Chapter 1

The Continuous and the Discrete in Ancient Greece, the Orient, and the European Middle Ages

1. Ancient Greece

THE PRESOCRATICS

The opposition between Continuity and Discreteness played a significant role in ancient Greek philosophy. This probably derived from the still more fundamental question concerning the One and the Many, an antithesis lying at the heart of early Greek thought.[1]

The opposition between One and Many seems to have been an animating principle in the thought of the Milesian philosophers **Thales** (*fl.* 585 B.C.), **Anaximander** (*fl.* 570 B.C.), and **Anaximenes** (*fl.* 550 B.C.). Monists all, they shared the belief that the world, manifold in appearance, could be reduced to a single underlying principle—although they disagreed as to what that principle was. The **Pythagoreans**[2] were, in essence, dualists, claiming that the world was built on the two ultimate principles of the limited and the unlimited, which in turn engender the whole series of opposites such as odd and even, one and many, still and moving. The Pythagoreans are also believed to have held that all things are made of number, from which it would seem to follow that they were atomists in some sense. But they can be considered genuine atomists only if the "numbers" they held to be constitutive of magnitude are indivisible atomic magnitudes in something like the sense of later atomism. Their discovery of incommensurable lines provides an instant refutation of a narrow version of atomism in which it is claimed that any continuous line is composed of a definite finite number of minimal indivisible unit lengths. Perhaps the alarm with which, according to tradition, the Pythagoreans reacted to this discovery is an indication that they did subscribe to this narrower atomism. But this was unclear even to Aristotle:

> *Nor is it in any way defined in what sense numbers are the causes of substances and of Being; whether as bounds, e.g. as points are the bounds of spatial magnitudes and as Eurytus determined which number belongs to which thing—e.g. this number to man, and this to horse—by using pebbles to copy the shapes of natural objects, like those who arrange numbers in the form of geometrical figures, the triangle and the square. Or is it because harmony is a ratio of numbers, and so is man and everything else?*[3]

Contemporary scholars are, appropriately perhaps, divided on the issue.

Heraclitus (*fl.* 500 B.C.), while essentially a monist, introduced into his monism two strikingly novel elements: these are the *Doctrine of Flux*, namely that all things are undergoing constant, if imperceptible change; and the *Unity of Opposites*: each object is constituted by opposing features. Heraclitus's "Flux" seems to mean continuous flux. The doctrine of the Unity of Opposites may derive from the observation that objects acquire contradictory attributes through a process of continuous change—as the

[1] See Stokes (1971).
[2] Pythagoras himself is believed to have been active between 540 and 520 B.C.
[3] Aristotle (1996), 1092b8

ground, initially dry, becomes wet after rainfall. On this basis Heraclitus may be counted a forerunner of the synechists—a "protosynechist", perhaps.

The Greek debate over the continuous and the discrete seems to have been ignited by the efforts of the Eleatic philosophers **Parmenides** (*b. c.* 515 B.C.), **Zeno** (*fl.* 460 B.C.) and **Melissus** (*fl.* 440 B.C.) to establish their doctrine of absolute monism. They were concerned to show that the divisibility of Being into parts leads to contradiction, so forcing the conclusion that the apparently diverse world is a static, changeless unity.[1] In his *Way of Truth* Parmenides asserts that Being is *homogeneous* and *continuous:*

> *It [Being] never was nor will be, since it is now, all together, one, continuous. ...Nor is it divided, since it all exists alike; nor is it more here and less there, which would prevent it from holding together,. but it is all full of Being. So it is all continuous; for what is neighbours what is.* [2]

These passages suggest that Parmenides be identified as a synechist. But in asserting the continuity of Being Parmenides is likely no more than underscoring its essential unity. For consider the later passage:

> *But look at things which, though far off, are securely present to the mind.; for you will not cut off for yourself what is from holding to what is, neither scattering everywhere in every way in order, nor drawing together.*[3]

Parmenides seems to be claiming that Being is more than merely continuous—that it is, in fact, a single whole, indeed an *indivisible* whole. The single Parmenidean existent is a continuum without parts, at once a continuum and an atom. If Parmenides was a synechist, his absolute monism prevented him from being a divisionist.

Parmenides' assertion that reality is a unique, partless continuum was reiterated by his disciple Melissus. However the latter's observation:

> *If there were a plurality, things would have to be of the same kind as I say the One is,*[4]

which was intended as a *reductio ad absurdum* of belief in a plurality of things, seems to have opened the door to the emergence of atomism. In the atomists' hands, Melissus's assertion became, in effect,

> *there is a plurality of things, all of the same character as the One.*

This "plurality of things" are the indivisible atoms from which, according to the atomists, reality is synthesized.

Zeno's *Dichotomy* and *Achilles* paradoxes both rest explicitly on the limitless divisibility of space and time. It has been supposed by some scholars, following Aristotle, that Zeno's *Arrow* paradox depends on the assumption that time, at least, is composed of atomic instants, but this view has been challenged. If indeed, as Barnes[5] and Furley[6] claim, none of Zeno's paradoxes of motion assume the atomic hypothesis, then it would be not unreasonable to number him among the divisionists. This is also consistent with the fact that he was a disciple of Parmenides.

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[1] That this was the Eleatic position may be inferred from Plato's *Parmenides.*
[2] Kirk, Raven and Schofield (1983), pp. 249-50.
[3] *Ibid.,* p. 262.
[4] *Ibid.,* p. 399.
[5] Barnes (1986).
[6] Furley (1967)..

The Eleatic arguments that plurality and change are illusions created an impasse for those philosophers—the protophysicists—concerned with the understanding of natural phenomena. They felt it essential to circumvent the Eleatic arguments, so preserving the multiplicity of the world evident to the senses, without deriving in the process a plurality from a pre-existing unity or allowing the generation or change of anything real. Two essentially different ways out of the impasse were found, one based on continuity, the other on discreteness. The first was the creation of **Anaxagoras** (500-428 B.C.), who conceived of matter, like magnitude, as being infinitely divisible, and who eliminated both generation and the derivation of plurality from unity by postulating *ab initio* an endless variety of primary substances in the form of infinitely divisible "seeds", all mixed together. Anaxagoras's theory of matter, the *homoimereia* or theory of homogeneous mixtures, was described by Lucretius with memorable scorn:

> *In speaking of the* homoimereia *of things Anaxagoras means that bones are formed of minute miniature bones, flesh of minute miniature morsels of flesh, blood by the coalescence of many drops of blood; gold consists of grains of gold; earth is a conglomeration of little earths, fire of fires, moisture of moistures. And he pictures everything else as formed in the same way. At the same time he does not admit any vacuum in things, or any limit to the splitting of matter…*[1]

From this last phrase it may be inferred that Anaxagoras can be counted among the divisionists. Further evidence for this is provided by his own assertion, as reported by Simplicius:

> *Neither is there a smallest part of what is small, but there is always a smaller, for it is impossible that what is should ever cease to be.*[2]

That Anaxagoras may even be considered a full-blown synechist follows from two other passages attributed to him:

> *But before these things were separated off, while all things were together, there was not even any colour plain; for the mixture of all things prevented it, of the moist and the dry, the hot and the cold, the bright and the dark.*[3]
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> *The things in the one world-order are not separated one from the other nor cut off with an axe, neither the hot from the cold nor the cold from the hot.*[4]

The second attempt at escaping the Eleatic dilemma, *atomism*, was first and foremost a physical theory. It was mounted by **Leucippus** (*fl.* 440 B.C.) and **Democritus** (b. 460–457 B.C.) who, in contrast with Anaxagoras, maintained that matter was composed of indivisible, solid, homogeneous, spatially extended corpuscles, all below the level of visibility. Leucippus was regarded by Aristotle as the true founder of atomism. In *On Generation and Corruption*, Book I, he asserts:

> *For some of the ancients* [i.e., Parmenides and Melissus] *thought that what is must necessarily be one and motionless, since the void is nonexistent and there could be no motion without a separately existing void, and again there could not be a plurality without something to separate them. And if someone thinks that the universe is not continuous but consists of divided pieces in contact with each*

[1] Lucretius (1955), p. 51
[2] Kirk, Raven and Schofield (1983), p. 360.
[3] *Ibid.* p. 358.
[4] *Ibid.* p. 371.

> *other, this is no different, they held, from saying that it is many, not one, and is void. For if it is divisible everywhere, there is no unit, and therefore no many, and the whole is void. If on the other hand it is divisible at one place and not another, this seems like a piece of fiction. For how far is it divisible, and why is one part of the whole like this—full—and another part divided? Again, it is necessary similarly that there be no motion… But Leucippus thought he had arguments which would assert what is consistent with sense-perception and not do away with coming into being or perishing or motion, or the plurality of existents. He agrees with the appearances to this extent, but he concedes, to those who maintain the One, that there would be no motion without void, and says that void is "non-existence", and nothing of "what is" is "not being"; for 'what is" in the strictest sense is a complete* plenum. *But this* plenum, *he says, is not one but many things of infinite number, and invisible owing to their minuteness. These are carried along in the void (for there* is *a void) and, when they come together, they cause coming-to-be and, when they dissolve, they cause passing-away. They act and are acted upon where they happen to come into contact (for there they are not one), and they generate when they are placed together and intertwined. From what is truly one no plurality could come into being, nor a unity from what is truly a plurality— this is impossible.* [1]

And in *Physics,* Book I:

> *Some gave in to both of these arguments—to the argument that all is one if what is signifies one thing, and to the argument from dichotomy—by positing atomic magnitudes.*[2]

As can be seen from the first passage, Leucippus's atomism—his theory of infinitely numerous invisible corpuscles moving in a void—is presented by Aristotle as an attempt in the first instance to reconcile the evidence of the senses with Eleatic metaphysics. The senses tell us that the world is not a unity, but a plurality. In that case, where is unity to be sought? According to (Aristotle's) Leucippus, this unity is to be found in his postulated indivisible atomic magnitudes, each of which is, on a minute scale, an embodiment of the Eleatic One. Their combinations and dispersions underlie the phenomena of coming to be and passing away. The final sentence of the quotation indicates that Leucippus did not regard extended continua as true unities, since he presumably accepted the evident fact that such continua can be divided, thereby generating (as observed above) a plurality. The second passage's mention of the argument from dichotomy has been taken by scholars[3] as indicating Leucippus's rejection of Zeno's putative divisionism.

Scholarly opinion is divided over the question whether Leucippus and, especially, Democritus considered their atoms to be *physically*, but not *theoretically* indivisible. The traditional view[4] (as presented, e.g. in Heath) was that at least the latter did not. Furley, in his *Two Studies of the Greek Atomists*[5], on the other hand, argues that neither Democritus nor Aristotle made any distinction between physical and theoretical indivisibility, and so the former would think of his indivisible magnitudes as being theoretically as well as physically indivisible. As Furley points out, the hypothesis that Leucippus and Democritus postulated theoretically indivisible atoms is confirmed by Simplicius:

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[1] *Ibid.* p. 407.
[2] *Ibid.* p. 408.
[3] See, e.g. Kirk, Raven and Schofield (1983), p. 408
[4] As presented, e.g. in Heath (1981).
[5] Furley (1967).

> *Leucippus and Democritus think that the cause of the indivisibility of the primary bodies is not merely their imperviousness but also their smallness and partlessness; Epicurus, later, does not think they are partless, but says that they are atomic because of their imperviousness.*[1]

Like the Eleatic One, Democritus's atoms were, Furley thinks, "absolutely solid, packed with being and nothing else". As plena atoms are impenetrable and so indivisible. But the universe as a whole is divisible since it consists of a multiplicity of existents separated by void.

But if Democritus was a geometric atomist (and the case will in all likelihood never be proven), he was almost certainly aware of the difficulties attendant on the idea of theoretical indivisibles. For witness the following well-known passage from Plutarch, believed to be a quotation of Democritus's own words, and which has been termed by scholars the *cone dilemma:*

> *If a cone is cut by sections parallel to its base, are we to say that the sections are equal or unequal? If we suppose that they are unequal, they will make the surface of the cone rough and indented by a series of steps. If the surfaces are equal, then the sections will be equal and the cone will become a cylinder, being composed of equal, instead of unequal, circles. This is a paradox.*[2]

A related fact is Archimedes' attribution (in *The Method*) to Democritus of the discovery[3] that the volume of a right circular cone is one third that of the circumscribed cylinder. Archimedes also claims that Democritus was unable to prove the result rigorously. While it is not known whether Democritus managed to resolve the cone dilemma, it is highly likely lthat he arrived at the volume formula by analyzing the cone into a collection of infinitesimally thin circular laminas. This use of infinitesimals anticipates Cavalieri's method of indivisibles[4] (see below).

Even if Democritus did not uphold the actual existence of geometric indivisibles such as lines or surfaces, his *material* atomism may well have suggested the geometric analogy, which, while metaphysically problematic, proved to be mathematically most fruitful.

THE METHOD OF EXHAUSTION

Antiphon the Sophist, a contemporary of Socrates, is believed to have made one of the earliest attempts to rectify the circle. According to Simplicius, his procedure involved the inscribing in a circle of a regular polygon, for example a triangle or square, and then successively doubling the number of sides. In this way,

> *Antiphon thought that the area (of the circle) would be used up, and we should some time have a polygon inscribed in the circle the sides of which, owing to their smallness, coincide with the circumference of the circle. And, as we can make a square equal to any polygon… we shall be in a position to make a square equal to a circle.* [5]

As Simplicius notes, this infringes the principle that magnitudes are divisible without limit. If Antiphon truly thought that a circle could actually coincide with an inscribed

[1] For Epicurus's views see below.
[2] Quoted in Sambursky (1963), p. 153.
[3] But not the rigorous proof, which in his treatise *On the Sphere and Cylinder* Archimedes ascribes to Eudoxus.
[4] See Chapter 2.
[5] Quoted in Heath (1981), vol. I, p. 222.

polygon of a sufficiently large number of sides, then he has to be considered an atomist. Millennia later, the idea that a curve can be considered as an assemblage of infinitesimal straight lines came to play an important role in the development of the calculus.

If Antiphon and Democritus were in fact geometric atomists, they constituted exceptions among the mathematicians of ancient Greece. For Greek geometry rested on the assumption that magnitudes are divisible without limit, and so the very practice of Greek mathematicians would incline them towards divisionism. In particular the method of exhaustion worked out by **Eudoxus** (408-355 B.C.) the germ of which is stated in the proposition opening Book X of **Euclid**'s (325–265 B.C.) *Elements*, clearly presupposes that any magnitude can be divided without limit:

> *If from any magnitude there be subtracted not less than its half, from the remainder not less than its half and so on continually, there will at length remain a magnitude less than any assigned magnitude of the same kind.*

Eudoxus is also believed to have created the general theory of proportions presented in Book V of the Elements. In Definition 4 of that book, magnitudes are decreed to have a ratio to one another just when they "are capable, when multiplied, of exceeding one another". This prescription effectively excludes infinitesimal magnitudes from consideration[1].

Archimedes (287-212 B.C.), the greatest mathematician of antiquity, made a number of important applications of the method of exhaustion. As a pivotal principle he employed what has come to be known as the *axiom of Archimedes,* an elaborated version of Definition 4 in Euclid's Book V:

> *Of unequal lines, unequal surfaces, or unequal solids, the greater exceeds the less by such a magnitude as is capable, if added (continually) to itself, of exceeding any magnitude of those which are comparable to one another.*

As has been pointed out, a prescription of this sort excludes infinitesimal magnitudes. Yet one of the central ideas in Archimedes' *Method* is that surfaces may be regarded as being composed of lines. How Archimedes intended this to be understood is not entirely clear. He does not speak of the number of lines in each figure as infinite, saying only that the figure is *made up of all* the lines in it. But this does suggest that he probably thought of these lines as indivisibles, infinitesimally narrow surface "elements". Further evidence for this is offered "by the highly suggestive fact that he was led to many new results by a process of balancing, in thought, elements of dissimilar figures, using the principle of the lever precisely as one would in weighing mechanically a collection of thin laminae or material strips."[2]

Important as the method of indivisibles was to Archimedes as a heuristic, however, he did not consider results discovered through its use as having been rigorously proved. Rigorous proof was invariably supplied by means of the method of exhaustion. For Archimedes, atomism pointed the way to (geometric) truth, but that truth could only be secured by rigorous derivation from synechist postulates.

PLATO

Plato (429–328 B.C.) may have accepted the existence of indivisible magnitudes. Aristotle, in the *Metaphysics*, reports:

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[1] Yet in Book III infinitesimals appear in the form of "horn" angles: angles between curved lines. Proposition 16 asserts that the angle between a circle and a tangent is less than any rectilineal angle.

[2] Boyer (1959), p. 50.

> *Plato steadily rejected this class of objects* [i.e., points]*as a geometrical fiction, but he recognized "the beginning of a line," and he frequently assumed this latter class, which he called the "indivisible lines".*[1]

Similar views were held by Plato's pupil Xenocrates, who postulated the existence of atomic magnitudes in an attempt to avoid the pitfalls of Zeno's paradoxes.

In the *Timaeus* Plato postulates that the Empedoclean elements earth, water, air and fire are made up of two basic geometric units, both right-angled triangles. One is the right-angled isosceles triangle, the half-square; the other the Pythagorean "1, √3, 2" triangle, or half-equilateral triangle. The half-square is used to build up cubes, the characteristic shape of earth particles; the half-equilateral is used to build up regular pyramids, octahedra, and icosahedra, the characteristic shapes of the particles of fire, air and water, respectively.

While Plato says nothing concerning the indivisibility or otherwise of his triangular units, a number of scholars, including Cornford and Furley, have argued that the theory implicitly requires the use of indivisible magnitudes. Furley[2] points out that in Plato's theory, while fire, air and water can be transformed into one another—for instance, the half-equilaterals forming an icosahedron of water may re-form into pyramids and thus become fire—earth, composed of half-squares, cannot become anything else. But from this it would seem to follow that at least the elementary half-squares must be indivisible. For a (divisible) square can be divided into twelve half-equilaterals and a smaller square as in the diagram:

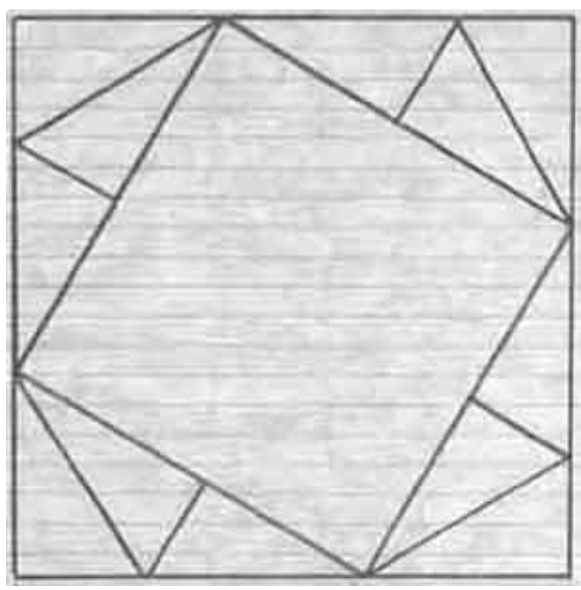

FIGURE 1

It follows that, unless the faces of a cube of earth are indivisible, the cube can be transformed into nine pyramids of fire and a smaller cube of earth, whch is certainly at

[1] Aristotle (1996), 992a 20.
[2] Furley (1967), p. 108 f.

odds with Plato's assertion that earth is irreducible to anything else. From this it would seem to follow that Plato's theory involves the existence of indivisible triangles.

What sort of indivisibility would these triangles have? If they are material, then their indivisibility would be no more than physical. On the other hand, if they are not material, but some kind of geometric abstraction (as some scholars have claimed), then their indivisibility would perforce be of a theoretical order. This question, and indeed the whole issue of Plato's "atomism", is still unresolved.

ARISTOTLE

It was **Aristotle** (384–322 B.C.) who first undertook the systematic analysis of continuity and discreteness. A thoroughgoing synechist, he maintained that physical reality is a continuous plenum, and that the structure of a continuum, common to space, time and motion, is not reducible to anything else. His answer to the Eleatic problem is a refinement of that of Anaxagoras, namely, that continuous magnitudes are potentially divisible to infinity, in the sense that they may be divided *anywhere*, though they cannot be divided *everywhere* at the same time.

Aristotle identifies continuity and discreteness as attributes applying to the category of Quantity[1]. As examples of continuous quantities, or *continua*, he offers lines, planes, solids (i.e., solid bodies), extensions, movement, time and space; among discrete quantities he includes number[2] and speech[3]. He also lays down definitions of a number of terms, including continuity:

> *Things are said to be "*together" *in place when the immediate and proper place of each is identical with that of the other and* "apart" (or *"severed"*) when this is not so . *They* are "in contact" *when their extremities are in this sense "together". One thing is "*in (immediate) succession*" to another if it comes after the point you start from in an order determined by position, of "form", or whatsoever it may be, and if there is nothing of its own kind between it and that to which it is said to be in immediate succession … "*Contiguous" *means in immediate succession and in contact. Lastly, the "*continuous*" is a subdivision of the contiguous; for I mean by one thing being continuous with another that those extremities of the two things in virtue of which they are in contact with each other become one and the same thing and (as the very name indicates) are "held together", which can only be if the two limits do not remain two but become one and the same. From this definition it is evident that continuity is possible in the case of such things as can, in virtue of their natural constitution, become one by coming into contact; and the whole will have the same sort of union as that which holds it together, e.g. by rivet or glue or contact or organic union.* [4]

In effect, Aristotle here defines continuity as a *relation* between entities rather than as an *attribute* appertaining to a single entity; that is to say, he does not provide an explicit definition of the concept of *continuum.* At the end of this passage he indicates that a single continuous whole can be brought into existence by "gluing together" two things which have been brought into contact, which suggests that the continuity of a whole should derive from the way its *parts* "join up". That this is indeed the case is

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[1] In Book VI of the *Categories.* Quantity (ποσον) is introduced by Aristotle as the category associated with *how much.* In addition to exhibiting continuity and discreteness, quantities are, according to Aristotle, distinguished by the feature of being *equal* or *unequal.*

[2] Here it must be noted that for Aristotle, as for ancient Greek thinkers generally, the term "number" – *arithmos* – means just "plurality".

[3] Aristotle points out that (spoken) words are analyzable into syllables or phonemes, linguistic "atoms" themselves irreducible to simpler linguistic elements.

[4] Aristotle (1980), V, 3.

revealed by turning to the account of the difference between continuous and discrete quantities offered in the *Categories:*

> *Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and space. For the parts of a number have no common boundary at which they join together. For example, ten consists of two fives, however these do not join together at any common boundary but are separate; nor do the constituent parts three and seven join together ay any common boundary. Nor could you ever in the case of number find a common boundary of its parts, but they are always separate. Hence number is one of the discrete quantities… . A line, on the other hand, is a continuous quantity. For it is possible to find a common boundary at which its parts join together—a point. And for a surface—a line; for the parts of a plane join together at some common boundary. Similarly in the case of a body one would find a common boundary—a line or a surface—at which the parts of the body join together. Time also and space are of this kind. For present time joins on to both past time and future time. Space again is one of the continuous quantities. For the parts of a body occupy some space, and they join together at a common boundary. So the parts of the space occupied by various parts of the body themselves join together at the same boundary as the parts of the body do. Thus space is also a continuous quantity, since its parts join together at one common boundary.*[1]

Accordingly for Aristotle quantities such as lines and planes, space and time are continuous by virtue of the fact that their constituent parts "join together at some common boundary". By contrast *no* constituent parts of a discrete quantity can possess a common boundary.

Let us attempt to turn Aristotle's notions into precise mathematical definitions. Suppose that we are given "quantities" $A, B, C,\ldots, U, V, X, Y, Z$. We suppose also that we have an inclusion relation $\subseteq$ between quantities: thus $U \subseteq A$ is understood to mean that U is included in A, or that U is a *subquantity* (or part) of A. We assume that for any quantity A there is a void subquantity $\varnothing$ with the property that $\varnothing \subseteq U$ for all subquantities U of A. Given subquantities U, V of a quantity A, we suppose that there are subquantities $U \cup V$, $U \cap V$ of A—the *join* and *meet*, respectively, of U and V, with the property that, for any subquantity X of A, $U \cup V \subseteq X$ if and only if $U \subseteq X$ and $V \subseteq X$, and $X \subseteq U \cap V$ if and only if $X \subseteq U$ and $X \subseteq V$. So $U \cup V$ is the "least" subquantity including both U and V, and $U \cap V$ is the "greatest" subquantity included in both U and V, or the *boundary* of U and V.

Let us call a quantity A *discrete* if, corresponding to any $U \subseteq A$, there is $V \subseteq A$ for which $U \cup V = A$ and $U \cap V = \varnothing$. U and V are then "constituent parts of A without a common boundary". By contrast we call a quantity A *continuous,* or a(n) (Aristotelian) *continuum*, provided that, whenever $U, V \subseteq A$ are such that $U \neq \varnothing$ and $V \neq \varnothing$ and $U \cup V = A$, then $U \cap V \neq \varnothing$. That is, any pair of "constituent parts" of A, they must have a nonvoid "common boundary". A continuum may also be characterized by the somewhat stronger property of *indecomposability*, namely, if $U \cup V = A$ and $U \cap V = \varnothing$, then $U = \varnothing$ or $V = \varnothing$. Indecomposability expresses the idea that a continuum "hangs together" or "coheres": a continuum cannot be "split" into nonvoid constituent parts without a common boundary. In other words, unlike a discrete entity*, a continuum is not composed of its parts.* [2]

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[1] Aristotle (1996a)*, Categories,* VI.

[2] We note that if quantities were to be identified with sets or classes in the sense of modern set theory then the classical law of excluded middle would imply that only the void set and singletons qualify as continua. It follows that, if quantities such as space and time are to be treated set-theoretically and yet remain continua

One of the central theses Aristotle is at pains to defend in *Physics VI* is the irreducibility of a continuum to discreteness—that a continuum cannot be "composed" of indivisibles or atoms, parts which cannot themselves be further divided. He begins his reasoning as follows:

> *Now if the terms "continuous", "in contact", and "in immediate succession" are understood as defined above—things being "continuous" if their extremities are one, "in contact" if their extremities are together, and "in succession" if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points, the line being continuous and the point indivisible. For two points cannot have identical extremities, since in an indivisible there can be no extremity as distinct from some other part; and (for the same reason) neither can the extremities be together, for a thing which has no parts can have no extremity, the extremity and the thing of which it is the extremity being distinct. Yet the points would have to be either* continuous *or* contiguous *if they were to compose a continuum. And the same reasoning applies in the case of any indivisible. As to the impossibility of their being continuous, the proof just given will suffice; and one thing can be contiguous with another only if whole is in contact with whole or part with part or part with whole. But since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they cannot compose a continuum, for a continuum is divisible into parts which are distinguishable from each other in the sense of being in different places.*[1]

In this last sentence Aristotle appears to be arguing that a number of indivisibles wholly in contact with one another would constitute another indivisible, and not a continuum, since a continuum is always divisible.

Having disposed of the possibility that a continuum could be made up of indivisibles either continuous or in contact with one another, Aristotle next turns to the question of whether a continuum such as length or time could be composed of indivisibles in succession. Once more he answers in the negative:

> *Again, one point, so far from being continuous or in contact with another point, cannot even be* in immediate succession *to it, or one "now" to another "now", in such a way as to make up a length or a space of time; for things are "in succession" if there is nothing of their own kind intermediate between them, whereas two points have always a line (divisible at intermediate points) between them, and two "nows" a period of time (divisible at intermediate "nows"). Moreover, if a continuum such as length or time could thus be composed of indivisibles, it could itself be resolved into its indivisible constituents. But, as we have seen, no continuum can be resolved into elements which have no parts.*

While it has been shown that continua such as length and time cannot be composed of successive points or instants with *nothing of the same kind between them*, there remains the possibility that there might lie between them something of a *different* kind, e.g. stretches of emptiness or "void" such as certain Pythagoreans supposed to separate the distinct points composing a line. (Aristotle's arguments against the existence of void in general are presented in Book IV of the *Physics*.) Against this sort of picture Aristotle argues:

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in an Aristotelian sense, the law of excluded middle must be abandoned, in short, we must use *intuitionistic* rather than classical logic. See Chapter 10 below.

[1] Aristotle (1980) VI, 1.

Nor can there be anything of any other kind between the points or between the moments: for if there could be any such thing it is clear that it must be indivisible or divisible, and if divisible, it must be divisible either into indivisibles or divisibles that are infinitely divisible, in which case it is a continuum. Moreover, it is plain that every continuum is divisible into parts that that are divisible without limit: for if it were divisible into indivisibles, we should have one indivisible in contact with another, since the extremities of things that are continuous with one another are one and are in contact.[1]

This somewhat cryptic argument deserves elucidation. Let us suppose with the Pythagoreans that a continuous line is composed of successive points separated by stretches of "void". Since any such void stretch, as a part of a continuum, cannot be indivisible, it is accordingly divisible either (*a*) into indivisible parts or (*b*) infinitely. Alternative (*a*) can be dismissed, for, if there were indivisible parts, in order to make up a continuum they would have to be in contact with another, which we have already seen to be impossible. This leaves alternative (*b*), but in that case our stretch of void is itself a (linear) continuum, and so the alleged "successive" points separated by it are not in fact successive, for there is now a continuum a line stretching between them which is itself divisible at intermediate points of the same kind.

Aristotle sometimes recognizes *infinite divisibility*—the property of being divisible into parts which can themselves be further divided, the process never terminating in an indivisible—as a consequence of continuity as he characterizes the notion in Book V. But on occasion he takes the property of infinite divisibility as *defining* continuity. It is this definition of continuity that figures in Aristotle's demonstration of what has come to be known as the *isomorphism thesis*, namely:

The same argument applies to magnitude, time and motion: either they are all composed of indivisible things and divided into indivisible things, or none of them is.[2]

Briefly, the isomorphism thesis asserts that either magnitude, time and motion are all continuous, or they are all discrete. Aristotle's demonstration of this thesis rests on two key postulates concerning motion:

1. When motion is taking place, something is moving from here and *vice-versa.*
2. A moving object cannot simultaneously be in the act of moving towards a given point and in the state of being already at it.

Here in a nutshell is Aristotle's argument : given components *L, M, N…* of a motion, and assuming that each of these is itself a motion, then, by postulate 2, after *L* has started and before it has finished, the moving object *P* is, by postulate 1, past the start and short of the finish of the corresponding distance *A.* It follows that *A* is divisible in correspondence with *L,* and so likewise are the distances *B, C, …* corresponding to *M, N, ….* . Aristotle also shows how the assumption that the distances *A, B, C, …* are indivisible leads to what he saw as absurdities concerning the motion. For if *L,* say, is a motion, then *P* would be moving over *A*, but since *A* lacks parts, *P's* movement over *A* leaves no "trace": in traversing *A* it "jumps" instantaneously from a state of rest to a state of rest. On the other hand, if *L* is not a motion then *P* would never be in motion but would accomplish the motion without moving. Accordingly both distance and motion must be divisible. As for time, Aristotle argues that it is divisible if both distance and motion are, and *vice-versa.* For, he says, if the whole of the length *A* is traversed in time *T,* a part of it would be traversed (at equal speed) in less than *T.* On the other

[1] *Ibid.*
[2] *Ibid.*

hand, if the whole time *T* were occupied in traversing *A,* then in part of the time a part of *A* would be covered.

While Aristotle held that magnitude, motion and time possessed a common (continuous) form, he had doubts as to whether *time* was an existent in the same sense as the first two. On this question we read in *Physics* Book IV, 10:

> *The following considerations might make one suspect that there is really no such thing as time, or at least that it has only an equivocal and obscure existence.*
>
> *(1) Some of it is past and no longer exists, and the rest is future and does not yet exist; and time, whether limitless or any given length of time we take, is entirely made up of the no-longer and not-yet; and how can we conceive of that which is composed of non-existents sharing in existence in any way?*
>
> *(2) Moreover, if anything divisible exists, then, so long as it is in existence, either all its parts or some of them must exist. Now time is divisible into parts, and some of these were in the past and some will be in the future, but none of them exists. The present "now" is not part of time at all, for a part measures the whole, and the whole must be made up of the parts, but we cannot say that time is made up of "nows".*

Aristotle thus questions the existence of time on the grounds that none of its parts can be said to exist: the past no longer exists, the future does not yet exist, and the present, while it may exist, is a sizeless instant and so cannot be considered a *part* of time. These perplexities concerning the nature of time continued to puzzle Aristotle's successors. In *Physics* Book IV, 11, Aristotle had defined time as "the number of motion in respect of 'before' and 'after' "—a definition to which his pupil Strato later objected, not unreasonably, on the grounds that the use of the term "number", as a discrete quantity, is inappropriate in connection with time, which is continuous.

. The question of whether magnitude is perpetually divisible into smaller units, or divisible only down to some atomic magnitude leads to the *dilemma of divisibility,*[1] a difficulty that Aristotle necessarily had to face in connection with his analysis of the continuum. In the dilemma's first, or *nihilistic* horn, it is argued as above that, were magnitude everywhere divisible, the process of carrying out this division completely would reduce a magnitude to extensionless points, or perhaps even to nothingness. The second, or *atomistic,* horn starts from the assumption that magnitude is not everywhere divisible and leads to the equally unpalatable conclusion (for Aristotle, at least) that indivisible magnitudes must exist.

Aristotle mounts his main attack on the atomistic horn of the dilemma in Book VI of the *Physics*, where, as we have already seen, he repudiates the idea that a continuum can be composed of indivisibles. His refutation of the dilemma's nihilistic horn, presented in Book I of *On Generation and Corruption,* rests on to two ideas: that the conception that it is the nature of a continuum to exist prior to its parts, and that a point is nothing more than a cut or division in a line, as the beginning or end—the limit—of a line segment. Precisely because points exist only as divisions or limits Aristotle denies them substantial reality; they are mere "accidents" arising from operations performed on substances or magnitudes. Points exist for Aristotle essentially in a *potential* mode, as marking out possible divisions in magnitudes. When a moving body moves continuously along a continuous path, he avers, the points over which it moves have no more than a potential existence, and are only actualized by the body coning to a halt and starting to move again[2]. Similarly, a point in a straight line is

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[1] Miller (1982).

[2] This is a crucial point for Aristotle in his refutation of Zeno's dichotomy paradox, since Aristotle concedes to Zeno only that in order to reach a goal a moving body must pass through a *potential* infinity of half-distances. If the body were to traverse an actual infinity of such distances, it would have to make an infinite number of

brought into existence only by dissecting the line. Aristotle refutes the dilemma's nihilistic horn by showing that even though unlimited division of a magnitude is possible and a point exists everywhere *potentially*, it does not follow that magnitude reduces to points. A magnitude can be "divided throughout" only by a process in which a subsection is divided into further subsections. There is never a stage at which the division is completed and the line is reduced to unextended constituents. For an actually existing point necessarily presupposes the existence of extended magnitudes which have been divided: until the division has actually been performed the point has no more than potential existence. Accordingly the division must be successive rather than simultaneous, and it occurs "at every point" not in the sense of actually existing points but in the sense of points which could mark further subdivisions.

In Book VI of the *Physics* Aristotle brings some of these ideas to bear in his refutation of Zeno's paradoxes. The first of these paradoxes, as reported by Aristotle, is the *Dichotomy*, in which the possibility of motion is denied

> *because, however near the moving object is to any given point, it will always have to cover the half, and then the half of that, and so on without limit until it gets there.*[1]

That is,

> *It is impossible for a thing to traverse or severally come into contact with illimitable things in a limited time.*

In repudiating this Aristotle argues that

> *there are two senses in which a distance or a period of time (or indeed any continuum) may be regarded as illimitable, viz., in respect of its divisibility or in respect of its extension. Now it is not possible to come into contact with quantitatively illimitable things in a limited time, but it is possible to traverse what is illimitable in its divisibility: for in this respect time itself is also illimitable. Accordingly, a distance which is (in this sense) illimitable is traversed in a time which is (in this sense) not limited but illimitable; and the contacts with the illimitable (points) are made at "nows" which are not limited but illimitable in number.*[2]

Here Aristotle traces the paradox as arising from Zeno's tacit assumption that in the course of the motion the number of contacts to be established accord in the case of the distance with its unlimited *divisibility* but in the case of the time with its limited *extension.* In reality, however, these contacts accord with unlimited divisibility in both cases. A definite distance and a definite period of time are (by the isomorphism thesis) divisible in exactly the same way: the distance at points into shorter distances, the time at nows into shorter periods. A point marks off a stage in the journey, and the corresponding now marks off the time taken to accomplish that stage. Neither the points nor the nows are limitable; but they both exist only in a potential sense.

Aristotle takes the third paradox, that of the *Arrow*, as asserting the impossibility for a thing to be moving during a period of time, because it is impossible for it to be moving at an indivisible instant. This is summarily dismissed on the grounds that time, as a continuum, cannot be composed of indivisible instants. Some modern scholars[3] claim that Aristotle has misunderstood the core of Zeno's argument (as plausibly reconstructed). In essence Aristotle takes the paradox as resting on the assumption that time is not infinitely divisible; but in fact Zeno's argument requires no assumption

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stops and starts. In other words, only an impossibly *discontinuous* motion (in Aristotle's sense) would convert this potential infinity into an actual one.

[1] Aristotle (1980) VI, 9.

[2] *Ibid.*, 2.

[3] E.g. Barnes (1986); Kirk, Raven and Schofield (1983).

concerning the structure of time (or space). All he requires for the validity of his inference is that what is true of something (in this case, to be at rest) *at every moment* of a period of time (whether or not moments are indivisible instants) is true of it *throughout* the period.

Aristotle considered (uninterrupted) *movement*, or, more generally, *change*, to be a prime example of the continuous. In Book V of the *Physics* he answers the question, "what constitutes the unity of a movement?", by asserting:

> *Not its indivisibility (for every movement is potentially divisible without limit), but its uninterrupted continuity. Thus if a movement is strictly one, it must be continuous, and if continuous, one. … For a movement to possess absolute unity and continuity (a) the movement must be specifically the same throughout the course, (b) the mobile must retain its numerical identity, and (c) the time occupied must [itself] be "one".*[1]

In the final section of Book VI he argues that no indivisible object can undergo movement or change in this unified sense, concluding:

> *It follows that a thing without parts cannot move, or indeed change at all. The only way in which it* could *move is if time were composed of "nows"; for in any "now" it would* have moved *or* have changed*, and so it would never move but always have moved. But the impossibility of this has already been demonstrated; time is not composed of "nows", nor a line of points, nor motion of jerks—for anyone who asserts that a partless thing can move is in fact saying that motion is composed of partless units, as if time were composed of "nows" or length were composed of points*.[2]

Another way of putting this is that an indivisible can move only in instantaneous "jumps" or "jerks".

As regards space, time, motion and extension, Aristotle was a thoroughgoing divisionist. But with regard to matter, or, at least, organic matter, his divisionism took a qualified form. This emerges in Book I , Ch. 4 of the *Physics* where, in criticizing Anaxagoras's theory of mixtures, with its arbitrarily small "seeds" of matter, he puts forward his view as why the "natural parts" of a thing must have determinate size:

> *Flesh, bone and the like are the parts of animals It is clear then that neither flesh, bone, nor any such thing can be great or small without limit.*[3]

He goes on to present his objections to Anaxagoras' assertion that everything contains all possible kinds of seed. Here as an example he takes water, in which according to Anaxagoras the seeds of flesh must be present:

> *For if flesh has been extracted from a given body of water and then more flesh is sifted from the remainder by repeating the process of separation: then, even if the successive extracts will continually decrease in quantity, still they cannot fall below a certain magnitude.* [4]

These quotations make it clear[5] that Aristotle does not admit the infinite divisibility of matter (or at least of organic matter) in an actual, physical sense. In a word, matter must, according to Aristotle, have *natural minima.* Aristotle does not develop this theory to any extent, but it assumed a more explicit form at the hands of later commentators such as Simplicius, Alexander of Aphrodosias (c. 200 A.D.), Themistius (4th cent. A.D.)

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[1] Aristotle (1980) V, 4.
[2] *Ibid.* VI, 10.
[3] *Ibid.* I, 4; 187b 18-21.
[4] This having been established above.
[5] Van Melsen (1952), p. 42.

and Philoponus (6th cent. A.D.) [1]. The *doctrine of natural minima* that emerged was based on the following theses [2]:

1. *Qua* mathematical extension, quantity is (potentially) infinitely divisible; physically, it is not.
2. Each type of substance has its *natural minimum*, beyond which it cannot be further divided.

This doctrine, which came to exert a considerable influence on the scholars of the middle ages, bears a superficial resemblance to atomism, but is in fact quite distinct.[3] While natural minima certainly are, and atoms may be, mathematically divisible, natural minima of a given type may, unlike atoms, be physically divisible into other substances. Moreover, unlike atoms, minima do not exist in an objective, independent sense, they are only *potential* parts of substances.

EPICURUS

As a thoroughgoing materialist, **Epicurus** (341–271 B.C.) could not accept the notion of potentiality on which Aristotle's theory of continuity rested, and so was propelled towards atomism in both its conceptual and physical senses. According to Simplicius,

> *Aristotle often refuted the doctrine of Democritus and Leucippus; because of these refutations, perhaps, as they were directed at the concept of the "partless", Epicurus, a later adherent of the doctrine of Democritus and Leucippus about the primary bodies, retained their imperviousness but dropped their partlessness, since they had been refuted on this ground by Aristotle.*[4]

Like Leucippus and Democritus, Epicurus felt it necessary to postulate the existence of physical atoms, but to avoid Aristotle's strictures he proposed that these should not be themselves conceptually indivisible, but should *contain* conceptually indivisible parts. Aristotle had shown that a continuous magnitude could not be composed of *points,* that is, indivisible units lacking extension, but he had not shown that an indivisible unit must necessarily lack extension. Epicurus met Aristotle's argument that a continuum could not be composed of such indivisibles by taking indivisibles to be partless units of magnitude possessing extension.

Epicurus's *Letter to Herodotus* contains a summary of his natural philosophy, and more particularly of his atomism. According to Epicurus the ultimate contents of the universe are bodies and space, or "void"; these are themselves irreducible and everything can be reduced to them. These ultimate bodies are

> *physically indivisible and unchangeable, if all things are not to be destroyed into non-being but are to remain durable in the dissolution of compounds—solid by nature, unable to be dissolved anywhere or anyhow. It follows that the first principles must be physically indivisible bodies.*[5]

In other words, real things cannot be "destroyed into non-being"; but unless there were a limit to physical divisibility this is what would happen; accordingly there is a limit to

[1] *Ibid.,* p. 47.
[2] Pyle (1997), pp. 216-217.
[3] *Ibid.,* p. 217.
[4] Quoted in Furley (1967) p. 111.
[5] Quoted *ibid.*, p. 7.

physical divisibility[1]. Two millenia after Epicurus, the English philosopher Samuel Clarke, in his correspondence with Leibniz, put Epicurus' argument for the existence of atoms in the following way:

> *If there be no perfectly solid atoms, then there is no matter at all in the universe. For, the further the division and the subdivision of the parts of any body is carried, before you arrive at parts perfectly solid and without pores; the greater is the proportion of the pores to solid matter in that body. If, therefore, carrying on the division* in infinitum, *you never arrive at parts perfectly solid and without pores; it will follow that all bodies consist of pores only, without any matter at all: which is a manifest absurdity.*[2]

The existence of atoms having been demonstrated, Epicurus goes on to investigate their properties. In *Two Studies in the Greek Atomists,* David Furley provides the following paraphrase of Epicurus' analysis:

> *(A) In a finite body such as the atom, there cannot be an infinite number of parts with magnitude, however small they may be.*
> *This implies:*
> *(B1) the body cannot be divided into smaller and smaller parts to infinity (if we admitted this, we would put the whole world upon an insecure foundation; when we tried to get a firm mental grasp on the atoms we should find it impossible, because our mental picture of them would crumble away until nothing was left); and*
> *(B2) the process of traversing in the imagination from one side to the other of a finite body cannot consist of an infinite number of steps, not even with progressively diminishing steps.*
> *We establish (A) on the following grounds:*
> *(C1) If someone asserts that there is an infinite number of parts with magnitude in an object, then that object must itself be infinite in magnitude; this is true however small the parts may be. Moreover:*
> *(C2) In the process of traversing an object in the imagination, one begins with the outermost distinguishable portion, and moves to the next; but this next must be similar to the first; hence it must be possible, in the view of one who asserts that there is an infinite number of such parts, to reach infinity in thought, when that object is totally grasped by the mind.*
> *We establish (C2) by the following analogy:*
> *(D1) The minimum* perceptible *quantity is like larger perceptible quantities except that no parts can be perceived in it. (D2) The fact that it is like larger perceptible quantities, in which parts* can *be perceived, may suggest that we can distinguish one part from another in the minimum, too. But this is false. If we perceive a second quantity, it must be at least equal to the first, since the first was a minimum. (D3) We measure perceptible objects by studying these minima in succession, beginning from the first. They do not touch each other part to part (since they have no parts), nor do they coincide in one and the same place. They are arranged in succession, and they form the units of perceptible magnitude; more of them form a larger magnitude, fewer of them a smaller magnitude.*
> *(E) Similarly with the minimum in the atom—though it is much smaller than the perceptible minimum. The similarity is to be expected, since we have already argued that the atom has magnitude by an analogy with perceptible things, thus in effect projecting the atom on the larger scale of perceptible things.*
> *(F) Furthermore, these minimal, partless extremities furnish the primary, irreducible unit of measurement, in terms of which we "see" the magnitude of atoms of different*

[1] *Ibid.,* p. 8.
[2] Alexander (1956), p. 54.

size when we study them in thought. So much can be inferred from the analogy with perceptible things (D3); but the latter of course are liable to change, and we must not be led by our analogy to think that atoms too are liable to change, in the sense of being put together out of separable parts. [1]

Epicurus was, as it were, faced with a choice between infinite divisibility and minimal parts. He must have seen that the former alternative would lead to positions inconsistent with experience: for instance, it would be necessary to be able to "reach infinity in thought". Aristotle, who also rejected actual infinity, had resolved the problem of infinite divisibility by introducing the subtle and somewhat elusive concept of potential infinity, and so contrived to avoid the postulation of minimal parts. But Epicurus, as a thorough-going materialist, rejected the idea of a potentiality which could never be actualized[2], a Becoming never brought to full Being. He would have regarded it as contradictory to assert that a finite magnitude is potentially divisible to infinity, and yet to deny that it consists of an infinity of parts: an entity just consists of those entities into which it is divided, whether potentially or actually. For him the potential had to be treated as if it was actual. This left him no option but to postulate of minimal units of magnitude.

Epicurus's physical atoms were *materially*, but not *conceptually* indivisible[3]. But while the atoms of Democritus *may* have been conceptually divisible *ad infinitum,* those of Epicurus have a finite number of minimal parts which *are* conceptually indivisible. Minimal parts of atoms may be considered as constituting the ultimate units of magnitude (the existence of which Aristotle, as a divisionist, had explicitly denied). This left Epicurus' theory vulnerable to another objection raised by Aristotle against atomism:

For if they [atoms] are all of one substance...why do not they become one when they come into contact, just as water does when it becomes water? [4]

In the Epicurean theory atoms are conceived as being composed of a finite number of minimal parts in contact. However, when two atoms come into contact they must, to avoid Aristotle's objection, remain distinct, and accordingly physically separable. But only finitely many minimal parts, perhaps just two, of the respective atoms touch. If two minimal parts can be physically separated, so can any finite number. Since the atom is composed of finitely many minimal parts, it is separable. To avoid this difficulty it has been speculated[5] that the Epicureans adopted the view that the minimal parts of an atom are essentially constituents of *that* atom, and have no separate existence outside it. Thus minimal parts of two *different* atoms coming into contact are separable, but from this it no longer follows that minimal parts of the *same* atom are separable[6].

On one point, however, the Epicurean theory is clear. The properties of atoms are reducible to the numbers and arrangements of ultimate units, and physics is thereby reduced to combinatorics. What, then, of geometry? As far as is known, Epicurus made no attempt to work out the consequences for geometry of his atomistic doctrine of

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[1] Furley (1967), pp. 8–10.

[2] Epicurus' position in this respect is similar to that of Cantor (see Chapter 4 below). Both can be said to have accepted the thesis that any potential infinity presupposes an actual infinity. But the consequences of this acceptance were quite different for the two. Epicurus, a finitist who repudiated actual infinity, was led necessarily to reject the potential infinite as well. But Cantor's whole world view reposed on the actual infinite, so for him the thesis served not to demonstrate the non-existence of the potential infinite, but rather to reveal it as a shadow cast by the substantial reality of the actual infinite.

[3] For a penetrating discussion of the problem of material indivisibility, see Pyle (1997), Appendix 1.

[4] Aristotle (2000a), *On Coming-to-Be and Passing Away.,* I, 8., 326a 32-33.

[5] Pyle (1997), Appendix 1.

[6] In this respect Epicurean minimal parts may be said to resemble the quarks that are currently presumed to be the ultimate building-blocks of matter: just as Epicurean minimal parts have no separate existence, quarks appear only in groups of two, three, or five.

magnitudes. Such an “Epicurean geometry”, with its ultimate units of magnitude, would have to meet the challenge of the existence of incommensurable magnitudes (e.g. the side and diagonal of a square), and resolve the seeming absurdity that, if a square is built up from miniature tiles as “units”, there are as many tiles along the diagonal as there are along the side, and so the diagonal should be equal in length to the side.[1] Furley suggests that it might be expected that Epicurus would regard geometry as irrelevant to the study of nature, since one of its basic principles—infinite divisibility—is contrary to the facts of nature. Some evidence for this surmise is provided by Proclus, in whose *Commentary on Euclid* the Epicureans are identified as “those who criticize the principles of geometry alone”.

Furley illustrates Epicurus’ theory through the analogy of a drawing made on a piece of graph paper by shading some squares of the grid and leaving others blank: the shaded squares then represent units of matter, the unshaded ones units of “void”. The squares are all considered as wholes, so there is no place for part of a square, or the diagonal of a square. If they are arranged in rows, the right edge of one is in contact with the left edge of the next. But the edges are not “parts” of the squares in the sense that one might fill in the edge of a void square while leaving the rest blank: the squares are indivisible wholes. The Epicurean atom was, according to Furley, supposed to exist within a three-dimensional grid of this kind. It is not necessary for the cells of the grid to be all of the same shape or size.

But why did Epicurus identify his minimum units of extension with *parts* of atoms, rather than with the atoms themselves? Furley’s conjecture is that it was a response to Aristotle’s analysis of motion, which had established that, if indivisible magnitudes actually exist, then the distance traversed by a moving body must be composed of indivisible minimal units. This made it necessary for Epicurus to consider, in addition to the moving atoms themselves, the places they successively occupied. It would then become clear that the units must all be equal, for otherwise absurd consequences would follow, such as that a (geometrically) indivisible space was too large or too small for a (geometrically) indivisible atom to fit into it. But from the equality of the minimal units, it would have to follow that either all atoms are identical in size, or else some atoms occupy more than one unit of spatial extension. Epicurus, Furley surmises, would have rejected the first alternative as not squaring with phenomena, and would accordingly have adopted the second.

THE STOICS AND OTHERS

In opposition to the atomists, the Stoic philosophers **Zeno of Cition** (fl. 250 B.C.) and **Chrysippus** (280–206 B.C.) upheld the Aristotelian position that space, time, matter and motion are all continuous. And, like Aristotle, they explicitly rejected any possible existence of void within the cosmos. The cosmos is pervaded by a continuous invisible substance which they called *pneuma* (Greek: “breath”). This pneuma—which was regarded as a kind of synthesis of air and fire, two of the four basic elements, the others being earth and water—was conceived as being an elastic medium through which impulses are transmitted by wave motion. All physical occurrences were viewed as being linked through tensile forces in the pneuma, and matter itself was held to derive its qualities form the “binding” properties of the pneuma it contains.

A major difficulty encountered by the Stoic philosophers was that of the nature of *mixture*, and, in particular, the problem of explaining how the pneuma mixes with material substances so as to “bind them together”. The atomists, with their granular conception of matter, did not encounter any difficulty here, since they could regard the mixture of two substances as an amalgam of their constituent atoms into a kind of

[1] These difficulties are similar to those encountered by the Islamic atomists in the 9th and 10th centuries: see below.

lattice or mosaic. But the Stoics, who regarded matter as continuous, had difficulty with the notion of mixture. For in order to mix fully two continuous substances, they would either have to interpenetrate in some mysterious way, or, failing that, these would each have to be subjected to an infinite division into infinitesimally small elements which would then have to be arranged, like finite atoms, into some kind of discrete pattern. The mixing of particles of finite size, no matter how small they may be, presents no difficulties. But this is no longer the case when we are dealing with a continuum, whose parts can be divided *ad infinitum*. Thus the Stoic philosophers were confronted with what was at bottom a *mathematical* problem.

Plutarch reports an attempted resolution of the cone dilemma by Chrysippus:

> *[Chrysippus] says that the surfaces will neither be equal nor unequal; the bodies, however, will be unequal, since their surfaces are neither equal nor unequal.* [1]

Sambursky[2] thinks that the first part of this quotation refers to the process of convergence to the limit; he also regards Chrysippus as having been the first to get a clear grasp of this idea. Sambursky suggests that, if we consider the infinite sequence of sections of the cone approaching the given one, "we have to discard the static concept of equal and unequal, taking into account that for each given difference in surfaces one can determine a distance which will yield a still smaller difference." It is Sambursky's contention that this is what Chrysippus intended to express by the phrase "neither equal nor unequal". Sambursky also contends that, provided one interprets "body" as the solid contained between parallel sections of the cone, the second part of the quotation is intended by Chrysippus to mean that, of the surfaces of three adjacent sections A_1, A_2, A_3, the volume defined by the surfaces A_1 and A_2 is unequal to that defined by A_2 and A_3, despite the fact that both $A_3 - A_2$, and $A_2 - A_1$ both converge to zero. Sambursky points out that, while it is most unlikely that Chrysippus formulated a rigorous proof of this proposition, it is necessary to ensure that in the limit process the volumes of adjacent sections do not become equal, which would lead to a cylinder instead of a cone and thus restore Democritus' dilemma.

Michael White[3] interprets Chrysippus as meaning that the two adjacent surfaces cannot be exactly equal, yet there is no discriminable quantity by which one exceeds the other. White suggests that the indiscriminably small difference between the surfaces may be represented as infinitesimals within Robinson's nonstandard analysis.[4]

Chrysippus also considered the problem raised by Aristotle concerning the reality of time. Aristotle had pointed out that only the "now" actually existed, but as a mere boundary between past and future, a mathematical point, it cannot be counted a part of time. As Sambursky[5] points out, the reduction of the "now" to a mathematical point could have been avoided by postulating the existence of indivisible temporal atoms. While atomists such as Xenocrates and Epicurus would have regarded this move as unexceptionable, Chrysippus and his fellow-Stoics, as synechists, had no choice but to reject it. Instead they suggested what Sambursky calls a "dynamic solution" to the problem: as reported by Plutarch,

> *The Stoics do not admit the existence of a shortest element of time, nor do they concede that the "now" is indivisible, but that which someone might assume and think of as present is according to them partly future and partly past. Thus nothing*

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[1] Quoted in Sambursky (1971), p. 93.
[2] *Ibid.,* p. 94 f.
[3] White (1992), Ch. 7.
[4] Another possibility is to formulate the problem within the framework of smooth infinitesimal analysis, where intuitionistic logic prevents infinitesimals from being in general equal or unequal to zero. See Chapter 10 below.
[5] Sambursky (1963), p. 151.

> *remains of the Now, nor is there left any part of the present, but what is said to exist now is partly spread over the future and partly over the past.* [1]

Plutarch also quotes Chrysippus's view on time:

> *He states most clearly that no time is entirely present. For the division of continua goes on indefinitely, and by this distinction time, too, is infinitely divisible; thus no time is strictly present but is defined only loosely.*[2]

Sambursky interprets this "looseness" of definition of the present as "the result of a limiting process of convergence consisting in an infinite approach to the mathematical Now both from the direction of the past and from the future". So, according to his account, "the present is given by an infinite sequence of nested time intervals shrinking towards the mathematical "now", and it is therefore to be regarded as a duration of only indistinctly defined boundaries whose fringes cover the immediate past and future." [3]

In his celebrated work *De Rerum Natura,* Epicurus's Roman disciple **Lucretius** (*c.*100 – 55 B.C.) offers a systematic exposition of the former's materialist atomism, arguing against the views of various divisionists, including Anaxagoras and the Stoics. Lucretius formulates what appears to be a new argument for the existence of minimal parts:

> *If there are no such least parts, even the smallest bodies will consist of an infinite number of parts, since they can be halved and their halves halved again without limit. On this showing, what difference will there be between the whole universe and the very least of things? None at all. For, however endlessly infinite the universe may be, yet the smallest things will equally consist of an infinite number of parts. Since true reason cries out against this and denies that the mind can believe it, you must needs give in and admit that there are least parts which themselves are partless.*[4]

In appealing to "true reason" Lucretius would appear to be invoking the Euclidean axiom that the whole is always greater than the part. And indeed, divisionism does seem to lead to a violation of that hallowed principle. For, under the hypothesis of infinite divisibility, complete division into parts of, say, a line, and its half, would in both cases yield infinities manifesting no "difference". Aristotle would have striven to avoid this unpalatable conclusion by taking refuge in potentiality and denying that complete division of a continuum could actually be carried out. But, as has already been observed, the materialist Epicurus rejected the notion of an unrealizable potentiality, and Lucretius followed suit. For them the complete division of a continuum must terminate after finitely many steps (as we would say), yielding minimal parts.

The neoplatonist philosopher **Damascius** (c. 462 – 540) was exercised by Aristotle's conundrum concerning the unreality of time. Arguing that the present is more than a mere instant, indeed has an extension, and so can be regarded as a *part* of time, he concluded that one of the parts of time does actually exist. Of the views of Damascius and the neoplatonist school on this question Simplicius reports:

> *I am impressed by how they solve Zeno's problem by saying that the movement is not completed with an indivisible bit, but rather progresses in a whole stride at*

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[1] Quoted in Sambursky (1963), p. 151.
[2] *Ibid.*
[3] *Ibid.* , pp. 151–2. Smooth infinitesimal analysis suggests another way of interpreting the Stoic conception of time.
[4] Lucretius (1955), p. 45

> *once. The half does not always precede the whole, but sometimes the movement as it were leaps over both whole and part. But those who said that only an* indivisible *now existed did not recognize the same thing happening in the case of time. For time always accompanies movement and as it were runs along with it, so that it strides along together with it in a whole continuous jump and does not progress one now at a time ad infinitum. This must be the case because motion obviously occurs in things, and because Aristotle shows clearly that nothing moves or changes in a now but only* has *moved or changed, whereas things do change and move in time. At any rate, the leap in movement is a* part *of the movement which occurs in the course of moving and will not be taken in the now; nor, being present, will it occur in the non-present. So that in which the present movement occurs is the present time, and it is infinitely divisible, just as the movement is, for each is continuous, and every continuum is infinitely divisible.* [1]

Here "Zeno's problem" refers to the dichotomy paradox. Certain of Aristotle's synechist successors, including, apparently, the Stoics, not content with Aristotle's resolution of the paradox by appeal to potentiality, introduced another device for circumventing it. They held that a motion could occur all at once, in a "leap", without the half-motion taking place before the whole. In that case the moving body vanishes from one position and reappears a little further on, without an intervening time lapse. The leap itself could still be thought of as infinitely divisible because the distance traversed would be infinitely divisible; another leap could be made across a shorter distance, in fact over any distance, however short. This idea of "motion by leaps" served to resolve Zeno's paradox: provided a body can leap in the manner indicated, it is not necessary for it to travel first the half-distance, and before that half of the half, ad infinitum, as Zeno's paradox threatens. Of course this "resolution" raises difficulties of its own, not the least being that it involves a body being in two places at once.

Damascius's idea seems to have been that time itself embodies such "leaps". Being divisible, these temporal leaps are not atoms, however. Damascius defines time as "the *measure* of the flow of being"; Sorabji (in "Atoms and Time Atoms") suggests that Damascius had in mind here a numerical, or discrete measure, like the hours in a day. If time is a discrete measure, then it will obviously contain leaps. On the other hand, the leaps can be called infinitely divisible, for the discrete stages recorded by the measure can be made arbitrarily close. Here we see the extension to the continuum of time of the idea, long familiar in the case of the linear continuum, of imposing a discrete measure in the form of an arbitrarily small unit.

2. Oriental and Islamic Views.

CHINA

Chinese natural philosophy seems to have inclined more to synechism than to atomism. In the *Chuang Tzu,* "The Book of Master Chuang", written around 290 B.C., are to be found a number of paradoxes, including some startlingly similar to those of Zeno. For example[2]

> *That which has no thickness cannot be piled up, but it can cover a thousand square miles in area.*

[1] Quoted in Sorabji (1982), pp. 74–5.
[2] Needham (1954–), vol. II, pp. 190-1.

The shadow of a flying bird has never yet moved.

There are times when a flying arrow is neither in motion nor at rest.

*If a stick one foot long is cut in half every day, it will still have something left after ten thousand generations***.**

The last of these would seem to be affirm a divisionist view and to deny the existence of atoms, at least of a material nature.

On the other hand[1] the idea of a geometric point appears in the Mohist Canon of c. 330 B.C. There we find the following definition of a point:

The line is separated into parts, and that part which has no remaining part (i.e., cannot be divided into still smaller parts) and thus forms the extreme end of a line is a point.

Further elaboration follows:

If you cut a length continually in half, you go forward until you reach the position that the middle (of the fragment) is not big enough to be separated into halves; and then it is a point. Cutting away the front part (of a line) and cutting away the back part, there will eventually remain an indivisible point in the middle. Or if you keep cutting into half, you will come to a stage at which there is an almost nothing, and since nothing cannot be halved, this can no more be cut.

It is to be noted that this characterization as uncuttable applies equally well to an infinitesimal as to a point.

Like the Islamic philosophers (see below), The Mohists also[2] seem to have considered the idea of atoms or instants of time, as witness these passages from the Mohist canon:

The "beginning" means an (instant of) time.

Time sometimes has duration and sometimes not, for the "beginning" point of time has no duration.

The following passages on the nature of cohesion, contact, and coincidence would not seem out of place in Aristotle:

A discontinuous line includes empty spaces.

The meaning of "empty" is like the spaces between two opposed pieces of wood. In these spaces there is no wood; that is, surfaces cannot be absolutely smooth and cannot therefore fully cohere.

Contact means two bodies mutually touching.

Lines placed in contact with each other will not necessarily coincide, since one may be longer or shorter than the other. Points placed in contact with one another will coincide because they have no dimensions. If a line is placed in contact with a point, they may or may not coincide; they will do so if the point is placed at the end of the line, for both have no thickness; they will not do so if the point is placed at the middle of the line, for

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[1] Needham, (1954-), 19(h).

[2] Needham (1954-), 26(b).

> *the line has length while the point has no length. If a hard white thing is placed in contact with another hard white thing, the hardness and whiteness will coincide mutually; since the hardness and whiteness are qualities diffused throughout the two objects, they may be considered to permeate the new larger object formed by the contact of the two smaller ones. But two material bodies cannot mutually coincide because of the mutual impenetrability of material solids.*

Qualities such as hardness or whiteness being conceived by the Mohists as "diffused throughout" or "permeating" material objects, that is, continuously, it would seem naturally to follow that they saw matter itself as continuous. Indeed, according to Needham, Chinese natural philosophy as a whole was, like the world-view of the Stoics, "dominated throughout by the concept of waves rather than of atoms". The two fundamental forces in the universe, the Yin and the Yang, were conceived by the Chinese as exerting their influence in oscillatory succession, the one waxing as the other wanes. Needham sums up the Chinese view as follows:

> *…the Chinese physical universe in ancient and medieval times was a perfectly continuous whole.* Chhi [matter-energy, similar to the pneuma of the Stoics] *condensed in palpable matter was not particulate in any important sense, but individual objects acted and reacted with all other objects in the world. Such mutual influences could be effective over very great distances, and operated in a wave-like or vibratory manner dependent in the last resort on the rhythmic alternation at all levels of the two fundamental forces, the Yin and the Yang. Individual objects thus had their intrinsic rhythms. And these were integrated like the sounds of individual instruments in an orchestra, but spontaneously, into the general pattern of harmony in the world.* [1]

This conception of the universe could not be further removed from atomism's flurry of particles in a void.

INDIA

From a very early period atomism played a role in Indian philosophical thought[2]. Generally speaking, it was subscribed to by thinkers of a realist tendency, such as the Jains, the Hinayana school of Buddhism, and the adherents of Nyaya-Vaisesika; and opposed by the idealists, most notably the Mahayana school of Buddhism and the Vedantists. The origins of atomism in India are shrouded in obscurity. Some scholars have claimed to find traces of atomism in the Upanishads. Others have sought to explain its emergence in India by drawing a parallel with the situation in ancient Greece: just as atomism emerged there as a response to Eleatic monism, so the analogous doctrine arose in India as a response to the doctrine of the eternal and immutable Brahman of the Upanishads.

The Indian idealists took issue with the atomist claim that atoms were both corporeal and yet also partless, holding this to be contradictory. For, they argued, the corporeal, being spatially extended, is composed of parts. Only the noncorporeal, for instance consciousness and sensation, is partless. For a dualist this would mean that reality is made up of two continua – a continuum of matter and a continuum of mind – with radically different properties: the one composed of parts and so divisible, and the other a partless unity, an Eleatic monad**.** But idealists, like materialists, are first and foremost monists, and the Indian idealists were no exception. The Vedantists in particular repudiated the idea that the world could be a plurality. In their unswerving pursuit of the ideal of unity, they took the unity of consciousness and the self as the

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[1] Needham (1954-), 26(b).
[2] See Gangopadhyaya (1980).

ultimate reality, regarding as illusory not merely the external world with its apparent multiplicities, but also the received notion that there exists a multiplicity of minds or consciousnesses.

ISLAMIC THOUGHT

Greek philosophy, and in particular Greek theories of the continuum, enjoyed a revival in Islamic thought from the 7th century A.D. Synechism and atomism once again did battle, with the latter eventually proving the more infuential within Islamic philosophy.

The dispute seems to have begun soon after 800 with the controversy between the philosophers **Nazzam** (died c. 846), a divisionist, and **Abu l-Hudahyl al-Allaf** (died c. 841), an atomist[1]. Like Damascius and the Stoics, Nazzam shielded his divisionist belief from Zenonian paradox by maintaining that motion takes place in divisible leaps. Nazzam had a number of interesting arguments against atomism. One of these had been earlier used by the Greeks for the same purpose. Since atomic movements take no time, they must all occur at the same speed. Now the Islamic atomists accounted for evident differences in (linear) speed by allowing an atom to linger for a varying number of time atoms in successive space atoms. But this raises diffculties in accounting for *rotatory* motion: the inner atoms of a rotating millstone, for example, must linger in their places, while the more rapidly moving outer atoms continue to progress, resulting in fragmentation or distortion of the millstone. Sorabji suggests how Nazzam's theory of divisible leaps may have overcome this difficulty. For, since all points in the millstone can leap simultaneously, it is never required that one point in the millstone remains still while others move. The divisibility of the leaps allow the points to rematerialize at precisely those distances required to preserve the millstone's shape. Of course motion conceived as continuous also avoids this "fragmentation" problem, but, in Nazzam's eyes such motion was subject to Zeno's paradox. Leaping motion alone could avoid both Zeno's paradox and the fragmentation problem.

Another argument of Nazzam's against atomism is of particular interest, because of its influence on medieval European discussions. Consider a square and one of its diagonals. If atoms are sizeless, then, Nazzam contends, from every sizeless atom on the diagonal a straight line can be drawn at right angles until it joins a sizeless atom on one of the two sides. When all such lines have been drawn, they will be parallel and no gaps will lie between them. Thus to each atom on the diagonal there corresponds exactly one atom one one of the two sides, and vice-versa. So there must be the same number of atoms along the diagonal of a square as along the two adjoining sides. In that case the absurd conclusion is reached that the route along the diagonal should be no quicker than the route along the two sides.

These and other arguments against material atomism due to **Avicenna** (980-1037) appear in the *Metaphysics* of **Algazel** (1058-1111), through which they came to exert a considerable influence on the thinkers of medieval Europe.

The most forceful champions of atomism among the Islamic philosophers were the Mutakallemim, or professors of the Kalam, of the 10th and 11th centuries. The views of this school are critically summarized in **Maimonides**' (1135-1204) *Guide for the Perplexed.* The summary takes the form of 12 propositions and commentaries, of which those concerning atomism are the first three—that all things are composed of atoms, that a vacuum exists, and that time is composed of time-atoms.

On the first of these propositions Maimonides comments:

> *"The Universe, that is, everything contained in it, is composed of very small parts [atoms} which are indivisible on account of their smallness; such an atom has no magnitude; but when several atoms combine, the sum has a magnitude, and thus*

[1] Sorabji (1982), pp. 37–87.

forms a body." … All these atoms are perfectly alike; they do not differ from each other in any point… [1]

On the second he observes:

The original also believe that there is a vacuum, i.e. one space, or several spaces that contain nothing, which are not occupied by anything whatsoever, and are devoid of all substance. This proposition is to them an indispensable sequel to the first. For if the Universe were full of such atoms, how could any of them move? For it is impossible to conceive that one atom should move into another. And yet the composition, as well as the decomposition of things, can only be effected by the motion of atoms. Thus the Mutakallemim are compelled to assume a vacuum, in order that the atoms may combine, separate, and move in that vacuum that does not contain any thing or any atom. [2]

On the existence of time-atoms and the consequences thereof Maimonides is at his most critical:

"Time is composed of time-atoms, i.e., of many parts, which on account of their short duration cannot be divided." This proposition also is a logical consequence of the first. The Mutakallemim undoubtedly saw how Aristotle proved that space, time and locomotion are of the same nature, that is to say, they can be divided into parts which stand in the same proportion to each other: if one of them is divided, the other is divided in the same proportion. They, therefore, knew that if time were continuous and divisible ad infinitum, *their assumed atom of space would of necessity likewise be divisible. Similarly, if it were supposed that space is continuous, it would necessarily follow, that the time-element, which they considered to be indivisible, could also be divided. … Hence they concluded that space was not continuous, but was composed of elements that could not be divided; and that time could likewise be reduced to time-elements, which were indivisible…Time would thus be an object of position and order.*

The Mutakallemim did not at all understand the nature of time… Now mark what conclusions were accepted by the Mutakallemim as true. They held that locomotion consisted in the translation of each atom of a body from one point to the next one; accordingly the velocity of one nobody in motion cannot be greater than that of another body. When, nevertheless, two bodies are observed to move during the same time through different spaces, the cause of the difference is not attributed by them to the fact that the body which has moved through a larger distance had a greater velocity, but to the circumstance that motion which in ordinary language is called slow, has been interrupted by more moments of rest, while the motion which ordinarily is called quick has been interrupted by fewer moments of rest. …

Maimonides derides the atomists' account of geometry:

Nor must you suppose that the aforegoing theory concerning motion is less irrational than the proposition resulting from this theory, that the diagonal of square is equal to one of its sides, and some of the Mutakallemim go so far as to declare that the square is not a thing of real existence. In short, the adoption of the first proposition would be tantamount to the rejection of all that has been proved in Geometry. The propositions in Geometry would, in this respect, be divided into two classes: some would be absolutely rejected; e.g., those which relate to properties of

[1] Maimonides (1956), p. 120.
[2] *Ibid.,* p. 121.

> *the incommensurability and the commensurability of lines and planes, to rational and irrational lines, and all other propositions in the tenth book of Euclid, and in similar works. Other propositions would appear to be only partially correct; e.g., the solution of the problem of dividing a line in two equal parts, if the line consists of an odd number of atoms; according to the theory of the Mutakallemim such a line cannot be bisected.* [1]

And the atomists' account of bodily rotation is scrutinized:

> *We ask them: "Have you observed a complete revolution of a millstone? Each point in the extreme circumference of the stone describes a large circle in the very same time in which a point nearer the centre describes a small circle; the velocity of the outer circle is therefore greater than that of the inner circle. You cannot say that the motion of the latter was interrupted by more moments of rest; for the whole moving body, i.e., the millstone, is one coherent body." They reply, "During the circular motion, the parts of the millstone separate from each other, and the moments of rest interrupting the motion of the portions nearer the centre are more than those which interrupt the motion of the outer portions." We ask again: "How is it that the millstone, which we perceive as one body, and which cannot be easily broken, even by a hammer, resolves into its atoms when it moves, and becomes again one coherent body, returning to its previous state as soon as it comes to rest, while no one is able to observe its breaking up?" Again their reply is based on the twelfth proposition, which is that the senses cannot be trusted, and that only the evidence of the intellect is admissible.* [2]

This last sentence bears witness to the philosophical gulf that had opened up between Epicureanism and Islamic philosophy: the Epicureans held that all knowledge derived from the senses, while their Islamic successors maintained the exact opposite. Yet for both atomism was a central tenet.

The commentaries on Aristotle by the Islamic philosopher **Averroes** (1126-1198) came to be widely disseminated in the West, where they exerted great influence. The doctrine of natural minima played a central role in Averroes' conception of continuous substance, in which the distinction between physical and mathematical divisibility is made quite clearly[3]. Witness, for example, this passage from his commentary on Aristotle's *Physics*:

> *A line as a line can be divided indefinitely. But such a division is impossible if the line is taken as made of earth*[4]

Averroes considered natural minima as actual parts of (continuous) substances, so possessing a kind of physical reality which makes them not dissimilar to atoms. The doctrine of natural minima thus allowed the atomistic principle to gain a foothold in the continuum theory.

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[1] *Ibid.*
[2] *Ibid.*, p. 122.
[3] Van Melsen (1952), p. 59.
[4] Quoted in *ibid.*, p. 59.

3. The Philosophy of the Continuum in Medieval Europe

The scholastic philosophers of Medieval Europe, in thrall to the massive authority of Aristotle, mostly subscribed in one form or another to the thesis, argued with great effectiveness by the Master in Book VI of the *Physics,* that continua cannot be composed of indivisibles. On the other hand, the avowed infinitude of the Deity of scholastic theology, which ran counter to Aristotle's thesis that the infinite existed only in a potential sense, emboldened certain of the Schoolmen to speculate that the actual infinite might be found even outside the Godhead, for instance in the assemblage of points on a line.

A few scholars of the time chose to follow Epicurus in upholding atomism reasonable and attempted to circumvent Aristotle's counterarguments. **Henry of Harclay** (c.1275-1317), for instance, claimed that a line is composed of an actual infinity of points in immediate juxtaposition, touching "whole to whole" without superposition. Even with the introduction of actual infinity, this view is open to the objection raised by Aristotle against the contiguity of points, and was attacked on that and related grounds.

The anti-Aristotelian **Nicholas of Autrecourt** (c.1300-69) was also an atomist, holding that space is composed of points and time of instants. Like Henry, he claimed *contra* Aristotle that points, "having their own position and mode of being", can touch "whole to whole" without superposition, and thereby constitute an extended magnitude. In the section entitled "Indivisibles" of his *Universal Treatise* he attempts to answer a number of Aristotle's objections to the atomistic thesis. With regard to motion he reiterates the Mutakallemim theory that motion takes place through atomic "jerks", and draws the conclusion that a body moving without stopping has attained the upper limit to velocity. Rejecting Aristotle's contention that a continuum is divisible into parts only potentially, not actually, he sums up his own view of the composition of the continuum in the following terms:

> *First, the continuum is not composed of parts which can always be further divided (and in this there would be a departure from common opinion). Secondly, a continuum demonstrable to sense or imagination is not composed of a finite number of points (and in this a departure would be made from the opinion of all those who have posited that a continuum is composed of indivisibles). And on this basis other propositions are true. For example, "For every magnitude which is given or which is pointed out to sense or imagination, there is a smaller one (at least, nothing seems to prevent this), and yet, along with this, it will be true to say that there is in a thing some magnitude such that a smaller cannot be found."*[1]

Nicholas's conclusions

> *For every magnitude pointed out to sense or imagination, there is a smaller one,*

and

> *there is in a thing some magnitude such that a smaller one cannot be found,*

seem contradictory at first sight. Andrew Pyle[2] has pointed out that the contradiction between these two propositions is only apparent, since the first asserts the reducibility of every sensible or imaginable magnitude, while the second denies that this holds for every magnitude *tout court.* The "irreducible" magnitude of the second proposition "does

[1] Nicholas of Autrecourt (1971), p. 82
[2] Pyle (1997), p. 208.

not come to sense or imagination as a finished being"[1] and is smaller than any reducible (that is, sensible or imaginable) magnitude. Pyle tentatively suggests that Nicholas's irreducible magnitude might be seen as an early instance of the idea of an *infinitesimal*. But Pyle, who holds the view that the infinitesimal is an incoherent concept, makes this suggestion only to underline what he sees as the ultimate untenability of Nicholas's program[2]. It seems to me, however, that, despite its obscurities, Nicholas's vision is closer in certain respects to the punctate account of the continuum given by Cantor and Dedekind in the 19th century[3]. For Nicholas actually asserts that " a continuum is composed of points; and a continuum, marked on a wall or elsewhere, whether sensed or imagined, is composed of *infinite* points."[4] (my emphasis). And he goes on to deny that the compounding of infinitely many points would necessarily lead to infinite extension. These are two key features of the Cantor-Dedekind theory.

The incipient atomism of the fourteenth century met with a determined synechist rebuttal. This was initiated by **John Duns Scotus** (c. 1266-1308). In his analysis[5] of the problem of "whether an angel can move from place to place with a continuous motion" he offers a pair of purely geometrical arguments against the composition of a continuum out of indivisibles. One of these arguments is a variant of Nazzam's: that, if the diagonal and the side of a square were both composed of points, then not only would the two be commensurable in violation of Book X of Euclid, they would even be equal. The other[6] starts from Euclid's second postulate that a circle of any diameter can be constructed with any point as centre. Two unequal circles are constructed about a common centre *A*. Supposing the larger circle to be composed of points, choose two contiguous such points *B* and *C*, and draw straight lines from *A* to *B* and *C* (figure 2).

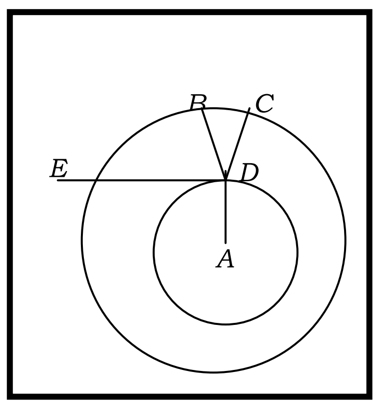

Figure 2

The lines then cut the circumference of the smaller circle at right angles, either at different points, or at a single point. If at different points, there will be just as many points on the larger circle as on the smaller, violating Euclid's fifth axiom that the whole is greater than the part. Suppose, on the other hand, that the straight lines *AB* and *AC* cut the smaller circle at a single point *D*. Then the tangent line *ED* to the

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[1]Nicholas of Autrecourt (1971), p. 82.

[2] In Pyle's words:

> *We might almost claim that Nicholas was one of the founders of the doctrine of the* infinitesimal, *that curious creature greater than nothing yet less than anything, an infinity of which make up a magnitude. However great its* heuristic *value in the history of mathematics this doctrine is quite incoherent and the infinitesimal—as was already apparent in Zeno's day—an impossible entity.*

[3] See Chapter 4 below.

[4] Nicholas of Autrecourt (1971), p. 80

[5] In his *Opus Oxoniense*. My source here is Murdoch's discussion of medieval atomism in §52 of Grant (1974).

[6] Grant (1974), p. 317.

smaller circle at *D* makes two right angles with *AB*, and also with *AC.* Hence ∠*ADE* together with ∠*BDE* is equivalent to two right angles, and likewise for ∠*ADE* together with ∠*CDE.* By Euclid's third postulate, all right angles are equal, so if we subtract the angle these two pairs have in common, namely, *ADE*, the remainders will be equal; consequently ∠*BDE* will be equal to ∠*CDE* and the part again equal to the whole.

William of Ockham (c. 1280-1349) brought a considerable degree of dialectical subtlety[1] to his analysis of continuity; it has been the subject of much scholarly dispute.[2] For Ockham the principal difficulty presented by the continuous is the infinite divisibility of space, and in general, that of any continuum [3]. The treatment of continuity in the first book of his *Quodlibet* of 1322-7 rests on the idea that between any two points on a line there is a third—perhaps the first explicit formulation of the property of density—and on the distinction between a *continuum* "whose parts form a unity" from a *contiguum* of juxtaposed things[4]. Ockham recognizes that it follows from the property of density that on arbitrarily small stretches of a line infinitely many points must lie, but resists the conclusion that lines, or indeed any continuum, consists of points. Concerned, rather, to determine "the sense in which the line may be said to consist or to be made up of anything."[5], Ockham continues:

> *Here we must express the true meaning of the problem, which is this: whether any parts of a line are indivisible: and if such be the meaning, I say that* no part of the line is indivisible, nor is any part of a continuum indivisible.[6]

In proving this he first notes that indivisible parts of a line would have to be at the same time points and minimum lengths; there can be only finitely many of these and so a line consists of finitely many points. The remainder of his proof proceeds along the lines of the similar demonstration in Duns Scotus.[7]

While Ockham does not assert that a line is actually "composed" of points, he had the insight, startling in its prescience, that a punctate and yet continuous line becomes a possibility when conceived as a dense array of points, rather than as an assemblage of points in contiguous succession.

The most ambitious and systematic attempt at refuting atomism in the 14th century was mounted by **Thomas Bradwardine** (c. 1290 – 1349). The purpose of his *Tractatus de Continuo* (c. 1330) was to "prove that the opinion which maintains continua to be composed of indivisibles is false."[8] This was to be achieved by setting forth a number of "first principles" concerning the continuum—akin to the axioms and postulates of Euclid's *Elements*—and then demonstrating that the further assumption that a continuum is composed of indivisibles leads to absurdities. In his stimulating analysis[9] analysis[9] of this work John Murdoch enumerates what he sees as the successes of its in regard to what its author hoped to establish. Among these Murdoch includes Bradwardine's improved Aristotelian definition of continuum; his strict definition of indivisible rendering it independent of the idea of extension; his rigorous demonstration on the basis of the two previous definitions that a continuum cannot be created through the juxtaposition or superposition of indivisibles; and his demonstration that the primary assumptions of Euclidean geometry presuppose the infinite divisibility of the geometric line. On the other hand Bradwardine does not seem to have grasped the

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[1] He seems to have refrained, however, from subjecting the continuum to his celebrated "razor".
[2] See, e.g. the papers of Murdoch and Stump in Kretzmann (1982).
[3] Burns (1916), p. 506.
[4] *Ibid.,* p. 510.
[5] *Ibid.,* p. 507.
[6] *Ibid.,* p. 507
[7] *Ibid.* p. 507-9.
[8] Murdoch (1957), p. 54.
[9] *Op. cit.*

possibility, suggested by Ockham, that a continuous line could be composed of a dense array of indivisibles.

Nicole Oresme (c. 1325-1382), the foremost French thinker of the 14th century, made a number of significant contributions to mathematics, introducing in particular the idea of representing uniformly accelerated motion by means of linear graphs. He also made translations from Latin into French, with commentaries, of several of the works of Aristotle, including *De Caelo*[1]. In his commentary on Book I of Aristotle's work Oresme observes that the terms "magnitude", "continuous body" and "continuum" are synonymous. He then comments on Aristotle's assertion "the continuous is that which is divisible into parts, which themselves are continuously divisible." Like Averroes, Oresme draws a distinction between the physical and mathematical divisibility of continua, but places a stronger emphasis on the potentially infinite nature of the latter form of divisibility:

> Divisible *is used in two ways: one way it means the real separation of the parts of anything, and the other way it means division conceptually in the mind. It is not to be thought that every magnitude or continuum is divisible in the first sense, for it is naturally impossible to divide the heavens as one divides a wooden log, separating one part from another. In dividing a log or a stone or another material or destructible object, one can reach a part so small that further division would destroy its substance. But any continuum or magnitude is continually divisible conceptually in the human mind, just as astrologers divide the heavens into degrees, the degrees into minutes, the minutes into seconds, the seconds into thirds, fourths, and then fifths. The imagination can proceed thus endlessly. In the same way, any object such as earth, water, a stone, a log, etc., has many parts, and each of its parts has many parts, and so on and on; just as each body has two halves and each half has two halves, proceeding thus endlessly even though by these divisions we arrive at parts so small that they are imperceptible to the senses. This is true of all continuous things like a line, a surface, a solid body, motion, time and similar things; for each of these has parts and we cannot say nor think a number of parts so great that it could not be greater, even a hundred or a thousand times greater, beyond any ratio, without any end or limit, however small the thing may be, even the thousandth part of a grain of millet.*[2]

The later emergence of the mathematical concept of *function* owes much to Oresme. The function concept is closely tied up with the idea of continuity, more exactly, with the idea of one attribute varying continuously, but not necessarily uniformly, with another. The ancient Greek thinkers had grasped that motion, for instance, could be described as a continuous variation of distance or position with time. So the idea of a variable *quantity*, of one quantity depending on another quantity, was accepted in Greek philosophy, as is indicated by the fact that Greek mathematicians such as Hippias and Archimedes employed the idea in the generation of curves. On the other hand motion itself was considered a quality and as such unquantifiable; indeed Aristotle had explicitly repudiated the idea of instantaneous velocity[3]. And the notion of of a variable quality, that is, of a quality (continuously) correlated with a quantity, seems not to have been regarded as an admissible concept in Greek science. But the 13th century in Europe saw the emergence of a theory of variable qualities in which the germ of the concept of function can be discerned—the so-called doctrine of the *latitude of forms*, Here the term *form* "refers in general to any quality which admits of variation and which involves the intuitive idea of intensity—that is, to such notions as velocity, acceleration, density"[4]. A form is accordingly what later became known as an *intensive*

[1] Oresme (1968).
[2] *Ibid.* pp. 45-47.
[3] Aristotle (1980), 234a
[4] Boyer (1959), p. 73.

quantity. The latitude of a form was "the degree to which the latter possessed a certain quality", and the central concern of the theory was the study of the manner in which these qualities could be intensified or diminished—the *intensio* or *remissio* of the form. As forms or intensive quantities the Scholastics considered not only what we would today call instantaneous velocity (although they lacked an exact definition of the notion), but also brightness, temperature, and density. They also distinguished between uniform and nonuniform rates of change, rates of rates of change, and the like.

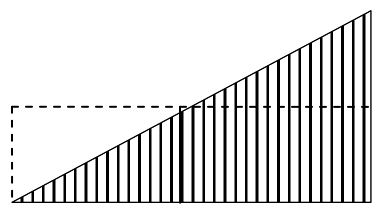

Figure 3

Now Oresme held that everything measurable is imaginable as continuous quantity, and he hit upon the brilliant idea of drawing a picture or graph of the way a measurable form could vary. In doing so he ushered in the idea of a function being represented by (continuous) curve, although he was unable to make effective use of it except in the case of linear functions. In particular he drew what amounts to a velocity-time graph for a body moving with uniform acceleration (figure 3).[1] In this diagram points on the base of the triangle represent instants of time, and the length of the each vertical line the velocity of the body at the corresponding instant. Oresme realized that the distance covered by the body is represented by the area of the triangle and, since this latter coincides with the area of the indicated rectangle, was able to infer the rule that, under uniformly accelerated motion, the average velocity is the arithmetic mean of the terminal and initial velocities.

The views on the continuum of **Nicolaus Cusanus** (1401-64), a champion of the actual infinite, leave a somewhat contradictory impression. In his treatise *Of Learned Ignorance* of 1440, he contrasts the indivisibility of the infinite line (a somewhat mystical conception) with the divisibility of any finite line:

> *A finite line is divisible, whereas an infinite line, in which the maximum is at one with the minimum, has no parts and is in consequence indivisible. The finite line, however, cannot be divided into anything but a line, for, as we have already seen, in dividing an extended object, we never reach a minimum point which is the smallest that can exist.*[2]

On the other hand in *De Mente Idiotae* of 1450, in answer to the question "What dost thou understand by an atom?" Cusanus responds:

> *Under mental consideration that which is continuous becomes divided into the ever divisible, and the multitude of parts progresses to infinity. But by actual division we arrive at an actually indivisible part which I call an atom. For an atom is a quantity, which on account of its smallness is actually indivisible.*[3]

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[1] Boyer and Merzbach (1989), p. 264f.
[2] Cusanus (1954), p. 36
[3] Quoted in Stones (1928).

Here Cusanus seems not to be contrasting the "mental" or "geometric" continua, which are infinitely divisible, and the "physical" continua of extended matter, which are not. Rather, he is saying that *any* continuum, be it geometric, perceptual, or physical, is subject to *two* types of successive division, the one *ideal*, the other *actual*. Ideal division "progresses to infinity"; actual division terminates in atoms after finitely many steps. This distinction is similar to that between physical and mathematical divisibility found in Oresme.

Cusanus's realist conception of the actual infinite is reflected in his quadrature of the circle[1]. He took the circle to be an *infinilateral* regular polygon, that is, a regular polygon with an infinite number of (infinitesimally short) sides. By dividing it up into a correspondingly infinite number of triangles, its area, as for any regular polygon, can be computed as half the product of the apothegm (in this case identical with the radius of the circle), and the perimeter. The idea of considering a curve as an infinilateral polygon was employed by a number of later thinkers, for instance, Kepler, Galileo and Leibniz.

[1] Boyer (1959), p. 91. The argument may well be of Greek origin.