The Axiom of Choice in the Foundations of Mathematics

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The principle of set theory known as the *Axiom of Choice* (**AC**) has been hailed as "probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago"[1] It has been employed in countless mathematical papers, a number of monographs have been exclusively devoted to it, and it has long played a prominently role in discussions on the foundations of mathematics.

In 1904 Ernst Zermelo formulated the Axiom of Choice in terms of what he called *coverings* (Zermelo [1904]). He starts with an arbitrary set M and uses the symbol M' to denote an arbitrary nonempty subset of M, the collection of which he denotes by M. He continues:

> *Imagine that with every subset M' there is associated an arbitrary element m_1', that occurs in M' itself; let m_1' be called the "distinguished" element of M'. This yields a "covering" γ of the set* M *by certain elements of the set M. The number of these coverings is equal to the product* [of the cardinalities of all the subsets M'] *and is certainly different from 0.*

The last sentence of this quotation—which asserts, in effect, that coverings always exist for the collection of nonempty subsets of any (nonempty) set—is Zermelo's first formulation of **AC**[2]. This is now usually stated in terms of *choice functions*: here a choice function on a collection $\mathscr{S}$ of nonempty sets is a map f

[1] Fraenkel, Bar-Hillel and Levy [1973], §II.4.

[2] Zermelo does not actually give the principle an explicit name at this point, however. He does so only in [1908], where he uses the term "postulate of choice".

with domain $\mathscr{S}$ such that $f(X) \in X$ for every $X \in \mathscr{S}$. Zermelo's first formulation of the Axiom of Choice then reads:

AC1 *Any collection of nonempty sets has a choice function.*

AC1 can also be reformulated in terms of relations, viz.

AC2 *for any relation R between sets A, B,*

$$\forall x{\in}A\exists y{\in}B\ R(x,y) \Rightarrow \exists f\!: A \to B\ \forall x{\in}A\ R(x, fx).$$

In his [1908] Zermelo offered a formulation of **AC** couched in somewhat different terms from that given in his earlier paper. Let us call a *choice set* for a family of sets $\mathscr{S}$ any subset $T \subseteq \cup\mathscr{S}$ for which each intersection $T \cap X$ for $X \in \mathscr{S}$ has exactly one element. Zermelo's second formulation of **AC** amounts to the assertion[3] that any family of mutually disjoint nonempty sets has a choice.

Zermelo asserts that "the purely objective character" of this principle "is immediately evident." In making this assertion meant to emphasize the fact that in this form the principle makes no appeal to the possibility of making "choices". It may also be that Zermelo had something like the following "combinatorial" justification of the principle in mind. Given a family $\mathscr{S}$ of mutually disjoint nonempty sets, call a subset $S \subseteq \cup\mathscr{S}$ a *selector* for $\mathscr{S}$ if $S \cap X \neq \varnothing$ for all $X \in \mathscr{S}$. Clearly selectors for $\mathscr{S}$ exist; $\cup\mathscr{S}$ itself is an example. Now one can imagine taking a selector S for $\mathscr{S}$ and "thinning out" each intersection $S \cap X$ for $X \in \mathscr{S}$ until it contains just a single element. The result[4] is a choice set for $\mathscr{S}$.

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[3] Zermelo's formulation reads literally:

> *A set S that can be decomposed into a set of disjoint parts A, B, C, ... , each containing at least one element, possess at least one subset S_1 having exactly one element with each of the parts A, B, C, ... , considered.*

[4] This argument, suitably refined, yields a rigorous derivation of **AC** in this formulation from Zorn's lemma.

Let us call Zermelo's 1908 formulation the *combinatorial* axiom of choice:

CAC [5] *Any collection of mutually disjoint nonempty sets has a choice set.*

It is to be noted that **AC1** and **CAC** for *finite* collections of sets are both provable (by induction) in the usual set theories.

As is well-known, Zermelo's original purpose in introducing **AC** was to establish a central principle of Cantor's set theory, namely, that every set admits a well-ordering and so can also be assigned a cardinal number. His introduction of the axiom, as well as the use to which he put it, provoked considerable criticism from the mathematicians of the day. The chief objection raised was to what some saw as its highly non-constructive, even idealist, character: while the axiom asserts the possibility of making a number of—perhaps even uncountably many—arbitrary "choices", it gives no indication whatsoever of how these latter are actually to be effected, of how, otherwise put, choice functions are to be *defined.* For this reason Bertrand Russell regarded the principle as doubtful at best. The French Empiricists Baire, Borel and Lebesgue, for whom a mathematical object could be asserted to exist only if it can be uniquely defined went further in explicitly repudiating the principle in the uncountable case.

On the other hand, a number of mathematicians came to regard the Axiom of Choice as being true *a priori.* These all broadly shared the view that for a mathematical entity to exist it was not necessary that it be uniquely definable. Zermelo himself calls **AC** a "logical principle" which "cannot … be reduced to a still simpler one" but which, nevertheless, "is applied without hesitation everywhere in mathematical deductions." Ramsey asserts that "the

[5] It is this formulation of **AC** that Russell and others refer to as the *multiplicative axiom*, since it is easily seen to be equivalent to the assertion that the product of arbitrary nonzero cardinal numbers is nonzero.

Multiplicative Axiom seems to me the most evident tautology"[6]. Hilbert employed **AC** in his defence of classical mathematical reasoning against the attacks of the intuitionists: indeed his ε-operators are essentially just choice functions. For him, "the essential idea on which the axiom of choice is based constitutes a general logical principle which, even for the first elements of mathematical inference, is indispensable."[7]

A particularly interesting analysis of the axiom of choice was formulated by Paul Bernays[8]. He saw **AC** as the result of a natural extrapolation of what he terms "extensional logic", valid in the realm of the finite, to infinite totalities. He considers formulation **AC2**, with the two sets *A* and *B* identical. In the special case in which *A* contains just two (or, more generally, finitely many elements), **AC2** is essentially just the usual distributive law for ∧ over ∨. Bernays now observes:

> *The universal statement of the principle of choice is then nothing other than the extension of an elementary-logical law* [i.e. the distributive law] *for conjunction and disjunction to infinite totalities, and the principle of choice constitutes thus a completion of the logical rules that concerns the universal and the existential judgment, that is, of the rules of existential inference, whose application to infinite totalities also has the meaning that certain elementary laws for conjunction and disjunction are transferred to the infinite.*

He goes on to remark that the principle of choice "is entitled to a special position only to the degree that the *concept of function* is required for its formulation." Most striking is his further assertion that the concept of function "in turn receives an adequate implicit characterization only through the principle of choice."

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[6] Ramsey [1926].

[7] Quoted in section 4.8 of Moore [1982].

[8] Bernays [1930-31], translated in Mancosu [1998]

What Bernays seems to be saying here is that in asserting the antecedent of **AC2**, in this case $\forall x\in A\exists y\in A\ R(x,y)$, one is implicitly asserting the existence of a function f: $A \to A$ for which $R(x,fx)$ holds for all x —that is, the consequent of **AC2.** On the surface, this seems remarkably similar to the justification of **AC** under constructive interpretations of the quantifiers: indeed, under (some of) those interpretations (discussed further below), the assertability of an alternation of quantifiers $\forall x\exists yR(x,y)$ means *precisely* that one is given a function f for which $R(x,fx)$ holds for all x. However, Bernays goes on to draw the conclusion that, for the concept of function arising in this way, "the existence of a function with a [given] property in no way guarantees the existence of a concept-formation through which a determinate function with [that] property is uniquely fixed." In other words, the existence of a function may be asserted without the ability to provide it with an explicit definition[9]. This is incompatible with stronger versions of constructivism.

Bernays and the constructivists both affirm **AC2** through the claim that its antecedent and its consequent *have the same meaning*. The difference is that, while Bernays in essence agrees with the constructive interpretation in treating the quantifier block $\forall x\exists y$ as meaning $\exists f\forall x$, he interprets the existential quantifier in the latter *classically*, so that in affirming "there is a function " it is not necessary, as under the constructive interpretation, actually to be *given* such a function.

Per Martin-Löf has recently[10] contrasted the constructive affirmability of Zermelo's 1904 formulation of the axiom of choice – which we shall take in the version **AC2,** and which Martin-Löf terms the *intensional* axiom of choice – with Zermelo's 1908 formulation, the combinatorial axiom of choice **CAC.**

Martin-Löf's discussion takes place within a simplified version of *constructive (dependent) type theory* (CTT), the system of constructive mathematics, based on intuitionistic logic, he introduced some years ago and which has

[9] This fact, according to Bernays, renders the usual objections against the principle of choice invalid, since these latter are based on the misapprehension that the principle " claims the possibility of a choice".
[10] Martin-Löf [2006].

become standard[11]. In CTT the primitive relation of identity of objects (necessarily of the same type) is *intensional.* In set theory, on the other hand, the identity relation is treated extensionally since two sets are identified if they have the same elements (Axiom of Extensionality). In CTT a set in the usual set-theoretic sense corresponds to a to an *extensional set*, that is, a set carrying an equivalence relation representing "extensional" equality of its elements.

That being the case, it is natural to formulate within CTT a version of **AC** for extensional sets**.** Martin-Löf calls this the *extensional* axiom of choice (**EAC**). To state this we need to introduce the notion of an extensional function. Thus let A and B be two sets carrying equivalence relations $=_A$ and $=_B$ respectively. A function $f: A \to B$ is called *extensional,* Ext(f), if $\forall xx' \in A\ (x =_A x' \to fx =_B fx')$. Then **EAC** may be stated: for any relation R between A and B,

$$\forall x \in A \exists y \in B\ R(x,y) \Rightarrow \exists f: A \to B\ [\mathrm{Ext}(f) \wedge \forall x \in A\ R(x, fx)].$$

Martin-Löf shows that, in CTT, **CAC** and **EAC** are equivalent.

Now the equivalence between **CAC** and **EAC,** is established within CTT where **AC2** is *already provable*[12]. There the equivalence between **CAC** and **EAC** is a nontrivial assertion. In *set theory*, on the other hand, not only are **CAC** and **EAC** equivalent, but they are themselves both equivalent to **AC2.** It becomes natural then to ask: can Martin-Löf's argument be presented within set theory without courting triviality?

I believe this can be done by noting that Martin-Löf also establishes the equivalence, in CTT, of **CAC** with the assertion that unique representatives can be picked from the equivalence classes of any given equivalence relation. Let us abbreviate this as **EQ.** In deriving **CAC** (actually the equivalent **EAC,** but no

[11] Martin-Löf [1975], [1982], [1984].
[12] For a proof see, e.g., Tait [1994].

matter) from **EQ,** Martin-Löf employs **AC2,** so establishing, in CTT, the implication

$$\mathbf{EQ + AC2 \Rightarrow CAC}$$

The problem thus boils down to giving a faithful version of the argument for this implication within set theory.

To do this, **AC2** must be furnished with a *constructively valid set-theoretical* formulation. This can be achieved by invoking the "propositions as types" doctrine (PAT)[13] underlying CTT. **CDTT** The central thesis of PAT is that each proposition is to be *identified* with the type, set, or assemblage of its proofs. As a result, such proof types, or sets of proofs, have to be accounted the *only* types, or sets. Strikingly, then, in the "propositions as types" doctrine, a type, or set, simply *is* the type, or set, of proofs of a proposition, and, reciprocally, a proposition *is* just the type, or set, of its proofs. In PAT logical operations on propositions are interpreted as certain mathematical operations on sets: in particular $\forall$ is interpreted as Cartesian product $\prod$ and $\exists$ as coproduct (disjoint union) $\coprod$.[14]

Under PAT, **AC2** may be taken to assert the existence, for any doubly-indexed family of sets $\{A_{ij} : i \in I, j \in J\}$, of a bijection

$$(+) \qquad \prod_{i\in I} \coprod_{j\in J} A_{ij} \cong \coprod_{f\in J^I} \prod_{i\in I} A_{if(i)} .$$

The requisite, indeed canonical, isomorphism is easily supplied in the form of the map

[13] See Tait [1994].

[14] Here $\coprod_{i\in I} A_i$ may be identified with $\bigcup_{i\in I} (A_i \times \{i\})$.

$$g \mapsto (\pi_1 \circ g, \pi_2 \circ g) = g^*,$$

where π_1, π_2 are the projections of ordered pairs onto their first and second coordinates.

Note that

(#) for $g \in \prod_{i \in I} \coprod_{j \in J} A_{ij}$, g^* is a pair of functions (e, f) with $f \in J^I$ and $e \in \prod_{i \in I} A_{if(i)}$.

Now **CAC** can be shown, in standard (intuitionistic) set theory, to be equivalent to the assertion that, for any doubly-indexed family of sets $\{A_{ij} : i \in I, j \in J\}$,

$$\prod_{i \in I} \bigcup_{j \in J} A_{ij} = \bigcup_{f \in J^I} \prod_{i \in I} A_{if(i)}.$$

which is in turn equivalent to

(*) $$\prod_{i \in I} \bigcup_{j \in J} A_{ij} \subseteq \bigcup_{f \in J^I} \prod_{i \in I} A_{if(i)}.$$

I shall present a natural derivation within set theory of (*) from **(+)** and **EQ**, so providing what seems to me a purely set-theoretical formulation of Martin-Löf's argument.

First observe that there is a natural epimorphism

$$\prod_{i \in I} \coprod_{j \in J} A_{ij} \twoheadrightarrow \prod_{i \in I} \bigcup_{j \in J} A_{ij}$$

given by

$$g \mapsto \pi_1 \circ g$$

Write ≈ for the equivalence relation on $\prod_{i\in I} \coprod_{j\in J} A_{ij}$ given by

$$g \approx h \Leftrightarrow \pi_1 \circ g = \pi_1 \circ h.$$

Each $k \in \prod_{i\in I} \bigcup_{j\in J} A_{ij}$ may be identified with the ≈-equivalence class $\{g: \pi_1 \circ g = k\} = \tilde{k}$. Using **EQ**, choose a system of unique representatives from the ≈-equivalence classes. This amounts to introducing a map

$$u: \prod_{i\in I} \bigcup_{j\in J} A_{ij} \to \prod_{i\in I} \coprod_{j\in J} A_{ij}$$

for which $u(k) \in \tilde{k}$, i.e.

(**) $\pi_1 \circ u(k) = k,$

for all $k \in \prod_{i\in I} \bigcup_{j\in J} A_{ij}$.

Now to establish (*), we take any $k \in \prod_{i\in I} \bigcup_{j\in J} A_{ij}$. Then under the natural bijection between $\prod_{i\in I} \coprod_{j\in J} A_{ij}$ and $\coprod_{f\in J^I} \prod_{i\in I} A_{if(i)}$ given in (+), $u(k)$ is correlated with the pair of maps

$$(\pi_1 \circ u(k)\ ,\ \pi_2 \circ u(k)),$$

i.e., using (**), with

$$(k\,,\,\pi_2 \circ u(k)).$$

Writing $f = \pi_2 \circ u(k)$, it follows from (#) that

$$f \in J^I \text{ and } k \in \prod_{i\in I} A_{if(i)}\,,$$

whence

$$k \in \bigcup_{f\in J^I} \prod_{i\in I} A_{if(i)}\,.$$

So we have derived (*).

What is really going here appears to be the following. Under the epimorphism

$$\prod_{i\in I} \coprod_{j\in J} A_{ij} \twoheadrightarrow \prod_{i\in I} \bigcup_{j\in J} A_{ij}$$

information is "lost", to wit, the identity , for a given member g of the domain of the epi, and an arbitrary $i \in I$, of the $j \in J$ for which $g(i) \in A_{ij}$. The map u furnished by **EQ** essentially resupplies that information. So starting with $k \in \prod_{i\in I} \bigcup_{j\in J} A_{ij}$, if one applies u to it, and then applies to the result the bijection given in (+), one winds up with a map $f \in J^I$ for which $k(i) \in A_{if(i)}$ for all $i \in I$. This is precisely what is demanded by (*).

In an intensional constructive framework such as CTT, the axiom of choice is compatible with intuitionistic logic, that is, with the non-affirmation of the law of excluded middle. But in 1975 Diaconescu showed[15] that, in extensional frameworks such as topos theory or set theory, the usual formulations of the axiom of choice *imply the law of excluded middle*, so making logic classical. And Martin-Löf's analysis shows that, in CTT, the imposition of (a form of) extensionality on the axiom of choice will enable Diaconescu's theorem to become applicable, again yielding classical logic[16]. That extensionality in some form is required to derive Diaconescu's theorem can be observed in a number of different ways in addition to Martin-Löf's penetrating analysis. Here are three.

1. *Second-order logic.* Let $\mathscr{L}$ be a second-order language with individual variables *x, y, z, ...,* predicate variables *X, Y, Z, ...* and second-order function variables *F, G, H, ...* . Here a second-order function variable *F* may be applied to a predicate variable *X* to yield an individual term *FX.* The scheme of sentences

AC* $\forall X[\Phi(X) \to \exists x X(x)] \to \exists F\, \forall X[\Phi(X) \to X(FX)]$

may be taken as the axiom of choice in $\mathscr{L}$.

We assume that the background logic of $\mathscr{L}$ is intuitionistic logic. Given certain mild further presuppositions, **AC** can be shown to imply **LEM**, the law of excluded middle that, for any for any proposition *A*, $A \vee \neg A$. These mild further presuppositions latter may be stated:

Predicative Comprehension $\exists X\, \forall x[X(x) \leftrightarrow \varphi(x)]$

[15] Diaconescu [1975].

[16] Note, however, that if the axiom of choice is formulated within set theory or topos theory in the "harmless" —indeed mathematically useless— way (+), it is perfectly compatible with intuitionistic logic.

Here φ is a formula not containing any bound predicate variables.

Extensionality of Functions $\forall X\, \forall Y\, \forall F[X \equiv Y \rightarrow FX = FY]$

Here $X \equiv Y$ is an abbreviation for $\forall x[X(x) \leftrightarrow Y(x)]$, that is, X and Y are *extensionally equivalent.*

In addition we assume the presence of two individuals 0 and 1. Their distinctness is expressed by means of the trivial presupposition $0 \neq 1$.

Now let A be a given proposition. By Predicative Comprehension, we may introduce predicate constants U, V together with the assertions

(1) $\forall x[U(x) \leftrightarrow (A \vee x = 0)]$ $\forall x[V(x) \leftrightarrow (A \vee x = 1)]$

Let $\Phi(X)$ be the formula $X \equiv U \vee X \equiv V$. Then clearly we may assert $\forall X[\Phi(X) \rightarrow \exists x X(x)]$ so **AC*** may be invoked to assert $\exists F\, \forall X[\Phi(X) \rightarrow X(FX)]$. Now we can introduce a function constant K together with the assertion

(2) $\forall X[\Phi(X) \rightarrow X(KX)]$.

Evidently we may assert $\Phi(U)$ and $\Phi(V)$, so it follows from (2) that we may assert $U(KU)$ and $V(KV)$, whence also, using (1),

$$[A \vee KU = 0\,] \wedge [A \vee KV = 1].$$

Using the distributive law (which holds in intuitionistic logic), it follows that we may assert

$$A \vee [KU = 0 \wedge KV = 1].$$

From the presupposition that $0 \neq 1$ it follows that

(3) $$A \vee KU \neq KV$$

is assertable. But it follows from (1) that we may assert $A \to U \equiv V$, and so also, using Extensionality of Functions, $A \to KU = KV$. This yields the assertability of $KU \neq KV \to \neg A$, which, together with (3) in turn yields the assertability of

$$A \vee \neg A,$$

that is, **LEM.**

Note that in deriving **LEM** from version **AC** essential use was made of the principles of Predicative Comprehension and Extensionality of Functions. It follows that, in systems of constructive mathematics affirming **AC** (but not **LEM**) *either the principle of Predicative Comprehension or the Principle of Extensionality of Functions must fail.* While the Principle of Predicative Comprehension can be given a constructive justification, no such justification can be provided for the principle of Extensionality of Functions. Functions on predicates are given intensionally, and satisfy just the corresponding Principle of Intensionality $\forall X\ \forall Y\ \forall F[X = Y \to FX = FY]$. The Principle of Extensionality can easily be made to fail by considering, for example, the predicates P: *rational featherless biped* and Q: *human being* and the function K on predicates which assigns to each predicate the number of words in its description. Then we can agree that $P \equiv Q$ but $KP = 3$ and $KQ = 2$.

2. *Hilbert's Epsilon Calculus.*. In the logical calculus developed by Hilbert in the 1920s the Axiom of Choice appears in the form of a postulate he called the *logical* ε*-axiom* or the *transfinite axiom.* To formulate this postulate he introduced, for each formula $\alpha(x)$, a term (an *epsilon term*) $\varepsilon_x\alpha$ or simply ε_α which, intuitively, is intended to name an indeterminate object satisfying $\alpha(x)$. The ε-axiom then takes the form

(ε) $\exists x\alpha(x) \to \alpha(\varepsilon_\alpha)$.

All that is known about ε_α is that, if anything satisfies α, it does[17]. Now since α may contain free variables other than x, the identity of ε_α depends, in general, on the values assigned to these variables. So ε_α may be regarded as the result of having chosen, for each assignment of values to these other variables, a value of x so that $\alpha(x)$ is satisfied. That is, ε_α may be construed as a choice function, and the ε-axiom accordingly seen as a version of **AC.**

An *ε-calculus* $\mathscr{P}_\varepsilon$ is obtained by starting with a system $\mathscr{P}$ of first-order predicate logic, augmenting it with epsilon terms, and adjoining as an axiom scheme the formulas (ε). It is known that when $\mathscr{P}$ is classical predicate logic, $\mathscr{P}_\varepsilon$ is *conservative* over $\mathscr{P}$, that is, each assertion of $\mathscr{P}$ demonstrable in $\mathscr{P}_\varepsilon$ is also demonstrable in $\mathscr{P}$: the move from $\mathscr{P}$ to $\mathscr{P}_\varepsilon$ does not enlarge the body of demonstrable assertions in $\mathscr{P}$. But for *intuitionistic* predicate logic the situation is otherwise.

In fact it is easy to see that, if $\mathscr{P}$ is taken to be intuitionistic predicate logic, then a number of first-order assertions undemonstrable within $\mathscr{P}$, for instance $\exists x(\exists x\alpha(x) \to \alpha(x))$, are provable within $\mathscr{P}_\varepsilon$. More interesting is the fact that certain purely *propositional* assertions undemonstrable within $\mathscr{P}$ are rendered provable within $\mathscr{P}_\varepsilon$.[18] These include Dummett's scheme $A \to B \vee B \to A$ and (hence) the intuitionistically invalid De Morgan law $\neg(A \wedge B) \to \neg A \vee \neg B$. But, curiously, the Law of Excluded Middle does *not* become demonstrable as a result of passing from intuitionistic $\mathscr{P}$ to $\mathscr{P}_\varepsilon$.

This is related to the fact (remarked on above) that in deriving **LEM** from **AC** one requires the principle of Extensionality of Functions. The analogous principle within the ε-calculus is the *Principle of Extensionality for ε-terms*:

[17] David Devidi has had the happy inspiration of calling ε_α "the thing most likely to be α."
[18] Bell [1993], [1993a].

(Ext) $\forall x[\alpha(x) \leftrightarrow \beta(x)] \rightarrow \varepsilon_{\alpha} = \varepsilon_{\beta}$.

An argument similar to the derivation of **LEM** from **AC** given above yields **LEM** from (Ext) within the intuitionistic ε-calculus.

It is interesting to note that the use of (Ext) can be avoided in deriving **LEM** in the intuitionistic ε-calculus if one employs *relative ε-terms,* that is, allows ε to act on *pairs* of formulas, each with a *single* free variable. Here, for each pair of formulas $\alpha(x)$, $\beta(x)$ we introduce the "relativized" ε-term $\varepsilon_x\alpha/\beta$ and the "relativized" ε-axioms

$$(1)\ \exists x\, \beta(x) \rightarrow \beta(\varepsilon_x\alpha/\beta) \qquad (2)\ \exists x\, [\alpha(x) \wedge \beta(x)] \rightarrow \alpha(\varepsilon_x\alpha/\beta).$$

That is, $\varepsilon_x\alpha/\beta$ may be thought of as an individual that satisfies β if anything does, and which in addition satisfies α if anything satisfies both α and β. Notice that the usual ε–term $\varepsilon_x\alpha$ is then $\varepsilon_x\alpha/x = x$. In the classical ε-calculus $\varepsilon_x\alpha/\beta$ may be defined by taking

$$\varepsilon_x\alpha/\beta = \varepsilon_y[[y = \varepsilon_x(\alpha \wedge \beta) \wedge \exists x\, (\alpha \wedge \beta)] \vee [y = \varepsilon_x\beta \wedge \neg\exists x\, (\alpha \wedge \beta)]].$$

□ut the relativized ε-scheme is not derivable in the intuitionistic ε-calculus since it can be shown to imply **LEM**. To see this, given a formula γ define

$$\alpha(x) \equiv x = 1 \qquad \beta(x) \equiv x = 0 \vee \gamma.$$

Write a for $\varepsilon_x\alpha/\beta$. Then we certainly have $\exists x\beta(x)$, so (1) gives $\beta(a)$, i.e.

$$(3) \qquad a = 0 \vee \gamma$$

Also $\exists x\, (\alpha \wedge \beta) \leftrightarrow \gamma$, so (2) gives $\gamma \rightarrow \alpha(a)$, i.e.

$$\gamma \to a = 1,$$

whence

$$a \neq 1 \to \neg\gamma,$$

so that

$$a = 0 \to \neg\gamma.$$

And the conjunction of this with (3) gives $\gamma \vee \neg\gamma$, as claimed.

3. *Weak set theories lacking the axiom of extensionality.* In Bell [forthcoming] a first order *weak set theory* **WST** is introduced which lacks the axiom of extensionality[19] and supports only minimal set-theoretic constructions. **WST** may be considered a fragment both of (intuitionistic) Δ_0-Zermelo set theory and Aczel's constructive set theory[20]. Like CTT, **WST** is too weak to allow the derivation of **LEM** from **AC.** But (again as with constructive type theories) beefing up **WST** with extensionality principles (even very moderate ones) enables the derivation to go through.

I end with some further thoughts on the status of the axiom of choice in constructive type theory and the "propositions as types" framework. We have observed above that AC interpreted à la "propositions as types" is (constructively) canonically true, while construed set- (or topos-) theoretically it is anything but, since so construed its affirmation yields classical logic. This prompts the question: what modification needs to be made to the "propositions-as-types" framework so as to yield the set- (or topos-) theoretic interpretation of AC? An answer (due to M.E. Maietti)[21] to this question can be furnished within the general framework of (variable) type theories through the use of so-called

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[19] Set theories (with classical logic) lacking the axiom of extensionality seem first to have been extensively studied in [4] and [10].
[20] Aczel and Rathjen [2001].
[21] Maietti [2005].

monotypes (or mono-objects), that is, types containing at most one entity or having at most one proof. In the category **Set** of ordinary sets, mono-objects are *singletons*, that is, sets containing at most one element.

Monotypes correspond to monic maps. This can be illustrated concretely by considering the categories **Indset** of *indexed* sets and **Set**$^{\to}$ of *bivariant* sets. The objects of **Indset** are indexed sets of the form $M = \{<i, M_i>: i \in I\}$ and those of **Set**$^{\to}$□maps $A \to B$ in **Set**, with appropriately defined arrows in each case. It can be shown that these two categories are equivalent. If we think of (the objects of) **Set** as representing simple or static types, then (the objects of) **Indset**, and hence also of **Set**$^{\to}$, represent variable types. It is easily seen that a monotype, or object, in **Indset**, is precisely an object M for which each M_i has at most one element. Moreover, under the equivalence between **Indset** and **Set**$^{\to}$, such an object corresponds to a monic map- object in **Set**$^{\to}$.

Now consider **Set**$^{\to}$ as a topos. Under the topos-theoretic interpretation in **Set**$^{\to}$, formulas correspond to monic arrows, which in turn correspond to mono-objects in **Indset.** Carrying this over entirely to **Indset** yields the sought modification of the "propositions-as-types" framework to bring it into line with the topos-theoretic interpretation of formulas, namely, to take formulas or propositions to correspond to *mono*-objects, rather than to *arbitrary* objects. Let us call this the "formulas-as-monotypes" interpretation.

Finally let us reconsider **AC** under the "formulas-as-monotypes" interpretation within **Set.** In the "propositions-as-types" interpretation as applied to **Set**, the universal quantifier $\forall i \in I$ corresponds to the product $\prod_{i \in I}$ and the existential quantifier $\exists i \in I$ to the coproduct, or disjoint sum, $\coprod_{i \in I}$. Now in the "formulas-as-monotypes" interpretation, under which formulas correspond to singletons, $\forall i \in I$ continues to correspond to $\prod_{i \in I}$, since the product of singletons is still a singleton. But the interpretation of $\exists i \in I$ is changed. In fact, the interpretation of $\exists i{\in}I\, A_i$ (with each A_i a singleton) now becomes $[\coprod_{i\in I} A_i]$, where for each set X, $[X] = \{u{:}\ u = 0 \wedge \exists x.\ x \in X\}$ is the *canonical singleton* associated with X.

It follows that, under the "formulas-as-monotypes" interpretation, the proposition $\forall i{\in}I\, \exists j{\in}J\, A_{ij}$ is interpreted as the singleton

(1) $$\prod_{i\in I} [\coprod_{j\in J} A_{ij}]$$

and the proposition $\exists f{\in}J^I\, \forall i{\in}I\, A_{if(i)}$ as the singleton

(2) $$[\coprod_{f\in J^I} \prod_{i\in I} A_{if(i)}].$$

Under the "formulas-as-monotypes" interpretation AC would be construed as asserting the existence of an isomorphism between (1) and (2).

Now it is readily seen that to give an element of (1) amounts to no more than affirming that, for every $i \in I$, $\bigcup_{j\in J} A_{ij}$ is nonempty. But to give an element of (2) amounts to specifying maps $f \in J^I$ and g with domain I such that $\forall i{\in}I\, g(i) \in A_{if(i)}$. It follows that to assert the existence of an isomorphism between (1) and (2), that is, to assert AC under the "formulas-as-monotypes" interpretation, is tantamount to asserting AC in its usual form, so leading in turn to classical logic. This is in sharp contrast with AC under the "propositions-as-types" interpretation, where its assertion is automatically correct and so has no nonconstructive consequences.

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