuscripts leading up to the *Principia* all employ the same geometrical limit reasoning as the *Principia*, and there is no evidence of other techniques used initially that were then suppressed.[59]

The question of whether Newton used the calculus is, more importantly, ill-formed, as Whiteside has emphasized (Whiteside, 1991). The geometrical approach of the *Principia* allowed Newton to handle problems regarding instantaneous variation of quantities now treated with the calculus. Huygens and various others had treated similar problems using techniques like Newton's. At the time the contrast between "the calculus" and these techniques was partly stylistic and partly substantive. Leibniz explicitly argued in favor of blind manipulation of symbols — "the imagination would

[59] "The published state of the *Principia* – one in which the geometrical limit-increment of a variable line segment plays a fundamental role – is exactly that in which it was written" (Whiteside, 1970, p. 119). Whiteside further notes that the few attempts at presenting the *Principia* in terms of fluxions are clearly from well after 1st edition and seem to have been quickly abandoned.

be freed from the perpetual attention to figures" — whereas for Newton the attention to figures was a crucial source of certainty for mathematics.[60] Substantively, the two approaches differed in degree of generality and ease of use in particular problems.[61] The contrast between "the calculus" and Newton's geometric reasoning is thus not as sharp as the question presumes.[62] But, in addition, the question is ill-formed because the *Principia* is not based on a single mathematical method. The synthetic, geometric style Newton chose cannot completely hide the innovative mathematical techniques that informed its composition. Contemporary readers were left frustrated by several lacunae where Newton appealed to unspecified algorithmic procedures, notably quadrature techniques (i.e., ways of finding the area under a curve), and by the overly concise presentation of new ideas, as in Lemma 2 of Book 2.[63] In this sense the *Principia* does establish Newton's mastery of techniques directly related to the priority dispute, even though he did not disclose how he arrived at the results.

The argument in favor of Theorem 3 of the *De Motu* described above illustrates Newton's geometric style of reasoning. The most striking contrast with ancient geometry is the treatment of quantities as generated via a continuous motion; this continuity underwrites the evaluation of "ultimate" ratios that hold as, for example, the point Q approaches P in Figure 1.3.1 above. Infinitesimal quantities were handled entirely in terms of limits, whose existence and uniqueness followed from the continuity of the motion generating the quantities. As an illustration, Lemma 10 concerns geometric quantities that characterize an arbitrary curved line and provides a generalization of Galileo's treatment of free fall — namely, that the position varies "ultimately" as the square of the time. The proof proceeds, roughly speaking, by establishing equalities between ratios of evanescent quantities, which vanish as a point along a curve flows back into an initial point, and ratios of finite quantities. The continuity of the generation of the curve guarantees, Newton argues, that a limiting value for the ratio of evanescent quantities exists, and it is given by the ratio of finite quantities.

Many of the Lemmas in Section 1 had appeared elsewhere, and were used implicitly by Newton's contemporaries such as Huygens. Newton identifies his main innovation as providing proofs of them using limit arguments rather than appealing to the method of indivisibles. Contrasted with Leibniz's symbolic calculus, Newton's geometric approach sacrifices generality but allows him to deal much more directly with the differential

[60] See Part VI of (Guicciardini, 2009) for a recent discussion; the quotation is from *Mathematische Schriften*, Volume 5, p. 393 (Leibniz 1962).

[61] This is not to say that a calculus based approach is uniformly easier than Newton's geometric approach; rather, for some problems Newton's methods are particularly illuminating and more powerful, whereas for others a calculus based approach is more fruitful.

[62] This is in part due to attributing too much to Leibnizian formulations. Leibniz and Newton *both* lacked various key concepts, such as that of a function, introduced by Euler. Projecting Euler's innovations back into Leibniz's work falsely enhances the contrast. See (Fraser, 2003) for an overview of eighteenth century innovations in mathematics and mechanics, emphasizing the contrast between the work of Euler, Lagrange, and their contemporaries and the earlier work of Leibniz and others.

[63] Newton does remark in a scholium to this Lemma 2 that he had corresponded with Leibniz regarding these methods, which he says "hardly differed from mine except in the forms of words and notations." In the third edition, Leibniz's name does not appear, and Newton instead emphasizes that he had discovered these methods in 1672.

properties of curves.[64] Newton also regarded the geometric approach as more intuitive, certain, and direct, as opposed to algebraic techniques which he once characterized as nauseating (*Mathematical Papers*, Volume 4, p. 277). The lemmas have proved controversial from the initial appearance of the *Principia*. Leibniz's marginal notes in his annotated copy of the *Principia* indicate that he doubted the truth of Lemmas 9, 10, and 11, and later Berkeley and others famously attacked the rigor of Newton's approach. Newton did not address foundational questions regarding the calculus that would become the focus of later debates. However, the collection of lemmas is not ad hoc as has been sometimes claimed; Pourciau (1998) argues that the 11 Lemmas taken together provide a geometric formulation of the main definitions and theorems of the calculus needed in the *Principia*.

Section 1 ostensibly provides the mathematical background needed for the rest of the book, but at several points Newton relies on sophisticated mathematics that was not common knowledge. The inverse problem for an arbitrary force law is solved in Propositions 1.41-42 up to quadrature. But the utility of this result is limited by the analytical techniques available to perform this quadrature, and Newton only states one explicit solution.[65] In Corollary 3 of Proposition 41 Newton applies the general result to an inverse cube force law (attractive or repulsive) to determine the trajectory, and notes that "All this follows ... by means of the quadrature of a certain curve, the finding of which, as being easy enough, I omit for the sake of brevity" (Newton, 1726, p. 532). The quadrature required certainly was not easy for Newton's contemporaries, and the construction of the trajectories Newton specified in the *Principia* was unilluminating.[66] This is just one example where Newton invoked analytical techniques that are left completely unspecified. After the first edition, Newton considered augmenting the *Principia* with mathematical appendices that would provide more of the techniques required throughout the text. This case also illustrates a second point: Newton's success in recognizing the crucial physical principles for understanding a problem was not always matched with the mathematical methods needed to exploit it.

Newton's discussion of Kepler's problem in Section 6 includes a mathematical lemma that is absolutely stunning.[67] Newton invokes the lemma to prove that there is no algebraic function that will give the position along an elliptical trajectory as

[64] The most serious limitation facing this geometrical approach is that it is difficult to distinguish between infinitesimals of different order. It is relatively straightforward to classify infinitesimals geometrically up to second order, but Newton himself had difficulty with third-order infinitesimals (see, e.g., the discussion of Prop. 2.10 in Guicciardini 1999, pp. 233-247).

[65] The difficulty persists to the present: in modern terms, this line of thought leads to a differential equation which one then integrates to find the trajectory, and the resulting integral is only analytically solvable for a few specific force laws.

[66] The difficulty Newton's contemporaries faced in reconstructing these "easy" results is illustrated by David Gregory's correspondence with Newton regarding this corollary, discussed in Chapter 12 of (Guicciardini, 2009). Guicciardini has noted in personal communication that the quadrature required for the inverse-square case is more complicated than the case of the inverse-cube force law, although he thinks that it would have been within Newton's grasp (even though there is no manuscript evidence that he could perform the required integration).

[67] Cf. the discussions of this lemma in (Guicciardini, 2009), Chapter 13, and (Pourciau, 2001), which we draw on here.

a function of the time; in modern terms, that there is no algebraic solution for x to Kepler's equation: $x - e \sin x = t$.[68] The lemma itself offers an argument for a more general result: that the areas of "oval" figures cannot be fixed by algebraic equations with a finite number of terms. Consider an arbitrary "pole" O in the interior of a given oval, with a ray rotating from a given initial point along the oval. Newton introduces the area swept out by the rotating ray geometrically via a point along the ray moving with a velocity proportional to the square of the distance between the pole and the ray's point of intersection with the oval. As the ray rotates about the pole O, the point representing the area swept out moves away from O in an infinite spiral. But any line intersects the spiral infinitely many times, implying that the spiral cannot be specified by an equation of finite degree. Newton concludes that there are hence no ovals such that the area cut off by straight lines can be found via an algebraic equation. Newton could still handle the problem, using the infinite series expansions that had been the key to his earliest innovations in mathematics. He was thus able to handle a much broader class of curves, including the spiral used in this argument, than Descartes had considered legitimate in the *Géométrie*.

Newton's contemporaries were divided in response to this argument, mainly due to the ambiguity regarding what he meant by "ovals."[69] But Newton's approach to the problem is strikingly innovative, in that it utilizes an unprecedented topological argument and proves a result regarding the existence of a particular kind of function. Arnol'd (1990) argues that Newton's topological, global approach anticipates the pioneering work of Poincaré two centuries later.

1.4 Impact

In the *Principia* Newton aimed to establish both that the force of gravity suffices to account for nearly all celestial motions and for many terrestrial phenomena, and to introduce a way of reasoning more securely from the phenomena in natural philosophy. Above we distinguished three steps in the argument for univeral gravitation: (1) that the motions of the planets, their satellites, and comets can be accounted for by the force of gravity, (2) that the force of gravity is *universal*, and (3) that gravity is a "fundamental force." We also characterized Newton's proposed new way of inquiry and his reasons for taking it to be superior to the hypothetical reasoning favored by his contemporaries.

Judged by whether Newton persuaded his contemporaries on these issues, the *Principia*'s first edition enjoyed limited success in England and nearly complete failure on the Continent.[70] Halley penned an admiring précis, enclosed with the copy given to James II and published in the *Philosophical Transactions*. Flamsteed and Halley competently utilized Newton's ideas in subsequent research, but many of the members of the Royal Society who apparently accepted the *Principia*'s claims did so without a

[68] A curve is defined to be algebraic if it is part of or identical with the solution of some polynomial.

[69] Huygens and Leibniz immediately considered possible counterexamples to the claim; see (Pourciau, 2001).

[70] See (Guicciardini, 2003) for a detailed study of the reception of the *Principia* among British and continental mathematicians, up to about 1736.

thorough command of the book.[71] Hooke accused Newton of plagiarism with regard to the inverse-square law, but he lacked the mathematical skill to follow Newton's reasoning in detail. Sir Christopher Wren and John Wallis certainly possessed the technique, but there are no extant records of their assessment or clear signs of influence. It fell to a younger generation of natural philosophers to build on Newton's work, in particular the Scottish mathematician David Gregory and talented young Englishman, Roger Cotes, who died at 33 shortly after editing the second edition. Cotes contributed to the second edition a clear account of Newton's argument for universal gravitation, a polemical reply to Continental critics. Prior to the appearance of the second edition, few if any natural philosophers in England other than Cotes, Flamsteed, Gregory, Halley and John Keill could understand Newton's main aims and assess critically whether he attained them.

On the Continent, Newton's *Principia* was, in effect, read alongside Descartes's *Principia Philosophia* and found to suffer by comparison. An influential early French review by the Cartesian, Régis, suggested that the *Principia* was a contribution to mathematics or mechanics, but not to physics:[72]

> The work of M. Newton is a mechanics, the most perfect that one could imagine, as it is not possible to make demonstrations more precise or more exact than those he gives in the first two books... . But one has to confess that one cannot regard these demonstrations otherwise than as only mechanical ... In order to make an opus as perfect as possible, M. Newton has only to give us a Physics as exact as his Mechanics.

The view that a more perfect opus must include a mechanical explanation of gravity in terms of action by contact was common in Cartesian circles. The debate between the two *Principia*'s was often framed, as in Fontenelle's *Elogium* (1730), as between two competing hypothetical accounts of celestial motion to be evaluated in terms of their explanatory power. Within this context the reliance on attraction without an underlying mechanism was seen as a crucial flaw.

The denial that the *Principia* was a 'physics' meant that it was understood as not providing a causal account of nature. Even so, the *Principia* provided an influential framework for addressing a variety of problems in rational mechanics. It immediately had an impact on existing research traditions in mechanics. In Paris, for example, Varignon recognized the significance of Newton's achievement and introduced the *Principia* to the Paris Academy of Sciences. He derived several of Newton's main results using a more Leibnizian mathematical style starting in 1700 (Blay, 1999). The initial reception of the *Principia*, however, focused mainly on the account of gravity and planetary motions.

Critics of Newtonian attraction attempted to embed aspects of the *Princpia*'s planetary dynamics within a vortex theory.[73] Leibniz proposed a vortex theory in 1689 that leads to motion along elliptical orbits obeying Kepler's area rule (Leibniz, 1689). Despite his claim to independent discovery, manuscripts uncovered by Bertoloni Meli reveal that this theory was developed in response to a close reading of the opening

[71] Cite sources for Flamsteed and Halley's later work.

[72] The review appeared in the *Journal de Savants* in 1688, and we quote the translation from (Cohen, 1980, p. 96).

[73] Aiton (1972) gives an authoritative account of the development of vortex theory during this period.

sections of Book 1 (Bertoloni Meli, 1997). As imitation is the sincerest form of flattery, Leibniz's efforts indicate his recognition of Newton's achievement. But Leibniz seems to have appreciated only part of what Newton had accomplished, in effect reading the *Principia* as very close to the first *De Motu* manuscript in that he gleaned the results already contained in the *De Motu* from the text but not much else. The important further results Newton developed in order to treat the complexities of real motions, including departures from Keplerian motion and lunar motion, seem to have escaped Leibniz's notice. Leibniz did not recognize the possibility for further empirical assessment of the theory based on treating deviations from Keplerian motion and the various loose ends in Book 3. But in terms of explanatory power and intelligibility, Leibniz clearly took his own account to be superior to Newtonian attraction. In addition to avoiding attraction, Leibniz argued (in correspondence with Huygens) that his theory, unlike Newton's, could account for the fact that the planets orbit in the same direction and in nearly the same plane.

The strong opposition to attraction spurred the savants of Paris and Basel to develop a number of competing vortex theories, following Leibniz's lead. In particular, Johann Bernoulli criticized most of the mathematical parts of book 2. This was capped by Bernoulli's 1730 apt criticism of Newton's treatment of torque.[74] In the second edition, Newton and Cotes pressed a different, powerful objection to vortex theory, independent of the treatment in Book 2: vortex theories had great difficulty in accounting for the motion of comets, especially retrograde comets. Active work on vortex theories declined in mid-century in light of the work by Clairaut, Euler, and others described below.

Leibniz's later critical comments on the *Principia*, following the acrimonious priority dispute regarding the calculus, focused almost exclusively on Newton's metaphysics and his objectionable reliance on action-at-a-distance.[75] There is still a tendency to regard this issue as the crucial factor in the initial critical response to the *Principia*. Newton almost certainly found this way of framing the debate infuriating, and his responses to Leibniz emphasize the force of empirical considerations.

But there were also important early criticisms on terms that Newton would have accepted, namely regarding empirical rather than explanatory success. During the first half of the eighteenth century it was by no means a foregone conclusion that Newton's theory would be vindicated empirically. While Huygens, like Leibniz, regarded action-at-a-distance as "absurd," he acknowledged the force of Newton's argument for the claim that the inverse-square force law governed the behavior of celestial bodies. However, he rejected the inductive generalization to universal gravity and the conception of natural philosophy as aiming to discover fundamental forces. Like Leibniz, Huygens developed a vortex theory that could account for inverse square celestial forces without leading to truly universal gravitation. Huygens's argument turned on showing that no

[74] The criticism appeared in (Bernoulli, 1730), his prize-winning entry in the 1730 Royal Academy of Sciences competition. For a treatment of his detailed criticisms of most of the other results in Book 2 prior to 1730, due primarily to Johann Bernoulli; see Chapter 8 of Guicciardini (1999).

[75] See, in particular, (Bertoloni Meli, 1997) for an excellent treatment of the interplay between Leibniz and Newton.

central body was necessary to generate planetary vorteces. [76]

But, more importantly, Huygens recognized that there is a crucial empirical contrast between universal gravity and his preferred vortex theory. Research Huygens had performed using pendulum clocks to determine longitude at sea seemed to confirm his theory rather than Newton's. Huygens's theory assumed that the clocks slowed down at the equator due to the centrifugal effects due to the Earth's rotation alone, while Newton also added a correction due to mutual attraction of all the particles within the Earth (see Schliesser and Smith, ms). This developed into a lengthy controversy regarding the shape of the Earth (see Terrall, 2002). Newton's and Huygens's theories both implied that the Earth would have an oblate shape, flattened at the poles to different degrees; by contrast, the Cassinis — a famous Italian family of astronomers at the Parisian observatory — claimed that the Earth has an oblong shape.

The controversy could be settled by pendulum measurements far apart, as close as possible to the North pole and the Equator. It took a half century to settle the dispute in Newton's favor after the publication of the results of Maupertuis' expedition to Lapland and La Condamine's to what is now Peru. Maupertuis' book on the topic (Maupertuis 1738) appeared shortly after Voltaire's book-length defense of Newton (Voltaire 1738), aided by Émilie du Châtelet, who later published an influential translation and brilliant commentary on the *Principia*), and together they helped turn the tide in favor of Newtonianism in France.[77] Adam Smith, who followed French developments closely, believed that the "Observations of Astronomers at Lapland and Peru have fully confirmed Sir Isaac's system" (Smith 1982, p. 101).

By 1739 there was no similarly decisive evidence in favor of universal gravity forthcoming from astronomy. The astronomers of Newton's generation did not have the mathematical tools needed to make substantive improvements in accuracy based on Newton's theory. From the second edition of the Principia onward, Newton suggested that the explanation of the Great Inequality in the motion of Jupiter and Saturn could be based on their mutual interaction; this provided a stimulus for much technical work for several generations of mathematicians. Within predictive astronomy the *Principia* had the strongest immediate impact on cometary theory. Newton was the first to treat cometary motions as law-governed, enabling predictions of their periodic return. Based on Newton's methods, Halley published a study of the orbital elements of 24 sets of cometary observations (*Synopsis astronomiae cometicae*, 1705) and argued that comets seen in 1531, 1607, and 1682 were periodic returns of one and the same comet.[78] Halley predicted a return in 1758, but the exact time of the return and the expected position of the comet were uncertain. In addition to the inexactness and small number of observations used to determine the orbit, the determination of the orbit was extremely difficult due to the perturbing effects of Jupiter and Saturn.

[76] Huygens's essay, "Discourse on the Cause of Gravity," was first published in 1690, and is reprinted in Vol. 21 of the *Ouevres Complétes* (Huygens, 1888–1950).

[77] For a detailed discussion of the reception and influence of Newtonianism in France, see (Shank, 2008).

[78] Halley announced this conclusion to the Royal Society of London in 1696, following correspondence with Newton. A revised and expanded version of the *Synopsis* was published posthumously.

The time before the comet's return barely sufficed to develop the necessary methods to calculate the orbit based on Newton's theory.[79] Alexis-Claude Clairaut carried out the first numerical integration to find the perihelion of Halley's comet, an incredibly daunting calculation. In November 1758, rushing to beat the comet itself, he predicted that the comet's perihelion would be within a month of mid-April 1759. It was observed to reach perihelion on March 13th. Clairaut argued that this was an important vindication of Newtonian gravitation, but there was vigorous debate within the Paris Academy regarding the accuracy of his calculation.

Clairaut's calculation of the comet's orbit was based on the approximate solution to the three-body problem he had found a decade earlier.[80] Clairaut and his contemporaries, most importantly Leonhard Euler and Jean le Rond d'Alembert, advanced beyond Newton's qualitative treatment of the three-body problem (in 1.66 and its corollaries) by using analytical methods to construct a perturbative expansion. These analytical approaches relied on a number of post-*Principia* innovations in mathematics, in particular the understanding of trigonometric series, and it is doubtful whether Newton's geometrical methods could have led to anything like them. One of the most striking limitations of the *Principia*'s mathematical style is the apparent limitation to functions of a single independent variable.[81] Within Newton's lifetime, Varignon, Hermann, and Johann Bernoulli had begun formulating Newtonian problems in terms of the Leibnizian calculus. Euler, Clairaut, and d'Alembert drew on this earlier work, but unlike the earlier generation they were able to make significant advances on a number of problems Newton had not been able to treat quantitatively.

In 1747 Euler challenged the inverse-square force law due to an anomaly in the motion of the lunar apsides. Newton suggested that this motion could be accounted for by the perturbing effect of the Sun, but close reading of the *Principia* reveals that the calculated perturbative effect was only one-half of the observed motion. Euler preferred a vortex theory, and used the discovery of this anomaly to criticize the supposition of an exact inverse-square attraction. Clairaut and d'Alembert had both developed perturbative techniques to apply to the motion of the lunar apsides earlier, in 1746. Initially they both reached the same conclusion as Euler (Newton's theory was off by one-half), and considered modifying the inverse-square law. But that proved unnecessary; in 1748 Clairaut carried out a more careful calculation and discovered to his great surprise that the terms he had earlier regarded as negligible exactly eliminated the anomaly (see Wilson, 1995). Euler hailed this result as providing the most decisive confirmation of the inverse law: "...the more I consider this happy discovery, the more important it seems to me.... For it is very certain that it is only since this discovery that one can regard the law of attraction reciprocally proportional to the squares of the distances as solidly established; and on this depends the entire theory of astronomy," (Euler to Clairaut, 29 June 1751; quoted in Waff 1995, p. 46).

[79] This brief summary relies on the clear account given in (Waff, 1995*b*).

[80] For overviews of eighteenth century work on the three body problem, see (Waff, 1995*a*; Wilson, 1995)

[81] This independent variable is usually time, but Newton also treats time as a dependent variable in Prop 2.10, in the second and third editions. Generalizing to functions of a multiple variables was needed for the concepts of partial differentiation and the calculus of variations.

The techniques developed in the 1740s made it possible to assess the implications of universal gravity for a number of open problems in celestial mechanics. Newton suggested (in the second and third editions, 3.13) that the observed inequalities in the motion of Jupiter and Saturn could be accounted for as a consequence of their gravitational interaction. But Flamsteed and Newton's efforts to treat the problem quantitatively were not successful, and the Paris Academy sponsored three consecutive prize essays from 1748-52 regarding the inequalities. Clairaut and d'Alembert were actively working on the three-body problem at this time, and served as members of the prize commission (and hence were ineligible to enter). The leading competitors in these contests — Euler, Daniel Bernoulli, and Roger Boscovich — made important contributions to the problem, but a full treatment was only achieved by Laplace in 1785 (Wilson, 1995).

Analytical techniques developed to treat the three-body problem were also applied to the Earth-Moon-Sun system. Newton had proposed (in Prop. 3.39) that precession of the equinoxes is caused by the gravitational attraction of the Sun and the Moon on the Earth's equatorial bulge.[82] The British Astronomer Royal James Bradley discovered a further effect called nutation in the 1730s, and published his results in 1748. Nutation refers to a slight variation in the precession of the equinoxes, or wobble in the axis of rotation due to the changing orientation of the lunar orbit with respect to the Earth's equatorial bulge. Bradley's observations provided strong evidence for the gravitational effects of the Moon on the Earth's motion, which was almost immediately bolstered by d'Alembert's analytical solution describing nutation (d'Alembert 1749). This successful account of precession and nutation provided evidence for the inverse-square law almost as impressive as Clairaut's calculation. But in addition, d'Alembert's innovations in the course of applying gravitational theory to this problem were important contributions in their own right. Paraphrasing Laplace, d'Alembert's work was the seed that would bear fruit in later treatments of the mechanics of rigid bodies.[83]

By mid-century, universal gravitation was deeply embedded in the practice of celestial mechanics. Treating the solar system as a system of point-masses interacting via Newtonian gravitation led to tremendous advances in understanding the physical factors that play a role in observed motions. These advances stemmed in part from developing more powerful mathematical techniques in order to assess the implications of universal gravity for situations that Newton had been unable to treat quantitatively. Just as it is easy to overestimate the empirical case in Newton's favor in 1687, modern readers often mistakenly treat the *Principia* as containing all of modern rational mechanics. But in fact Newton does not even touch on a number of problems in mechanics that had been discussed by his contemporaries, such as the motion of rigid bodies, angular motion, and torque. Several parts of Book 3, including the account of the tides and the shape of the Earth, were flawed as a result. This is not a simple

[82] Precession of the equinoxes refers to a measurable, periodic variation in the positions of stars in equatorial coordinates, understood in Newton's time (and now) as a result of the motion of the rotational axis of the Earth.

[83] See (Wilson, 1987) for a detailed discussion of d'Alembert (1749), with a clear account of the contrast between Euler's approach and the impact of d'Alembert's work on Euler's (1752) "New Principles of Mechanics."

oversight that could be easily corrected. Extending and developing Newton's ideas to cover broader domains has been an ongoing challenge in mechanics ever since.

An important line of thought in the development of rational mechanics was the effort to assimilate and extend the ideas of the *Principia*. But eighteenth century rational mechanics drew on other, independent lines of thought as well. Pierre Varignon advocated a distinctive approach to mechanics in his *Project of a New Mechanics*, which appeared in the same year as the *Principia*. Newton's work was assimilated to an existing line of research in mechanics, a tradition that had much broader scope. Newton's great contemporaries on the continent — primarily Huygens, Leibniz, and Johann and Jacob Bernoulli — had all made important contributions to a set of long-standing problems in mechanics that Newton did not discuss. These problems involved the behavior of elastic, rigid, and deformable bodies rather than point masses, and their treatment required concepts such as stress, torque, and contact forces. For example, Huygens (1673) found the center of oscillation for a pendulum bob based on what Leibniz would later call conservation of *vis viva*. Jacob Bernoulli treated this problem using the "law of the lever" rather than Huygens's principle, and then extended these ideas to the study of elastic bodies in the 1690s. This line of work was entirely independent of Newton, and Truesdell (1968) argues that the impact of Bernoulli's ideas was nearly as significant as the *Principia* itself. The members of the Basel school treated the *Principia* as posing a challenge, to either re-derive Newton's results on their own terms or to find his errors. Several of the problematic claims in Book 2 acted as an impetus to particular research areas. Newton's treatment of the efflux problem in Proposition 2.36, for example, partially spurred Daniel Bernoulli's development of hydrodynamics.

The rich interplay of these ideas eventually led to formulations of mechanics such as Euler's *Mechanica* (1736), and his later 1752 paper announcing a "New principle of mechanics." This new principle was the statement that $\mathbf{F} = m\mathbf{a}$ applies to mechanical systems of all kinds, discrete or continuous, including point masses and bodies of finite extent. Euler immediately applied this principle to the motion of rigid bodies. There were numerous innovations in Euler's formulations of mechanics, but we emphasize this principle as a warning to those apt to read the work of Euler and others back into Newton. Editions of the *Principia* published during this time, by the Minim friars Le Seur and Jacquier and by Marquise du Châtelet, presented the *Principia* in Eulerian terms and showed how to reformulate some of Newton's results using the symbolic calculus. But by this point the *Principia* itself had largely disappeared from view; it was not required reading for those active in analytic mechanics, and there were better contemporary formulations of the underlying principles of mechanics. The common label "Newtonian mechanics" for these later treatments, while not entirely unjustified, fails to acknowledge the important conceptual innovations that had occurred in the eighteenth century and the ultimate source of these innovations in the work of Huygens, Leibniz, and the Bernoullis.

1.4.1 Cause of Gravity

One of the central questions of eighteenth century philosophy was the nature and cause of gravity. In discussing these matters we should distinguish among a) the force of gravity as a real cause (which is calculated as the product of the masses over the distance squared); b) the cause of gravity; c) "the reason for these [particular–ES] properties of gravity" (*Principia*, p. 943); and d) the medium, if any, through which it is transmitted. Much discussion about Newton conflates these matters. Of course, if the medium can explain all the properties of gravity then it is legitimate to conflate these.

Now one line of thought made popular by Newton in the General Scholium of the *Principia*, is to simply assert "it is enough that gravity really exists and acts according to the laws that we have set forth" (*Principia*, p. 943), while famously remaining agnostic about the causes that might explain it. (See Janiak 2007) On this view one could accept the reality of gravity in the absence of an explanation of it. The significance of this is that future research can be predicated on its existence without worrying about matters external to relatively autonomous ongoing inquiry. While Newton was not the first to defend such an attitude toward inquiry (it echoes his earlier stance in the controversy over his optical research, and during the 1660s members of the Royal Society had investigated experimentally and mathematically the collision rules with a similar stance), his had the most lasting impact.

In his famous correspondence with Leibniz (1715-16), Clarke asserts something similar to Newton's position, although Clarke's argument sometimes suggests a more instrumentalist stance, in which gravity is assumed in order to track and predict effects, namely the relative motion of bodies (Alexander, 1965). In his more revisionary project, Berkeley elaborated this instrumentalist re-interpretation of Newton. For Berkeley (and later Hume) Newton's mathematical science cannot assign causes–this is the job of the metaphysician (Berkeley 1744, p. 119-120 paragraphs 249-251; for discussion see Schliesser 2011). Yet most eighteenth century readers of Newton not only accepted gravity as a causally real force, but were also willing to entertain strikingly divergent positions regarding its causes. This was anticipated by Newton, who already in the first edition of the *Principia* listed at least three different possible mechanisms which could account for attraction (Scholium to 1.69, p. 588):

> I use the word "attraction" here in a general sense for any endeavor whatever of bodies to approach one another, whether that endeavor occurs as a result of the action of the bodies either drawn toward one another or acting on one another by means of spirits emitted or whether it arises from the action of aether or of air or of any medium whatsoever – whether corporeal or incorporeal — in any way impelling toward one another the bodies floating therein.

The "action" of bodies "drawn toward one another" can involve action at a distance. Some of the earliest readers of *Principia* thought that Newton was committed to action at a distance either modeled on Stoic sympathy (as Leibniz dismissively claimed) or on Epicurean innate gravity (as Bentley proposed in now lost letters to Newton). The Stoic sympathy and Epicurean gravity options that interpret attraction as resulting from the nature of bodies go against the previously-dominant view of mechanism, which only permitted contact of bodies as acceptable mechanism.

There is eighteenth century evidence for three accounts of the cause of gravity compatible with Newton's first sense. First, in his editor's preface to the *Principia*, Roger Cotes asserted that gravity was a "primary" quality of matter and put it on a par with impenetrability and other properties often taken to be essential qualities. However, in the third edition Newton made it clear that he did not accept this position, stating that he is "by no means affirming that gravity is essential to bodies"

(*Principia*, p. 796). Moreover, in famous responses to Bentley's letters, Newton explicitly denied "innate" gravity "as essential and inherent to matter" (Newton 2004, p. 102). Nevertheless, Cotes' interpretation became very influential, and was adopted by Immanuel Kant, among others.

A second one was modeled on Locke's superaddition thesis, that is, God could add mind-like qualities to otherwise passive matter. While gravity is not an essential quality of matter, it is certainly in God's power to endow matter with gravitational qualities at creation. This interpretation was encouraged by Newton in his exchange with Bentley and it was taken up by many of the Boyle lecturers that developed eighteenth century physical-theology. It was also made famous in the French-speaking world by a footnote added by the French translator of Locke's *Essay*. A third way was put forward by Newton himself in his posthumously published "Treatise of the System of the World." Curiously, Newton called attention to the existence of this popular, suppressed exposition of his views in the brief "preface" of the third Book in all three editions of the *Principia*, but it is unclear if he had a hand in having it published the year after his death. In the 'Treatise,' Newton offers a relational account of action at a distance that is compatible. On the view presented there, all bodies have a disposition to gravitate, but it is only activated in virtue of them having this common nature. While there is evidence that the 'Treatise' was read in the eighteenth century, the relational view seems not to have been very popular. But it is compatible with the position adopted by D'Alembert in the widely read *Preliminary Discourse* in describing Newton's achievement: "matter may have properties which we did not suspect" (d'Alembert 1751).

Some people attributed to Newton the view that he believed that gravitation is based on the direct will of God. This position was attributed to him by Fatio de Duillier and, perhaps more jokingly, by David Gregory (both of whom were considered as possible editors for a new edition of the *Principia* planned in the 1690s), who were both in his circle especially in the early years after the publication of the first edition of the *Principia*. The position is certainly consistent with Newton's last sense above (assuming God is immaterial), and there are other passages in Newton's writings that seem compatible with it. For example in a letter to Bentley, Newton writes, "Gravity must be caused by an agent acting constantly according to certain laws; but whether this agent be material or immaterial I have left to the consideration of my reader" (Newton 2004, p. 103).

Nevertheless, attributing gravity's cause to God's direct will appears at odds with a very famous passage in the General Scholium, where Newton articulates what he means by God's substantial and virtual omnipresence: "In him [God] all things are contained and moved, but he does not act on them nor they on him. God experiences nothing from the motion of bodies; the bodies feel no resistance from God's omnipresence."[84] Whatever Newton means by asserting both God's substantial and virtual omnipresence, he clearly states that God's presence does not interfere with the motions of bodies — either by offering resistance or impelling them. As David Hume aptly noted: "It was never the meaning of Sir Isaac Newton to rob second causes of

[84] Newton's footnote to the passage explains he is articulating God's dominion.

all force or energy; though some of his followers have endeavoured to establish that theory upon his authority."[85]

Finally, ether theories were very popular during the eighteenth century . Sometimes they were put forward in opposition to Newtonian action at a distance (e.g., by Euler). But we need to note two facts: first, ether theories had Newtonian precedent: Newton tentatively put forward ether accounts in the closing paragraph of the General Scholium and in a famous letter to Boyle known to eighteenth century readers. Newton's proposals were not exactly identical: in his letter to Boyle he conceived of an ether as a compressible fluid; in various Queries to the *Opticks*, Newton emphasizes the different densities of the ether around and between celestial bodies and he speculates about the need for short range repulsive forces within the ether. Second, thus, ether theories nearly always include action at a distance over relatively short ranges. One general problem with ether theories is that they require ethers to have neglible mass, which makes them very hard to detect, while being capable of great strength and rigidity in order to transmit light as fast as Rømer had calculated it goes. But Newton clearly did not rule out an immaterial ether composed of spirits of some sort.

[85] This appears in a footnote at the end of 7.1.25, in (Hume, 2000). Regarding Hume's terminology, God is a first cause whereas laws or forces are secondary causes that act within nature.

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