CONTINUITY

The usual meaning of the word *continuous* is "unbroken" or "uninterrupted": thus a continuous entity—a *continuum*—has no "gaps". We commonly suppose that space and time are continuous, and certain philosophers have maintained that all natural processes occur continuously: witness, for example, Leibniz's famous apothegm *natura non facit saltus*—"nature makes no jump". In mathematics the word is used in the same general sense, but has had to be furnished with increasingly precise definitions. So, for instance, in the later 18th century continuity of a function was taken to mean that infinitesimal (*q.v.*) changes in the value of the argument induced infinitesimal changes in the value of the function. With the abandonment of infinitesimals in the 19th century this definition came to be replaced by one employing the more precise concept of *limit*. So now a (real-valued) function is said to be continuous if, for any real number a, the limit of $f(x)$ as x tends to a exists and equals $f(a)$; in other words, if the difference between $f(x)$ and $f(a)$ remains as small as we please so long as x remains sufficiently near to a. In general, the property of continuity can be meaningfully ascribed to any function between *topological spaces*, that is, mathematical structures in which the concept of *neighbourhood* of a point is defined. Thus, a function f between topological spaces X and Y is said to be continuous if, for any point a in X, and any neighbourhood V of $f(a)$, there is a neighbourhood U of a whose image under f is entirely included in V. It is interesting to note that in recent years the concept of infinitesimal has been reformulated in a manner sufficiently rigorous as to enable the 18th century definition of continuity to be revived.

Sets of points may also be termed continuous. Thus, for example, continuity of the set of real numbers—to which the term *continuum* is then also applied—means that it has the following two properties: 1) between any two of its members there is a third; 2) when divided into two classes in such a way that all the members of the first precede all the members of the second, there is a definite *boundary point* between the two classes, that is, either the first contains a largest member, or the second a least. It is the second of these properties that distinguishes the continuum of the reals from the noncontinuum of the rationals: although the latter clearly has the first of these properties, it fails to have the second because, for instance, the set of rationals whose squares are less than 2 has no greatest member, while the set of rationals whose squares exceed 2 has no least.

Bibliography

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