The Axiom of Choice and the Law of Excluded Middle in Weak Set Theories

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In constructive mathematics the axiom of choice (**AC**) has a somewhat ambiguous status. On the one hand, in intuitionistic set theory, or the local set theory associated with a topos ([2]) it can be shown to entail the law of excluded middle (**LEM**) ([3], [5]). On the other hand, under the "propositions-as types" interpretation which lies at the heart of constructive predicative type theories such as that of Martin-Löf [9], the axiom of choice is actually *derivable* (see, e.g. [11]), and so certainly cannot entail the law of excluded middle. This incongruity has been the subject of a number of recent investigations, for example [6], [7], [9], [12]. What has emerged is that for the derivation of **LEM** from **AC** to go through it is sufficient that sets (in particular power sets), or functions, have a degree of extensionality which is, so to speak, built into the usual set theories but is incompatible with constructive type theories Another condition, independent of extensionality, ensuring that the derivation goes through is that any equivalence relation determines a quotient set. **LEM** can also be shown to follow from a suitably extensionalized version of **AC**. The arguments establishing these intriguing results have mostly been formulated within a type-theoretic framework. It is my purpose here to formulate and derive analogous results within a comparatively straightforward *set-theoretic* framework. The core principles of this framework form a theory – *weak set theory* **WST** – which lacks the axiom of extensionality[1] and supports only minimal set-theoretic constructions. **WST** may be considered a fragment both of (intuitionistic) Δ_0-Zermelo set theory and Aczel's constructive set theory ([1]). In particular **WST** is, like constructive type theories, too weak to allow the derivation of **LEM** from **AC.** But we shall see that , as with constructive type theories, beefing up **WST** with extensionality principles (even very moderate ones) or quotient sets enables the derivation to go through.

Let **L** be the first-order language of (intuitionistic) set theory which, in addition to the usual identity and membership symbols = and $\in$ also contains a binary operation symbol

[1] Set theories (with classical logic) lacking the axiom of extensionality seem first to have been extensively studied in [4] and [10].

$\langle\ ,\ \rangle$ permitting the formation of ordered pairs[2]. At certain points various additional predicates and operation symbols will be introduced into **L.** The restricted quantifiers $\exists x{\in}a$ and $\forall x{\in}a$ are defined as usual, that is, as $\exists x(x \in a \wedge \ \ldots)$ and $\forall x(x \in a \to \ \ldots)$ respectively. A formula is *restricted* if it contains only restricted quantifiers.

Weak set theory **WST** is the theory in **L** with the following basic axioms (in which the free variables are understood to be universally quantified, and similarly below):

Unordered Pair $\exists u \forall x[x \in u \Leftrightarrow x = a \vee x = b]$

Ordered Pair $\langle a,b\rangle = \langle c,d\rangle \Leftrightarrow a = c \wedge b = d$

Binary Union $\exists u \forall x[x \in u \Leftrightarrow x \in a \vee x \in b]$

Cartesian Product $\exists u \forall x[x \in u \Leftrightarrow \exists y \in a \exists z \in b\ (x = \langle y,z\rangle)]$

Restricted Subsets $\exists u \forall x[x \in u \Leftrightarrow x \in a \wedge \varphi]$

where in this last axiom φ is any restricted formula in which the variable u is not free.

We introduce into **L** new predicates and operation symbols as indicated below and adjoin to **WST** by the following "definitional" axioms:

$a \subseteq b \Leftrightarrow \forall x[x \in a \Rightarrow x \in b] \qquad a \approx b \Leftrightarrow \forall x[x \in a \Leftrightarrow x \in b] \quad Ext(a) \Leftrightarrow \forall x \in a \forall y \in a[x \approx y \Rightarrow x = y]$

$x \in a \cup b \Leftrightarrow x \in a \vee x \in b \qquad x \in \{a,b\} \Leftrightarrow x = a \vee x = b \qquad \{a\} = \{a,a\} \quad x\ r\ y \Leftrightarrow \langle x,y\rangle \in r$

$y \in \{x \in a : \varphi(x)\} \Leftrightarrow y \in a \wedge \varphi(y)$ (φ restricted)

$\neg x \in 0 \qquad 1 = \{0\} \qquad 2 = \{0,1\}$

$x \in a \times b \Leftrightarrow \exists u \in a \exists v \in b(x = \langle u,v\rangle) \quad x \in a + b \Leftrightarrow \exists u \in a \exists v \in b[x = \langle u,0\rangle \vee x = \langle v,1\rangle]$

$f : a \to b \quad \Leftrightarrow \quad f \subseteq a \times b \wedge \forall x \in a \exists y \in b(x\ f\ y) \wedge \forall x \forall y \forall z[(x\ f\ y \wedge x\ f\ z) \Rightarrow y = z]$

$Fun(f) \Leftrightarrow \exists a \exists b(f : a \to b)$

$f : a \to b \wedge x \in a \Rightarrow x\ f\ f(x)$

$f : a \to b \wedge g : b \to c \Rightarrow g \circ f : a \to c \wedge \forall x \in a[(g \circ f)(x) = g(f(x))]$

$f : a \twoheadrightarrow b \quad \Leftrightarrow \quad f : a \to b \wedge \forall y \in b \exists x \in a[y = f(x)]$

$\pi_1 : a + b \to a \cup b \wedge \forall x \in a[\pi_1(\langle x,0\rangle) = x] \wedge \forall y \in b[\pi_1(\langle y,1\rangle) = y]$

$\pi_2 : a + b \to 2 \wedge \forall x \in a[\pi_2(\langle x,0\rangle) = 0] \wedge \forall y \in b[\pi_2(\langle y,1\rangle) = 1]$

$Eq(s,a) \Leftrightarrow s \subseteq a \times a \wedge \forall x \in a(x\ s\ x) \wedge \forall x \in a \forall y \in a(x\ s\ y \Rightarrow y\ s\ x) \wedge$
$\qquad \forall x \in a \forall y \in a \forall z \in a[(x\ s\ y \wedge y\ s\ z) \Rightarrow x\ s\ z]$

$Comp(r,s) \Leftrightarrow \forall x \forall x' \forall y[(x\ s\ x' \wedge x'\ r\ y) \Rightarrow x\ r\ y]$

$Comp(r) \Leftrightarrow \forall x \forall x' \forall y[(x \approx x' \wedge x'\ r\ y) \Rightarrow x\ r\ y]$

$Extn(f,s) \Leftrightarrow Fun(f) \wedge \forall x \forall x'[x\ s\ x' \wedge \exists y \exists y'(x\ f\ y \wedge x'\ f\ y') \Rightarrow f(x) = f(x')]$

$Ex(f) \Leftrightarrow Fun(f) \wedge \forall x \forall x'[x \approx x' \wedge \exists y \exists y'(x\ f\ y \wedge x'\ f\ y') \Rightarrow f(x) = f(x')]$

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[2] While the ordered pair <*u,v*> could be defined in the customary way as $\{\{u\}, \{u,v\}\}$, here it is taken as a primitive operation —as it is in type theory—both for reasons of simplicity and to emphasize the fact that for our purposes it does not matter how (or indeed whether) it is defined set-theoretically.

Most of these definitions are standard. The functions π_1 and π_2 are projections of ordered pairs onto their 1st and 2nd coordinates respectively: clearly, for $u, v \in a + b$ we have

(**proj**) $$u = v \Leftrightarrow [\pi_1(u) = \pi_1(v) \wedge \pi_2(u) = \pi_2(v)].$$

The relation ≈ is that of *extensional equality.* Ext(*a*) expresses the *extensionality* of the members of the set *a. Eq*(*s,a*) asserts that *s* is an equivalence relation on *a.* If *r* is a relation between *a* and *b,* and *s* an relation on *a, Comp*(*r,s*) expresses the *compatibility* of *r* with *s,* and *Comp*(*r*) the compatibility of *r* with extensional equality. If f: $a \to b$, and *s* is an equivalence relation on *a*, *Etxn*(*f,s*) expresses the idea that *f* treats the relation *s* as if it were the identity relation: we shall then say that *f* is *s-extensional. Ex*(*f*) asserts that *f* is *extensional* in the sense of treating extensional equality as if it were identity.

In addition to the axioms of WST, We formulate the following axioms additional to those of **WST** (recalling that in stating axioms that all free variables are universally quantified):
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Extensionality $Ext(\{a, b\})$

Extsub(1) $a \subseteq 1 \wedge b \subseteq 1 \Rightarrow Ext(\{a, b\})$

Extsub(2) $a \subseteq 2 \wedge b \subseteq 2 \Rightarrow Ext(\{a, b\})$

While **Extensionality** asserts that all extensionally equal sets are identical, **Extsub(1)** and **Extsub(2)** are weak versions confining the assertion just to subsets of 1 or 2. Explicitly, **Extsub(1)** and **Extsub(2)** assert that each doubleton composed of subsets of 1, or of 2, is extensional. Notice that since $1 \subseteq 2$, **Extsub(1)** is a consequence of **Extsub(2).**

Following [1], we define a set *a* to be a *base* if every relation with domain *a* includes a function with domain *a,* i.e.

Base(*a*) $$\forall b \forall r[r \subseteq a \times b \wedge \forall x \in a \exists y \in b(x\ r\ y) \Rightarrow \exists f : a \to b \forall x \in a(x\ r\ f(x))].$$

We call *a* an *extensional base* if every relation with domain *a* compatible with extensional equality includes an *extensional* function with domain *a*, i.e.

Extbase(*a*):

$$\forall b \forall r[r \subseteq a \times b \wedge Comp(r) \wedge \forall x \in a \exists y \in b(x\ r\ y) \Rightarrow \exists f : a \to b[Ex(f) \wedge \forall x \in a(x\ r\ f(x))]]$$

We use these notions to state a number of versions of the axiom of choice (again recalling that in stating axioms that all free variables are universally quantified):

(Intensional) Axiom of Choice AC $Base(a)$

Weak Axiom of Choice 1 WAC(1) $a \subseteq 1 \wedge a' \subseteq 1 \wedge b \subseteq 1 \wedge b' \subseteq 1 \Rightarrow Base(\{\langle a, a'\rangle, \langle b, b'\rangle\})$

Weak Axiom of Choice 2 WAC(2) $a \subseteq 2 \wedge a' \subseteq 2 \Rightarrow Base(\{a, a'\})$

Universal Extensional Axiom of Choice UEAC

$$Eq(s,a) \wedge r \subseteq a \times b \wedge Comp(r,s) \wedge \forall x \in a \exists y \in b(x\ r\ y) \Rightarrow \exists f : a \to b[Extn(f,s) \wedge \forall x \in a(x\ r\ f(x))]$$

Extensional Axiom of Choice EAC $Extbase(a)$

In asserting that every set is a base, **AC** means, as usual, that a choice function always exists under the appropriate conditions on an arbitrarily given relation. **WAC(1)** and **WAC(2)** restricts the existence of such choice functions to relations whose domains are doubletons of a certain form[3]. **UEAC** asserts that, in the presence of an equivalence relation *s* with which a given relation *r* is compatible, the choice function can be taken to be *s*-extensional. **EAC** is the special case of **UEAC** in which the equivalence relation is that of extensional equality. In view of the fact that **AC** can be seen to be the special case of **UEAC** in which the equivalence relation is the identity relation, **AC** is sometimes known as the *intensional* axiom of choice.

Our next axiom is

Quotients $Eq(s,a) \to \exists u \exists f[f : a \twoheadrightarrow u \wedge \forall x \in a \forall y \in a[f(x) = f(y) \leftrightarrow x\ s\ y]]$

This axiom asserts that each equivalence relation determines a quotient set. In **WST + Quotients**, we introduce operation symbols $\bullet/\bullet$, $[\bullet]_\bullet$ and adjoin the "definitional" axiom

(Q)
$$Eq(s,a) \Rightarrow [\forall x \in a([x]_s \in a/r) \wedge \forall u \in a/s\, \exists x \in a(u = [x]_s) \wedge \forall x \in a \forall y \in a[[x]_s = [y]_s \Leftrightarrow x\ s\ y]].$$

[3] Using the fact that subsets of 2 are in natural bijective correspondence with pairs of subsets of 1, it can be shown that **WAC(1)** and **WAC(2)** are in fact equivalent. Nevertheless for our purposes it will be useful to have both principles.

Here $a/_s$ is the *quotient* of a by s and, for $x \in a$, $[x]_s$ is the *image* of x in a/s.

Reminding the reader that our background logic is intuitionistic, we finally introduce the following logical scheme:

Restricted Excluded Middle REM

$\varphi \vee \neg\varphi$ for any restricted formula φ

We are going to prove the following results.

Theorem 1. REM is derivable in: (a) **WST + Extsub(1) + WAC(1);** (b) **WST + Extsub(2) + WAC(2);** and (c) **WST + EAC.**

Theorem 2. REM is derivable in **WST + AC + Quotients**

Theorem 3. AC $\Leftrightarrow$ **UEAC** is derivable in **WST + Quotients**.

Thus, while in the absence of extensionality for doubletons of subsets of 1 or 2, the intensional axiom of choice does not entail the law of excluded middle, with that degree of extensionality the law of excluded middle becomes a consequence of very weak versions of the intensional axiom of choice. (*A fortiori* **REM** is derivable in **WST + Extensionality + AC.**) Moreover, the extensional axiom of choice entails the law of excluded middle without additional extensionality assumptions. And finally, when quotients are present the intensional axiom of choice is no weaker than its universal extensional version and entails the law of excluded middle without additional extensionality assumptions.

Proof of Theorem 1.

(a) We argue in **WST + Extsub(1) + WAC(1)**. Given an arbitrary restricted formula φ, we define $s = \{x \in \{0\}: \varphi\}$ and

$$a = \{\langle s, \{0\}\rangle, \langle \{0\}, s\rangle\}.$$

It is then easily shown that

$$\langle u, v\rangle \in a \Rightarrow \exists x \in 2[(x = 0 \Rightarrow 0 \in u) \wedge (x = 1 \Rightarrow 0 \in v)].$$

So **WAC(1)** gives f: $a \rightarrow 2$ such that , for $\langle u, v\rangle \in a$

(1) $f(\langle u,v\rangle) = 0 \rightarrow 0 \in u$

(2) $f(\langle u,v\rangle) = 1 \Rightarrow 0 \in v$.

Since *f* maps to 2, we have

$$[f(\langle s,\{0\}\rangle) = 0 \vee f(\langle s,\{0\}\rangle) = 1] \wedge [f(\langle \{0\},s\rangle) = 0 \vee f(\langle \{0\},s\rangle) = 1].$$

From this, together with (1) and (2), it follows that

$$[0 \in s \vee f(\langle s,\{0\}\rangle) = 1] \wedge [f(\langle \{0\},s\rangle) = 0 \vee 0 \in s],$$

whence, using the distributive law,

$$0 \in s \vee [f(\langle s,\{0\}\rangle) = 0 \wedge f(\langle \{0\},s\rangle) = 1].$$

Writing $\psi(s)$ for the second disjunct in this last formula, it becomes

(3) $0 \in s \vee \psi(s)$.

Now from $s \subseteq 1$ we deduce

$$0 \in s \Rightarrow s \approx \{0\},$$

whence using **Extsub(1),**

$$0 \in s \Rightarrow s = \{0\}.$$

Hence

$$\begin{aligned}[0 \in s \wedge \psi(s)] &\Rightarrow [s = \{0\} \wedge \psi(s)] \\ &\Rightarrow \psi(\{0\}) \\ &\Rightarrow 0 = 1.\end{aligned}$$

Since clearly $0 \neq 1$, we conclude that

$$\psi(s) \Rightarrow \neg 0 \in s$$

and (3) then yields

(4) $0 \in s \vee \neg 0 \in s$.

But obviously $0 \in s \Leftrightarrow \varphi$, so (4) gives $\varphi \vee \neg\varphi$, as required.

(b) [4] We argue in **WST + Extsub(2) + WAC(2)**. Given a formula φ, define

$$a = \{x \in 2 : x = 0 \vee \varphi\}, \quad b = \{x \in 2 : x = 1 \vee \varphi\}.$$

Since $0 \in a$ and $1 \in b$, we have

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[4] Using the observation in the previous footnote that **WAC(1)** and **WAC(2)** are equivalent, it can be seen that **(b)** actually follows from **(a).** However the direct proof given here, based on that given in [5], is, in the author's view, considerably more illuminating.

$$\forall x \in \{a,b\}\exists y \in 2.y \in x,$$

and so **WAC(2)** applied to the relation

$$r = \{\langle x,y\rangle \in \{a,b\} \times 2 : y \in x\})$$

yields a function f: $\{a, b\} \to 2$ for which $\forall x \in \{a,b\}.f(x) \in x$. It follows that $f(a) \in a \wedge f(b) \in b$, whence

$$[f(a) = 0 \vee \varphi] \wedge [f(b) = 1 \vee \varphi].$$

Applying the distributive law, we then get

$$\varphi \vee [f(a) = 0 \wedge f(b) = 1].$$

whence

(1) $$\varphi \vee f(a) \neq f(b).$$

Now clearly $\varphi \Rightarrow a \approx b$, and from this and **Extsub(2)** we deduce $\varphi \Rightarrow a = b$, whence

(2) $$\varphi \Rightarrow f(a) = f(b).$$

It follows that $f(a) \neq f(b) \Rightarrow \neg\varphi$, and we conclude from (1) that

$$\varphi \vee \neg\varphi,$$

as required.

(c) Here the argument in **WST+ EAC** is the same as that given in (**b**) except that in deriving (2) above we invoke **EAC** in place of **Extsub(2).** To justify this step it suffices to show that *Comp*(r), where r is the relation defined in the proof of **(b)**. This, however, is clear. ■

Proof of Theorem 2.

For $b \subseteq a$, we say that b is *detachable* (in a) if

$$\forall x \in a[x \in b \vee x \notin b].$$

An *indicator* for b (in a) is a function g: $a \times 2 \to 2$ satisfying

$$\forall x \in a[x \in b \Leftrightarrow g(\langle x,0\rangle) = g(\langle x,1\rangle)].$$

It is easy to show that a subset is detachable if and only if it has an indicator. For if $b \subseteq a$ *is* detachable, then g: $a \times 2 \to 2$ defined by

$$g(\langle x,0\rangle) = g(\langle x,1\rangle) = 0 \quad \text{if } x \in b$$
$$g(\langle x,0\rangle) = 0 \wedge g(\langle x,1\rangle) = 1 \quad \text{if } x \notin b$$

is an indicator for *b*. Conversely, for any function *g*: *a* × 2 → 2, we have $g(\langle x,0\rangle) = g(\langle x,1\rangle) \vee g(\langle x,0\rangle) \neq g(\langle x,1\rangle)$, so if *g* is an indicator for *b*, we infer $\forall x \in a[x \in b \vee x \notin b]$, and *u* is detachable.

Now we show in **WST + AC + Quotients** that every subset of a set has an indicator, and is hence detachable. Given $b \subseteq a$, let *s* be the binary relation on *a* + *a* given by:

$$s = \{\langle\langle x,0),(x,0\rangle\rangle : x \in a\} \cup \{\langle\langle x,1),\langle x,1\rangle\rangle : x \in a\} \cup \{\langle\langle x,0),\langle x,1\rangle\rangle : x \in b\} \cup \{\langle\langle x,1),\langle x,0\rangle\rangle : x \in b\}.$$

It is easily checked that *Eq*(*s, a* + *a*). Also, it is clear that, for *z, z'*∈ *a* + *a*,

(1) $$z\ s\ z' \Rightarrow \pi_1(z) = \pi_1(z')$$

and, for $x \in a$,

(2) $$x \in b \Leftrightarrow \langle x,0\rangle\ s\ \langle x,1\rangle.$$

Invoking axiom (**Q**) above, we introduce the quotient $(a+a)/s$ of *a* + *a* by *s* and the image $[u]_s$ of an element *u* of *a* + *a* in $(a+a)/s$ for which we then have

(3) $$\forall z \in (a+a)/s\ \exists u \in a + a(z = [u]_s)$$

and

(4) $$\forall u \in a + a \forall v \in a + a[[u]_s = [v]_s \Leftrightarrow u\ s\ v].$$

Applying **AC** to (3) yields a function $f : (a+a)/s \to a + a$ for which

(5) $$\forall z \in (a+a)/s\ z = [f(z)]_s).$$

Clearly *f is* one-one, that is, we have

(6) $$f(z) = f(z') \Leftrightarrow z = z'.$$

Next, observe that, for *i* = 0, 1, and $x \in a$,

(7) $$\pi_1([f(\langle x,i\rangle)]_s) = x.$$

For from (5) we have $[\langle x,i\rangle]_s = [f([\langle x,i\rangle]_s)]_s$, whence by (4) $\langle x,i\rangle\ s\ f([\langle x,i\rangle]_s)$. Hence by (1) $\pi_1(\langle x,i\rangle) = \pi_1(f([\langle x,i\rangle]_s))$. (7) now follows from this and the fact that $\pi_1(\langle x,i\rangle) = x$.

We have also

(8) $$x \in b \Leftrightarrow f([\langle x,0\rangle]_s) = f([\langle x,1\rangle]_s).$$

For we have

$$x \in b \Leftrightarrow \langle x,0\rangle \; s \; \langle x,1\rangle \quad \text{using (2)}$$

$$\Leftrightarrow [\langle x,0\rangle]_s = [\langle x,1\rangle]_s \quad \text{using (4)}$$

$$\Leftrightarrow f([\langle x,0\rangle]_s) = f([\langle x,1\rangle]_s) \quad \text{using (6).}$$

Now define g: $a \times 2 \to 2$ by

$$g(\langle x,i\rangle) = \pi_2(f([\langle x,i\rangle]_s)).$$

We claim that g is an indicator for b. This can be seen from the following equivalences:

$$x \in b \Leftrightarrow f([\langle x,0\rangle]_s) = f([\langle x,1\rangle]_s) \quad \text{(by (8))}$$

$$\Leftrightarrow \pi_1(f([\langle x,0\rangle]_s)) = \pi_1(f([\langle x,1\rangle]_s)) \wedge \pi_2(f([\langle x,0\rangle]_s)) = \pi_2(f([\langle x,1\rangle]_s)) \quad \text{(by (\textbf{proj}))}$$

$$\Leftrightarrow \pi_2(f([\langle x,0\rangle]_s)) = \pi_2(f([\langle x,1\rangle]_s)) \quad \text{(using (7)}$$

$$\Leftrightarrow g(\langle x,0\rangle) = g(\langle x,1\rangle).$$

So we have shown that **WST + AC + Quotients** every subset of a set has an indicator, and is accordingly detachable. This latter fact easily yields **REM.** For, given restricted φ, for any a, the set $b = \{x \in a: \varphi\}$ is then a detachable subset of a, from which $\forall x \in a(\varphi \vee \neg\varphi)$ immediately follows. By taking $a = \{x\}$ we get $\varphi \vee \neg\varphi$. ■

Proof of Theorem 3. It suffices to derive **UEAC** from **AC** in **WST + Quotients.** Assuming $Eq(s, a)$, we use **AC** as in the proof of Theorem 2 to obtain a function $p: a/s \to a$ such that $u = [p(u)]_s$ for all $u \in a/s$. From this we deduce $[x]_s = [p([x]_s)]_s$, whence

(1) $$x \; s \; p([x]_s)$$

for all $x \in a$.

Assuming the antecedent of **UEAC ,** viz.,

$$Eq(s,a) \wedge r \subseteq a \times b \wedge Comp(r,s) \wedge \forall x \in a \exists y \in b(x \; r \; y),$$

define the relation $r' \subseteq a/s \times b$ by

$$u \; r' \, y \Leftrightarrow p(u) \; r \; y .$$

Now use **AC** to obtain a function $g: a/s \to b$ for which $\forall u \in a/s (u \; r' \; g(u))$, i.e.

(2) $$\forall u \in a/s (p(u) \; r \; g(u)).$$

Define f: $a \to b$ by

$$f(x) = g([x]_s).$$

Then by (2)

$$\forall x \in a(p([x]_s)\ r\ g([x]_s)) .$$

From this, (1) and *Comp*(*r, s*) it follows that $\forall x \in a(x\ r\ g([x]_s))$, i.e.

(3) $$\forall x \in a(x\ r\ f(x)) .$$

Moreover, for all *x, x′* ∈ *a,* we have

$$x\ s\ x' \Rightarrow [x]_s = [x']_s \Rightarrow f(x) = g([x]_s) = g([x']_s) = f(x'),$$

whence *Extn*(*f, s*). This , together with (3), establishes the consequent of **UEAC.** ■

Concluding remarks.

1. The proof of Theorem 1 (a) is an adaptation of the proof of the analogous result given in [2], pp. 144-146, and that of (b) is based on that given in [5].

2. The proof of Theorem 2 is an adaptation to a set-theoretical context of the argument in [3] that, in a topos satisfying the axiom of choice, all subobjects are complemented. By weakening **Quotients** to the assertion **Quotients(1 + 1)** that quotient sets are determined just by equivalence relations on the set 1 + 1, the proof of Theorem 2 shows that **REM** is derivable in the theory **WST + AC + Quotients(1 + 1).**

3. Quotients can be derived within **WST** augmented by the full extensional *power set axiom*

Extpow $\exists u[Ext(u) \wedge \forall x[x \in u \Leftrightarrow x \subseteq a]]$

So (cf. [7]) adding *extensional* power sets to **WST + AC** yields **REM.**

4. Another version of the axiom of choice, easily derivable in **WST** from **AC** is:

AC* $\forall x \in a[x \subseteq b \wedge \exists y(y \in x)] \Rightarrow \exists f : a \to b\ \forall x \in a(f(x) \in x)].$

If one adds to **WST** the *nonextensional* power set axiom, viz.

Pow $\exists u \forall x[x \in u \Leftrightarrow x \subseteq a]$**,**

then **AC*** becomes derivable from **AC**. Note that while **Extpow** entails **REM, Pow** is logically "harmless", that is, it has no nonconstructive logical consequences such as **LEM.**

The extensional version of **AC***, viz.

EAC* $\forall x \in a[x \subseteq b \wedge \exists y(y \in x)] \Rightarrow \exists f : a \to b[Ex(f) \wedge \forall x \in a(f(x) \in x)].$

is derivable in **WST** from **EAC.** In **WST + Pow, EAC** and **EAC*** are equivalent.

5. A version of the axiom of choice considered in [9] is what we shall call

Rep(resentatives)

$$Eq(s,a) \Rightarrow \exists f[f : a \to a \wedge \forall x \in a(xsf(x)) \wedge \forall x \in a \forall y \in a[xsy \Leftrightarrow f(x) = f(y)]].$$

Rep asserts that unique representatives can be chosen from the equivalence classes of any equivalence relation. Obviously, in **WST**, **Rep** implies **Quotients**. Moreover, the proof of Theorem 2 is easily adapted to show that, in **WST**, **Rep** yields **REM**. In **WST**, **AC + Quotients** entails **Rep**, and, in **WST + Pow**, conversely.

6. Finally, consider the following versions of **AC** which are closely related to that introduced by Zermelo [13] (see also [9]), namely

ACZ $[\forall x \in a \exists y(y \in x) \wedge \forall x \in a \forall y \in a[\exists z(z \in x \wedge z \in y) \Rightarrow x = y]] \Rightarrow \exists u \forall x \in a \exists! y(y \in x \wedge y \in u)$

EACZ $[\forall x \in a \exists y(y \in x) \wedge \forall x \in a \forall y \in a[\exists z(z \in x \wedge z \in y) \Rightarrow x \approx y]] \Rightarrow \exists u \forall x \in a \exists! y(y \in x \wedge y \in u)$

These are construals of the assertion that, given any collection of mutually disjoint nonempty sets, there is a set intersecting each member of the collection in exactly one element. Clearly **EACZ** implies **ACZ**; the former is readily derivable from **EAC** and the latter from **AC.** Since **REMS** is not a consequence of **AC,** it cannot, a fortiori, be a consequence of **ACZ.** But, like **EAC, EACZ** can be shown to yield **REM.** We sketch the argument, which is similar to the proof of Thm. 2(b).

Given a restricted formula φ, define

$$b = \{x \in 2 : x = 0 \vee \varphi\}, \quad c = \{x \in 2 : x = 1 \vee \varphi\}$$

and $a = \{b, c\}$. A straightforward argument shows that a satisfies the antecedent of **EACZ**. So, if this last is assumed, its consequent yields a u with exactly one element in common with b and with c. Writing d and e for these elements, one easily shows that

(*) $$\varphi \vee d \neq e.$$

Now since it is also easily shown that $\varphi \Rightarrow d = e$, it follows that $d \neq e \Rightarrow \neg\varphi$, and this, together with (*) yields $\varphi \vee \neg\varphi$.

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