In spacetime physics any set $\mathcal{C}$ of events—a causal set—is taken to be partially ordered by the relation $\leq$ of possible causation: for $p, q \in \mathcal{C}$, $p \leq q$ means that $q$ is in $p$’s future light cone. In her groundbreaking paper *The internal description of a causal set: What the universe looks like from the inside*, Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by “sets evolving over $\mathcal{C}$”—that is, in essence, by the topos $\mathcal{S}et^{\mathcal{C}}$. To enable what she has done to be more easily expressed within the framework presented here, I will reverse the causal ordering, that is, $\mathcal{C}$ will be replaced by $\mathcal{C}^{\text{op}}$, and the latter written as $P$—which will, moreover, be required to be no more than a preordered set. Specifically, then: $P$ is a set of events preordered by the relation $\leq$, where $p \leq q$ is intended to mean that $p$ is in $q$’s future light cone—that $q$ could be the cause of $p$, or, equally, that $p$ could be an effect of $q$. In that case, for each event $p$, the set $p^{\downarrow} = \{q : q \leq p\}$ may be identified as the causal future of $p$, or the set of potential effects of $p$. In requiring that $\leq$ be no more than a preordering—in dropping, that is, the antisymmetry of $\leq$—I am, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Accordingly I fix a preordered set $(P, \leq)$, which I shall call the universal causal set. Markopoulou, in essence, suggests that viewing the universe “from the inside” amounts to placing oneself within the topos of presheaves $\mathcal{S}et^{\mathcal{P}}_{\text{op}}$. Here I am going to show how $\mathcal{S}et^{\mathcal{P}}_{\text{op}}$ may be effectively replaced by a certain model of intuitionistic set theory, with (I hope) illuminating results.
Let us suppose that we are given a relation $\vdash$ between events $p$ and assertions $\varphi$: think of $p \vdash \varphi$ as meaning that $\varphi$ holds as a result of event $p$. Assume that the relation $\vdash$ is persistent in the sense that, if $p \vdash \varphi$ and $q \leq p$, then $q \vdash \varphi$: once an assertion holds, it continues to hold in the future. (The basic assertions we have in mind are of the form: “such and such is (or was) the case at such-and such a time (event)”.

Given an assertion $\varphi$, the set $[\varphi] = \{p: p \vdash \varphi\}$ “measures” the degree or extent to which $\varphi$ holds: the larger $[\varphi]$ is, the “truer” $\varphi$ is. In particular, when $[\varphi] = P$, $\varphi$ is ‘universally” or “absolutely” true, and when $[\varphi] = \emptyset$, $\varphi$ is “universally” or “absolutely” false. These $[\varphi]$ may accordingly be thought of as “truth values”, with $P$ corresponding to “absolute truth” and $\emptyset$ to absolute falsity.

Because of the persistence property, each $[\varphi]$ has the property of being “closed under potential effects”, or “causally closed”, that is, satisfies $p \in [\varphi]$ and $q \leq p \rightarrow q \in [\varphi]$. A subset of $P$ with this property is called a sieve. Sieves serve as generalized “truth values” measuring the degree to which assertions hold. The set $\hat{P}$ of all sieves, or truth values has a natural logico-algebraic structure —that of a complete Heyting algebra, or frame. This concept is defined in the following way.

A lattice is a partially ordered set $L$ with partial ordering $\leq$ in which each two-element subset $\{x, y\}$ has a supremum or join—denoted by $x \lor y$—and an infimum or meet—denoted by $x \land y$. A lattice $L$ is complete if every subset $X$ (including $\emptyset$) has a supremum or join—denoted by $\lor X$—and an infimum or meet—denoted by $\land X$. Note that $\lor \emptyset = 0$, the least or bottom element of $L$, and $\land \emptyset = 1$, the largest or top element of $L$. 
A *Heyting algebra* is a lattice \( L \) with top and bottom elements such that, for any elements \( x, y \in L \), there is an element—denoted by \( x \Rightarrow y \)—of \( L \) such that, for any \( z \in L \),

\[
z \leq x \Rightarrow y \text{ iff } z \land x \leq y.
\]

Thus \( x \Rightarrow y \) is the *largest* element \( z \) such that \( z \land x \leq y \). So in particular, if we write \( \neg x \) for \( x \Rightarrow 0 \), then \( \neg x \) is the largest element \( z \) such that \( x \Rightarrow z = 0 \): it is called the *pseudocomplement* of \( x \). A *Boolean algebra* is a Heyting algebra in which \( \neg\neg x = x \) for all \( x \), or equivalently, in which \( x \lor \neg x = 1 \) for all \( x \).

If we think of the elements of a (complete) Heyting algebra as “truth values”, then \( 0, 1, \land, \lor, \neg, \Rightarrow, \forall, \exists \) represent “true”, “false”, “and”, “or”, “not” and “implies”, “there exists” and “for all”, respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic*. In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

\[
(*) \quad x \land \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \land y_i)
\]

And conversely, in any complete lattice satisfying (*)

\[
\Rightarrow \text{ by } x \Rightarrow y = \bigvee\{z : z \land x \leq y\}
\]

turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame*.

In the frame \( \hat{P} \subseteq \subseteq \), joins and meets are just set-theoretic unions and intersections, and the operations \( \Rightarrow \) and \( \neg \) are given by
Frames do duty as the “truth-value algebras” of the (current) language of mathematics, that is, set theory. To be precise, associated with each frame $H$ is a structure $V^H$—the universe of $H$-valued sets—with the following features.

- Each of the members of $V^H$—the $H$-sets—is a map from a subset of $V^H$ to $H$.
- Corresponding to each sentence $\sigma$ of the language of set theory (with names for all elements of $V^H$) is an element $[\sigma]^H \in H$ called its truth value in $V^H$. These “truth values” satisfy the following conditions. For $a, b \in V^H$,

\[
[b \in a] = \bigvee_{c \in \text{dom}(a)} [b = c] \land a(c) \quad [b = a] = \bigvee_{c \in \text{dom}(a) \land \text{dom}(b)} ([c \in b] \leftrightarrow [c \in a])
\]

\[
[\sigma \land \tau] = [\sigma] \land [\tau], \text{ etc.}
\]

\[
[\exists x \phi(x)] = \bigvee_{a \in V^H} [\phi(a)]
\]

\[
[\forall x \phi(x)] = \bigwedge_{a \in V^H} [\phi(a)]
\]

A sentence $\sigma$ is valid, or holds, in $V^H$, written $V^H \models \sigma$, if $[\sigma] = 1$, the top element of $H$.
- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in $V^H$. In this sense $V^H$ is an $H$-valued model of IZF. Accordingly the category $\mathcal{S}_H$ of sets constructed within $V^H$ is a topos: in fact $\mathcal{S}_H$ can be shown to be equivalent to the topos of canonical sheaves on $H$.
- There is a canonical embedding $x \mapsto \hat{x}$ of the usual universe $V$ of sets into $V^H$ satisfying
\[ \lfloor u \in x \rfloor = \bigvee_{y \in x} \lfloor u = y \rfloor \] for \( x \in V, u \in V^H \)

\[ x \in y \iff V^H \models \hat{x} \in \hat{y}, \quad x = y \iff V^H \models \hat{x} = \hat{y} \] for \( x, y \in V \)

\[ \varphi(x_1, \ldots, x_n) \iff V^H \models \varphi(\hat{x}_1, \ldots, \hat{x}_n) \] for \( x_1, \ldots, x_n \in V \) and restricted \( \varphi \)

(Here a formula \( \varphi \) is \textit{restricted} if all its quantifiers are restricted, i.e. can be put in the form \( \forall x \in y \) or \( \exists x \in y \).)

We observe that \( V^{(2)} \) is essentially just the usual universe of sets.

It follows from the last of these assertions that the canonical representative \( \hat{H} \) of \( H \) is a Heyting algebra in \( V^H \). A particularly important \( H \)-set is the \( H \)-set \( \Phi_H \) defined by

\[ \text{dom}(\Phi_H) = \{ a : a \in H \}, \quad \Phi_H(a) = a \] for \( a \in H \).

Then \( V^H \models \Phi_H \subseteq \hat{H} \). Also, for any \( a \in H \) we have \( \lfloor \hat{a} \in \Phi_H \rfloor = a \), and in particular, for any sentence \( \sigma \), \( \lfloor \sigma \rfloor = \lfloor \lfloor \hat{\sigma} \rfloor \in \Phi_H \rfloor \). Thus

\[ V^H \models \sigma \iff V^H \models \lfloor \sigma \rfloor \in \Phi_H ; \]

in this sense \( \Phi_H \) represents the “true” sentences in \( V^H \). \( \Phi_H \) is called the \textit{canonical truth set} in \( V^H \).

Now let us return to our causal set \( P \). The topos \( \mathcal{J} \mathcal{E} \mathcal{L}^{(P)} \) of sets in \( V^{(P)} \) is, as I have observed, equivalent to the topos of canonical sheaves on \( \hat{P} \), which is itself, as is well known, equivalent to the topos \( \mathcal{J} \mathcal{E} \mathcal{L}^{\text{pre}} \) of presheaves on \( P \). My proposal is then, that we work in \( V^{(P)} \) rather than, as did Markopoulou, within \( \mathcal{J} \mathcal{E} \mathcal{L}^{\text{pre}} \). That is, describing what the universe looks like “from the inside” will amount to reporting the view from \( V^{(P)} \). For simplicity let me write \( H \) for \( P \).
The “truth value” $[\sigma]$ of a sentence $\sigma$ in $V(H)$ is a sieve of events in $P$, and it is natural to think of the events in $[\sigma]$ as those at which $\sigma$ “holds”. So one introduces the forcing relation $\models_P$ in $V(H)$ between sentences and elements of $P$ by

$$p \models_P \sigma \iff p \in \llbracket \sigma \rrbracket.$$ 

This satisfies the standard so-called Kripke rules, viz.,

- $p \models_P \varphi \land \psi \iff p \models_P \varphi$ & $p \models_P \psi$
- $p \models_P \varphi \lor \psi \iff p \models_P \varphi$ or $p \models_P \psi$
- $p \models_P \varphi \rightarrow \psi \iff \forall q \leq p [q \models_P \varphi \rightarrow q \models_P \psi]$
- $p \models_P \neg \varphi \iff \forall q \leq p \ q \not\models_K \varphi$
- $p \models_P \forall x \varphi \iff p \models_P \varphi(a)$ for every $a \in V(P)$
- $p \models_P \exists x \varphi \iff p \models_P \varphi(a)$ for some $a \in V(P)$.

Define the set $K \in V(H)$ by $\text{dom}(K) = \{\hat{p} : p \in P\}$ and $K(\hat{p}) = p \downarrow$. Then, in $V(H)$, $K$ is a subset of $\hat{P}$ and for $p \in P$, $\llbracket \hat{p} \in K \rrbracket = p \downarrow$. $K$ is the counterpart in $V(P)$ of Markopoulou’s evolving set $\text{Past}$. ($\hat{P}$, incidentally, is the $V(H)$-counterpart of the constant presheaf on $P$ with value $P$—which Markopoulou calls $\text{World}$.) The fact that, for any $p, q \in P$ we have

$$(*) \qquad q \models_P \hat{p} \in K \iff q \leq p$$

may be construed as asserting that the events in the causal future of a given event are precisely those forcing (the canonical representative of) that event to be a member of $K$. Or, equally, the events in the causal past of a
given event are precisely those forced by that event to be a member of \( K \).

For this reason we shall call \( K \) the *causal set* in \( V^{(H)} \).

If we identify each \( p \in P \) with \( p \downarrow \in H \), \( P \) may then be regarded as a subset of \( H \) so that, in \( V^{(H)} \), \( \widehat{P} \) is a subset of \( \widehat{H} \). It is not hard to show that \( V^{(H)} \models K = \Phi_H \cap \widehat{P} \). Moreover, it can be shown that, for any sentence \( \sigma \), \([\sigma] = [\exists p \in K. p \leq [\sigma]]\), so that, with moderate abuse of notation,

\[
V^{(H)} \models [\sigma \leftrightarrow \exists p \in K. p \models \sigma].
\]

That is, in \( V^{(H)} \), a sentence holds precisely when it is forced to do so at some “causal past stage” in \( K \). This establishes the centrality of \( K \)—and, correspondingly, that of the “evolving” set \( \text{Past} \) in determining the truth of sentences “from the inside”, that is, inside the universe \( V^{(H)} \).

Markopoulou also considers the complement of \( \text{Past} \)—i.e., in the present setting, the \( V^{(H)} \)-set \( \neg K \) for which \([\neg \exists p \in K \downarrow] = \neg \exists \{q : \forall r \leq q. r \notin p \} \). Markopoulou calls (mutatis mutandis) the events in \( \neg p \downarrow \) those beyond \( p \)’s causal horizon, in that no observer at \( p \) can ever receive “information” from any event in \( \neg p \downarrow \). Since clearly we have

\[
(q \models p \iff \neg \exists p \in \neg K \iff q \in \neg p \downarrow),
\]

it follows that the events beyond the causal horizon of an event \( p \) are precisely those forcing (the canonical representative of) \( p \) to be a member of \( \neg K \). In this sense \( \neg K \) reflects, or “measures” the causal structure of \( P \).

In this connection it is natural to call \( \neg p \downarrow = \{q : \forall r \leq q \exists s \leq r.s \leq p \} \) the causal horizon of \( p \): it consists of those events \( q \) for which an observer placed at \( p \) could, in its future, receive information from any event in the future of an observer placed at \( q \). Since

\[
(q \models p \iff \neg \exists p \in \neg K \iff q \in \neg p \downarrow),
\]
it follows that the events within the causal horizon of an event are precisely those forcing (the canonical representative of) \( p \) to be a member of \( \neg \neg K \).

It is easily shown that \( \neg K \) is empty (i.e. \( V^{[H]} \models \neg K = \emptyset \)) if and only if \( P \) is directed downwards, i.e., for any \( p, q \in P \) there is \( r \in P \) for which \( r \leq p \) and \( r \leq q \). This holds in the case, considered by Markopoulou, of discrete Newtonian time evolution—in the present setting, the case in which \( P \) is the opposite \( \mathbb{N}^{op} \) of the totally ordered set \( \mathbb{N} \) of natural numbers. Here the corresponding complete Heyting algebra \( H \) is the family of all downward-closed sets of natural numbers. In this case the \( H \)-valued set \( K \) representing \( \text{Past} \) is neither finite nor actually infinite in \( V^{[H]} \).

To see this, observe first that, for any natural number \( n \), we have \( \llbracket \neg (\hat{n} \in \neg K) \rrbracket = \mathbb{N} \). It follows that \( V^{[H]} \models \neg \neg \forall n \in \hat{\mathbb{N}}. n \in K \). But, working in \( V^{[H]} \), if \( \forall n \in \hat{\mathbb{N}}. n \in K \), then \( K \) is not finite, so if \( K \) is finite, then \( \neg \forall n \in \hat{\mathbb{N}}. n \in K \), and so \( \neg \neg \forall n \in \hat{\mathbb{N}}. n \in K \) implies the non-finiteness of \( K \).

But, in \( V^{[H]} \), \( K \) is not actually infinite. For (again working in \( V^{[H]} \)), if \( K \) were actually infinite (i.e., if there existed an injection of \( \hat{\mathbb{N}} \) into \( K \)), then the statement

\[
\forall x \in K \exists y \in K. x > y
\]

would also have to hold in \( V^{[H]} \). But calculating that truth value gives:
\[ \forall x \in K \exists y \in K. x > y \]
\[ = \bigcap_{m \in \mathbb{N}} \left( \bigcup_{n \in \mathbb{N}} n \downarrow \cap \left[ \bigcup_{\hat{m} \in \mathbb{N}} \bigcap_{\hat{n} \in \mathbb{N}} \bigcup_{\hat{m} < \hat{n}} \bigcup_{\hat{n} \downarrow} \right] \bigcup_{n < m} n \downarrow \right) \]
\[ = \bigcup_{m \in \mathbb{N}} \left( \bigcup_{n : n < m} n \downarrow \right) \]
\[ = \bigcap_{m \in \mathbb{N}} \left( m \downarrow \Rightarrow (m + 1) \downarrow \right) \]
\[ \cap \left( m \downarrow \Rightarrow \emptyset \right) \]

So \( \forall x \in K \exists y \in K. x > y \) is false in \( V(H) \) and therefore \( K \) is not actually infinite. In sum, the causal set \( K \) in is \textit{potentially, but not actually infinite}.

In order to formulate an observable causal \textit{quantum theory} Markopoulou considers the possibility of introducing a \textit{causally evolving algebra of observables}. This amounts to specifying a presheaf \( \mathcal{A} \) of \( C^* \)-algebras on \( P \), which, in the present framework, corresponds to specifying a set \( \mathcal{A} \) in \( V(H) \) satisfying

\[ V(H) \models \mathcal{A} \text{ is a } C^*-\text{algebra}. \]

The “internal” \( C^* \)-algebra \( \mathcal{A} \) is then subject to the intuitionistic internal logic of \( V(H) \): \textit{any} theorem concerning \( C^* \)-algebras—provided only that it be constructively proved—automatically applies to \( \mathcal{A} \). Reasoning with \( \mathcal{A} \) is more direct and simpler than reasoning with \( \mathcal{A} \).

This same procedure of “internalization” can be performed with any causally evolving object: each such object of type \( \mathcal{T} \) corresponds to a set \( S \) in \( V(H) \) satisfying

\[ V(H) \models S \text{ is of type } \mathcal{T}. \]

Internalization may also be applied in the case of the presheaves \textit{Antichains} and \textit{Graphs} considered by Markopoulou. Here, for each event \( p \), \textit{Antichains}(\( p \)) consists of all sets of causally unrelated events in \textit{Past}(\( p \)), while \textit{Graphs}(\( p \)) is the set of all graphs supported by elements of \textit{Antichains}(\( p \)). In the present framework \textit{Antichains} is represented by the \( V(H) \)-set \( \text{Anti} = \{ X \subseteq \hat{P} : X \text{ is an antichain} \} \) and \textit{Graphs} by the \( V(H) \)-set \( \text{Grph} \)
\[= \{G : \exists X \in A . G \text{ is a graph supported by } A\}. \text{Again, both } \text{Anti} \text{ and } \text{Grph} \text{ can}\]

\[\text{be readily handled using the internal intuitionistic logic of } \mathcal{V}^{[H]}\].

**Cover schemes** or **Grothendieck topologies** may be used to force certain conditions to prevail in the associated models. (This corresponds to the process of **sheafification**.) A **cover scheme** on \(P\) is a map \(\mathbf{C}\) assigning to each \(p \in P\) a family \(\mathbf{C}(p)\) of subsets of \(p\downarrow = \{q : q \leq p\}\), called \((\mathbf{C})\text{-covers of } p\), such that, if \(q \leq p\), any cover of \(p\) can be sharpened to a cover of \(q\), i.e.,

\[
S \in \mathbf{C}(p) \& q \leq p \to \exists T \in \mathbf{C}(q)[\forall t \in T \exists s \in S(t \leq s)].
\]

A cover \(S\) of an event \(p\) may be thought of as a “sampling” of the events in \(p\)'s causal future, a “survey” of \(p\)’s potential effects, in short, a **survey of \(p\)**. Using this language the condition immediately above becomes: *for any survey \(S\) of a given event \(p\), and any event \(q\) which is a potential effect of \(p\), there is a survey of \(q\) each event in which is the potential effect of some event in \(S\)*.

There are three naturally defined cover schemes on \(P\) we shall consider. First, each sieve \(A\) in \(P\) determines two cover schemes \(\mathbf{C}_A\) and \(\mathbf{C}^A\) defined by

\[
S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \quad S \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A \subseteq S
\]

If \(p \in A\), any part of \(p\)'s causal future thus counts as a \(\mathbf{C}_A\text{-survey of } p\), and any part of \(p\)'s causal future extending the common part of that future with \(A\) counts as a \(\mathbf{C}^A\text{-survey of } p\). Notice that then \(\emptyset \in \mathbf{C}_A(p) \leftrightarrow p \in A\) and \(\emptyset \in \mathbf{C}^A(p) \leftrightarrow p\downarrow \cap A = \emptyset\).

Next, we have the **dense cover scheme** **Den** given by:
That is, $S$ is a dense survey of $p$ provided that for every potential effect $q$ of $p$ there is an event in $S$ with a potential effect in common with $q$.

Given a cover scheme $\mathbf{C}$ on $P$, a sieve $I$ will be said to encompass an element $p \in P$ if $I$ includes a $\mathbf{C}$-cover of $p$. Thus a sieve $I$ encompasses $p$ if it contains all the events in some survey of $p$. Call $I$ $\mathbf{C}$-closed if it contains every element of $P$ that it encompasses, i.e. if

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \rightarrow p \in I .$$

The set $\widehat{\mathbf{C}}$ of all $\mathbf{C}$-closed sieves in $P$, partially ordered by inclusion, can be shown to be a frame—the frame induced by $\mathbf{C}$—in which the operations of meet and $\Rightarrow$ coincide with those of $\widehat{P}$. Passing from $V^{(P)}$ to $V^{(C)}$ is the process of sheafification: essentially, it amounts to replacing the forcing relation $\models_P$ in $V^{(P)}$ by the new forcing relation $\models_C$ in $V^{(C)}$. For atomic sentences $\sigma$ these are related by

$$p \models_C \sigma \iff \exists S \in \mathbf{C}(p) \forall s \in S. \ s \models_P \sigma ;$$

i.e., $p$ $\mathbf{C}$-forces the truth of a sentence just the truth of that sentence is $P$-forced by every event in some $\mathbf{C}$-survey of $p$.

The frame induced by the dense cover scheme $\textbf{Den}$ in $P$ turns out to be a complete Boolean algebra $B$. For the corresponding causal set $K_B$ in $V^{[B]}$ we find that

$$q \models_B \widehat{p} \in K_B \iff q \in \neg \neg p \downarrow$$

$$\iff q \text{ is in } p \text{'s causal horizon.}$$
Comparing this with (*) above, we see that moving to the universe $V(B)$—“Booleanizing” it, so to speak—*amounts to replacing causal futures by causal horizons*. When $P$ is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of $P$, $B$ is the two-element Boolean algebra $2$, and $V(B)$ reduces to the universe $V$ of “static” sets. In this case, then, the effect of “Booleanization” is to *render the universe timeless*.

The universes associated with the cover schemes $C^A$ and $C_A$ seem also to have a rather natural physical meaning. Consider, for instance, the case in which $A$ is the sieve $p^\downarrow$—the causal future of $p$. In the associated universe $V(\widehat{C^A})$ the corresponding causal set $K^A$ satisfies, for every event $q$

$$ q \models_{C^A} \hat{p} \in K^A. $$

Comparing this with (*), we see that in $V(\widehat{C^A})$ that every event has been “forced” into $p$’s causal future: in short, that $p$ now marks the “beginning” of the universe as viewed from inside $V(\widehat{C^A})$.

Similarly, we find that the causal set $K_A$ in the universe $V(\widehat{C_A})$ satisfies, for every event $q$,

$$ q \models_{\widehat{C_A}} \hat{p} \in \neg K_A; $$

a comparison with (†) above reveals that, in $V(\widehat{C_A})$, every event has been “forced” beyond $p$’s causal horizon. In effect, $p$ has become a *singularity*. 