Causal Sets and Frame-Valued Set Theory

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In spacetime physics any set $\mathscr{C}$ of events—a *causal set*—is taken to be partially ordered by the relation $\leq$ of *possible causation*: for $p, q \in \mathscr{C}$, $p \leq q$ means that q is in p's future light cone. In her groundbreaking paper *The internal description of a causal set: What the universe looks like from the inside,* Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by "sets evolving over $\mathscr{C}$" —that is, in essence, by the topos $\mathscr{S}et^{\mathscr{C}}$ of presheaves on $\mathscr{C}^{op}$. To enable what she has done to be the more easily expressed within the framework presented here, I will reverse the causal ordering, that is, $\mathscr{C}$ will be replaced by $\mathscr{C}^{op}$, and the latter written as P—which will, moreover, be required to be no more than a *preordered* set. Specifically, then: P is a set of events preordered by the relation $\leq$, where $p \leq q$ is intended to mean that p is in q's future light cone—that q *could* be the cause of p, or, equally, that p *could* be an effect of q. In that case, for each event p, the set $p{\downarrow} = \{q\colon q \leq p\}$ may be identified as the *causal future* of p, or the set of *potential effects* of p. In requiring that $\leq$ be no more than a preordering—in dropping, that is, the antisymmetry of $\leq$—I am, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Accordingly I fix a preordered set $(P, \leq)$, which I shall call the *universal causal set.* Markopoulou, in essence, suggests that viewing the universe "from the inside" amounts to placing oneself within the topos of presheaves $\mathscr{S}et^{P^{op}}$. Here I am going to show how $\mathscr{S}et^{P^{op}}$ may be effectively replaced by a certain model of intuitionistic set theory, with (I hope) illuminating results.

Let us suppose that we are given a relation $\Vdash$ between events p and assertions φ: think of $p \Vdash \varphi$ as meaning that φ *holds* as a result of event p. Assume that the relation $\Vdash$ is *persistent* in the sense that, if $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$: once an assertion holds, it continues to hold in the future. (The basic assertions we have in mind are of the form: “such and such is (or was) the case at such-and such a time (event)”.)

Given an assertion φ, the set $[\![\varphi]\!] = \{p: p \Vdash \varphi\}$ “measures” the degree or extent to which φ holds: the larger $[\![\varphi]\!]$ is, the “truer” φ is. In particular, when $[\![\varphi]\!] = P$, φ is ‘universally” or “absolutely” true, and when $[\![\varphi]\!] = \varnothing$, φ is “universally” or “absolutely” false. These $[\![\varphi]\!]$ may accordingly be thought of as “truth values”, with P corresponding to “absolute truth” and $\varnothing$ to absolute falsity.

Because of the persistence property, each $[\![\varphi]\!]$ has the property of being “closed under potential effects”, or “causally closed”, that is, satisfies $p \in [\![\varphi]\!]$ and $q \leq p \rightarrow q \in [\![\varphi]\!]$. A subset of P with this property is called a *sieve*. Sieves serve as generalized “truth values” measuring the degree to which assertions hold. The set $\widehat{P}$ of all sieves, or truth values has a natural logico-algebraic structure —that of a *complete Heyting algebra*, or *frame*. This concept is defined in the following way.

A *lattice* is a partially ordered set L with partial ordering $\leq$ in which each two-element subset $\{x, y\}$ has a supremum or *join*—denoted by $x \vee y$—and an infimum or *meet*—denoted by $x \wedge y$. A lattice L is *complete* if every subset X (including $\varnothing$) has a supremum or *join*—denoted by $\bigvee X$— and an infimum or *meet*—denoted by $\bigwedge X$. Note that $\bigvee\varnothing = 0$, the least or *bottom* element of L, and $\bigwedge\varnothing = 1$, the largest or *top* element of L.

A *Heyting algebra* is a lattice L with top and bottom elements such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$—of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$: it is called the *pseudocomplement* of x. A *Boolean algebra* is a Heyting algebra in which $\neg\neg x = x$ for all x, or equivalently, in which $x \vee \neg x = 1$ for all x.

If we think of the elements of a (complete) Heyting algebra as "truth values", then 0, 1, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\bigvee$, $\bigwedge$ represent "true", "false", "and", "or", "not" and "implies", "there exists" and "for all", respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic.* In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$(*) \qquad x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$$

And conversely, in any complete lattice satisfying (*), defining the operation $\Rightarrow$ by $x \Rightarrow y = \bigvee\{z: z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame.*

In the frame $\widehat{P}$ $\leq$ is $\subseteq$, joins and meets are just set-theoretic unions and intersections, and the operations $\Rightarrow$ and $\neg$ are given by

$I \Rightarrow J = \{p : I \cap p{\downarrow} \subseteq J\}$ $\qquad\qquad \neg I = \{p : I \cap p{\downarrow} = \varnothing\}$.

Frames do duty as the "truth-value algebras" of the (current) *language of mathematics*, that is, *set theory.* To be precise, associated with each frame H is a structure $V^{(H)}$—the *universe of H-valued sets*—with the following features.

- Each of the members of $V^{(H)}$—the *H-sets*—is a map from a subset of $V^{(H)}$ to H.
- Corresponding to each sentence σ of the language of set theory (with names for all elements of $V^{(H)}$) is an element $[\![\sigma]\!] = [\![\sigma]\!]^H \in H$ called its *truth value in* $V^{(H)}$. These "truth values" satisfy the following conditions. For $a, b \in V^{(H)}$,

$$[\![b \in a]\!] = \bigvee_{c \in dom(a)} [\![b = c]\!] \wedge a(c) \qquad [\![b = a]\!] = \bigvee_{c \in dom(a) \cup dom(b)} ([\![c \in b]\!] \Leftrightarrow [\![c \in a]\!])$$

$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \wedge [\![\tau]\!], \text{ etc.}$$

$$[\![\exists x \varphi(x)]\!] = \bigvee_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

$$[\![\forall x \varphi(x)]\!] = \bigwedge_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

  A sentence σ is *valid* , or *holds,* in $V^{(H)}$, written $V^{(H)} \vDash \sigma$, if $[\![\sigma]\!] = 1$, the top element of H.
- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in $V^{(H)}$. In this sense $V^{(H)}$ is an *H-valued model* of *IZF.* Accordingly the category $\mathcal{Set}^{(H)}$ of sets constructed within $V^{(H)}$ is a topos: in fact $\mathcal{Set}^{(H)}$ can be shown to be equivalent to the topos of canonical sheaves on H.
- There is a canonical embedding $x \mapsto \hat{x}$ of the usual universe V of sets into $V^{(H)}$ satisfying

$$[\![u \in \hat{x}]\!] = \bigvee_{y \in x} [\![u = \hat{y}]\!] \text{ for } x \in V, u \in V^{(H)}$$

$$x \in y \leftrightarrow V^{(H)} \vDash \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \vDash \hat{x} = \hat{y} \text{ for } x, y \in V$$

$$\varphi(x_1, \ldots, x_n) \leftrightarrow V^{(H)} \vDash \varphi(\hat{x}_1, \ldots, \hat{x}_n) \text{ for } x_1, \ldots, x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

We observe that $V^{(2)}$ is essentially just the usual universe of sets.

It follows from the last of these assertions that the canonical representative $\widehat{H}$ of H is a Heyting algebra in $V^{(H)}$. A particularly important H- set is the H-set Φ_H defined by

$$dom(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \text{ for } a \in H .$$

Then $V^{(H)} \vDash \Phi_H \subseteq \widehat{H}$. Also, for any $a \in H$ we have $[\![\hat{a} \in \Phi_H]\!] = a$, and in particular, for any sentence σ, $[\![\sigma]\!]$ = $[\![\widehat{[\![\sigma]\!]} \in \Phi_H]\!]$. Thus

$$V^{(H)} \vDash \sigma \leftrightarrow V^{(H)} \vDash \widehat{[\![\sigma]\!]} \in \Phi_H ;$$

in this sense Φ_H represents the "true" sentences in $V^{(H)}$. Φ_H is called the *canonical truth set* in $V^{(H)}$.

Now let us return to our causal set *P*. The topos $\mathcal{Set}^{(\widehat{P})}$ of sets in $V^{(\widehat{P})}$ is, as I have observed, equivalent to the topos of canonical sheaves on $\widehat{P}$, which is itself, as is well known, equivalent to the topos $\mathcal{Set}^{P^{op}}$ of presheaves on *P*. My proposal is then, that we work in $V^{(\widehat{P})}$ rather than, as did Markopoulou, within $\mathcal{Set}^{P^{op}}$. That is, describing what the universe looks like "from the inside" will amount to reporting the view from $V^{(\widehat{P})}$. For simplicity let me write *H* for $\widehat{P}$.

The "truth value" $[\![\sigma]\!]$ of a sentence σ in $V^{(H)}$ is a sieve of events in *P*, and it is natural to think of the events in $[\![\sigma]\!]$ as those at which σ "holds". So one introduces the *forcing* relation $\Vdash_P$ in $V^{(H)}$ between sentences and elements of *P* by

$$p \Vdash_P \sigma \leftrightarrow p \in [\![\sigma]\!].$$

This satisfies the standard so-called Kripke rules, viz.,

- $p \Vdash_P \varphi \wedge \psi \leftrightarrow p \Vdash_P \varphi \;\;\&\;\; p \Vdash_P \psi$
- $p \Vdash_P \varphi \vee \psi \leftrightarrow p \Vdash_P \varphi \;\text{ or }\; p \Vdash_P \psi$
- $p \Vdash_P \varphi \rightarrow \psi \;\leftrightarrow\; \forall q \leq p[\, q \Vdash_P \varphi \;\rightarrow\; q \Vdash_P \psi]$
- $p \Vdash_P \neg\varphi \;\leftrightarrow\; \forall q \leq p \;\; q \nVdash_K \varphi$
- $p \Vdash_P \forall x\, \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for every $a \in V^{(\hat{P})}$
- $p \Vdash_P \exists x\, \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for some $a \in V^{(\hat{P})}$.

Define the set $K \in V^{(H)}$ by dom(*K*) = $\{\hat{p} : p \in P\}$ and $K(\hat{p}) = p\!\downarrow$. Then, in $V^{(H)}$, *K* is a subset of $\widehat{P}$ and for $p \in P$, $[\![\hat{p} \in K]\!] = p\!\downarrow$. *K* is the counterpart in $V^{(\hat{P})}$ of Markopoulou's evolving set *Past***.** ($\widehat{P}$, incidentally, is the $V^{(H)}$- counterpart of the constant presheaf on *P* with value *P*—which Markopoulou calls *World*.) The fact that, for any $p, q \in P$ we have

(*) $$q \Vdash_P \widehat{p} \in K \;\leftrightarrow\; q \leq p$$

may be construed as asserting that *the events in the causal future of a given event are precisely those forcing (the canonical representative of) that event to be a member of K.* Or, equally, *the events in the causal past of a* *given event are precisely those forced by that event to be a member of K.* For this reason we shall call K the *causal set in* $V^{(H)}$.

If we identify each $p \in P$ with $p\downarrow \in H$, P may then be regarded as a subset of H so that, in $V^{(H)}$, $\widehat{P}$ is a subset of $\widehat{H}$. It is not hard to show that $V^{(H)} \vDash K = \Phi_H \cap \widehat{P}$. Moreover, it can be shown that, for any sentence σ, $[\![\sigma]\!] = [\![\exists p \in K. p \leq \widehat{[\![\sigma]\!]}]\!]$, so that, with moderate abuse of notation,

$$V^{(H)} \vDash [\sigma \leftrightarrow \exists p \in K.\ p \Vdash \sigma].$$

That is, in $V^{(H)}$, *a sentence holds precisely when it is forced to do so at some "causal past stage" in K.* This establishes the centrality of K—and, correspondingly, that of the "evolving" set *Past*— in determining the truth of sentences "from the inside", that is, inside the universe $V^{(H)}$.

Markopoulou also considers the *complement* of *Past*—i.e., in the present setting, the $V^{(H)}$-set $\neg K$ for which $[\![\widehat{p} \in \neg K]\!] = [\![p \notin K]\!] = \neg p\downarrow = \{q : \forall r \leq q. r \not\leq p$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p\downarrow$ those *beyond p's causal horizon*, in that no observer at p can ever receive "information" from any event in $\neg p\downarrow$. Since clearly we have

$$(\dagger) \qquad q \Vdash_P \widehat{p} \in \neg K \ \leftrightarrow\ q \in \neg p\downarrow,$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of* $\neg K$. In this sense $\neg K$ reflects, or "measures" the causal structure of *P.*

In this connection it is natural to call $\neg\neg p\downarrow = \{q : \forall r \leq q \exists s \leq r. s \leq p]$ the *causal horizon* of p: it consists of those events q for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q. Since

$$q \Vdash_P \widehat{p} \in \neg\neg K \ \leftrightarrow\ q \in \neg\neg p\downarrow,$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of* $\neg\neg K$.

It is easily shown that $\neg K$ is *empty* (i.e. $V^{(H)} \vDash \neg K = \emptyset$) if and only if P is *directed downwards*, i.e., for any $p, q \in P$ there is $r \in P$ for which $r \leq p$ and $r \leq q$. This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which P is the opposite $\mathbb{N}^{op}$ of the totally ordered set $\mathbb{N}$ of natural numbers. Here the corresponding complete Heyting algebra H is the family of all downward-closed sets of natural numbers. In this case the H-valued set K representing *Past is neither finite nor actually infinite in* $V^{(H)}$.

To see this, observe first that, for any natural number n, we have $[\![\neg(\hat{n} \in \neg K)]\!] = \mathbb{N}$. It follows that $V^{(H)} \vDash \neg\neg\forall n \in \widehat{\mathbb{N}}.\ n \in K$. But, working in $V^{(H)}$, if $\forall n \in. \widehat{\mathbb{N}}\ \ n \in K$, then K is not finite, so if K is finite, then $\neg\forall n \in \widehat{\mathbb{N}}.\ n \in K$, and so $\neg\neg\forall n \in \widehat{\mathbb{N}}.\ n \in K$ implies the non-finiteness of K.

But, in $V^{(H)}$, K is not actually infinite. For (again working in $V^{(H)}$), if K were actually infinite (i.e., if there existed an injection of $\widehat{\mathbb{N}}$ into K), then the statement

$$\forall x \in K\, \exists y \in K.\ x > y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$
\begin{aligned}
&[\![\forall x \in K \exists y \in K . x > y]\!] \\
&\quad = \bigcap_{m \in \mathbb{N}^{op}} [m \downarrow \Rightarrow \bigcup_{n \in \mathbb{N}^{op}} n \downarrow \cap [\![\hat{m} > \hat{n}]\!]] \\
&\quad = \bigcap_{m} [m \downarrow \Rightarrow \bigcup_{n<m} n \downarrow] \\
&\quad = \bigcap_{m} [m \downarrow \Rightarrow (m+1) \downarrow] \\
&\quad = \bigcap_{m} (m+1) \downarrow = \varnothing
\end{aligned}
$$

So $\forall x \in K\ \exists y \in K.\ x > y$ is false in $V^{(H)}$ and therefore K is not actually infinite. In sum, the causal set K in is *potentially, but not actually infinite.*

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables.* This amounts to specifying a presheaf $\mathcal{A}$ of C^*-algebras on P, which, in the present framework, corresponds to specifying a set $\mathscr{A}$ in $V^{(H)}$ satisfying

$$V^{(H)} \vDash \mathscr{A} \textit{ is a } C^*\textit{-algebra.}$$

The "internal" C^*-algebra $\mathscr{A}$ is then subject to the intuitionistic internal logic of $V^{(H)}$: *any* theorem concerning C^*-algebras—provided only that it be constructively proved—automatically applies to $\mathscr{A}$. Reasoning with $\mathscr{A}$ is more direct and simpler than reasoning with $\mathcal{A}$.

This same procedure of "internalization" can be performed with any causally evolving object: each such object of type $\mathscr{T}$ corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \vDash S \textit{ is of type } \mathscr{T}.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event *p, Antichains*(*p*) consists of all sets of causally unrelated events in *Past*(*p*), while *Graphs*(*p*) is the set of all graphs supported by elements of *Antichains*(*p*). In the present framework *Antichains* is represented by the $V^{(H)}$–set $Anti = \{ X \subseteq \widehat{P} : X \textit{ is an antichain}\}$ and *Graphs* by the $V^{(H)}$–set *Grph*

= {G: $\exists X \in A$.*G is a graph supported by A*}. Again, both *Anti* and *Grph* can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Cover schemes or *Grothendieck topologies* may be used to force certain conditions to prevail in the associated models. (This corresponds to the process of *sheafification.*) A *cover scheme* on P is a map **C** assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p\!\downarrow\, = \{q\colon q \le p\}$, called (**C-**)*covers of p,* such that, if $q \le p$, any cover of p can be sharpened to a cover of q, i.e.,

$$S \in \mathbf{C}(p) \,\&\, q \le p \rightarrow \exists T \in \mathbf{C}(q)[\forall t \in T \exists s \in S(t \le s)].$$

A cover S of an event p may be thought of as a "sampling" of the events in p's causal future, a "survey" of p's potential effects, in short, a *survey of p.* Using this language the condition immediately above becomes: *for any survey S of a given event p, and any event q which is a potential effect of p, there is a survey of q each event in which is the potential effect of some event in S.*

There are three naturally defined cover schemes on P we shall consider. First, each sieve A in P determines two cover schemes $\mathbf{C}_A$ and $\mathbf{C}^A$ defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \qquad\qquad S \in \mathbf{C}^{\mathbf{A}}(p) \leftrightarrow p\!\downarrow \cap A \subseteq S$$

If $p \in$ A, any part of p's causal future thus counts as a $\mathbf{C}_A$–survey of *p,* and any part of *p's* causal future extending the common part of that future with A counts as a $\mathbf{C}^A$–survey of *p*. Notice that then $\varnothing \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\varnothing \in \mathbf{C}^A(p) \leftrightarrow p\!\downarrow \cap\, A = \varnothing$.

Next, we have the *dense cover scheme* **Den** given by:

$$S \in \mathbf{Den}(p) \leftrightarrow \forall q \leq p \exists s \in S \exists r \leq s(r \leq q):$$

That is, S is a dense survey of p provided that for every potential effect q of p there is an event in S with a potential effect in common with q.

Given a cover scheme **C** on P, a sieve I will be said to *encompass* an element $p \in P$ if I includes a **C-**cover of p. Thus a sieve I encompasses p if it contains all the events in some survey of p. Call I **C**-*closed* if it contains every element of P that it encompasses, i.e. if

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \rightarrow p \in I .$$

The set $\widehat{\mathbf{C}}$ of all **C-**closed sieves in P, partially ordered by inclusion, can be shown to be a frame—the frame *induced* by **C**—in which the operations of meet and $\Rightarrow$ coincide with those of $\widehat{P}$. Passing from $V^{(\widehat{P})}$ to $V^{(\widehat{\mathbf{C}})}$ is the process of *sheafification:* essentially, it amounts to replacing the forcing relation $\Vdash_P$ in $V^{(\widehat{P})}$ by the new forcing relation $\Vdash_{\widehat{\mathbf{C}}}$ in $V^{(\widehat{\mathbf{C}})}$. For atomic sentences σ these are related by

$$p \Vdash_{\widehat{\mathbf{C}}} \sigma \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S.\ s \Vdash_P \sigma ;$$

i.e., p **C-***forces the truth of a sentence just the truth of that sentence is P-forced by every event in some* **C**-*survey of p.*

The frame induced by the dense cover scheme **Den** in P turns out to be a complete Boolean algebra B. For the corresponding causal set K_B in $V^{(B)}$ we find that

$$q \Vdash_B \widehat{p} \in K_B \leftrightarrow q \in \neg\neg p{\downarrow}$$

$\leftrightarrow$ *q is in p's causal horizon.*

Comparing this with (*) above, we see that moving to the universe $V^{(B)}$—"Booleanizing" it, so to speak—*amounts to replacing causal futures by causal horizons.* When P is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of P, B is the two-element Boolean algebra **2,** and $V^{(B)}$ reduces to the universe V of "static" sets. In this case, then, the effect of "Booleanization" is to *render the universe timeless.*

The universes associated with the cover schemes $\mathbf{C}^A$ and $\mathbf{C}_A$ seem also to have a rather natural physical meaning. Consider, for instance, the case in which A is the sieve $p\downarrow$—the causal future of p. In the associated universe $V^{(\widehat{\mathbf{C}^A})}$ the corresponding causal set K^A satisfies, for every event q

$$q \Vdash_{\widehat{\mathbf{C}^A}} \widehat{p} \in K^A .$$

Comparing this with (*), we see that in $V^{(\widehat{\mathbf{C}^A})}$ that every event has been "forced" into p's causal future: in short, that p now marks the "beginning" of the universe as viewed from inside $V^{(\widehat{\mathbf{C}^A})}$.

Similarly, we find that the causal set K_A in the universe $V^{(\widehat{\mathbf{C}_A})}$ satisfies, for every event q,

$$q \Vdash_{\widehat{\mathbf{C}_A}} \widehat{p} \in \neg K_A ;$$

a comparison with (†) above reveals that, in $V^{(\widehat{\mathbf{C}^A})}$, every event has been "forced" beyond p's causal horizon. In effect, p has become a *singularity.*