Chapter 3

The 18th and Early 19th Centuries : The Age of Continuity

1. The Mathematicians

EULER

The leading practitioner of the calculus, indeed the leading mathematician of the 18th century, was **Leonhard Euler** (1707–83). While Euler's genius has been described as being of "equal strength in both of the main currents of mathematics, the continuous and the discrete"[1], philosophically he was a thoroughgoing synechist. Rejecting Leibnizian monadism, he favoured the Cartesian doctrine that the universe is filled with a continuous ethereal fluid and upheld the wave theory of light over the corpuscular theory propounded by Newton.

Euler's philosophical views may be gleaned from his *Letters to a German Princess*, written during 1760–62. He writes:

> *Two things, then, must be admitted: first, the space through which the heavenly bodies move is filled with a subtile matter; secondly, rays are not an actual emanation from the sun and other luminous bodies, in virtue of which part of their substance is violently emitted from them, according to the doctrine of* Newton.
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> *That subtile matter which fills the whole space in which the heavenly bodies revolve is called* ether. *Of its extreme subtilty no doubt can be entertained... It is also, without doubt, possessed of elasticity, by means of which it has a tendency to expand itself in all directions, and to penetrate into spaces where there would otherwise be a vacuum: so that if by some accident the ether were forced out of any space, the surrounding fluid would instantly rush in and fill it again.*[2]

Euler rejected the Newtonian doctrine that forces, in particular gravitation, could act at a distance, and upheld the idea that the effect of such forces is transmitted continuously in some way through the ether. In an early letter he writes:

> *Gravity, then, is not an intrinsic property of body: it is rather the effect of a foreign force, the source of which must be sought for out of the body. This is geometrically true, though we know not the foreign forces which occasion gravity...*
>
> *I have already remarked that these forces may very probably be caused by the subtile matter which surrounds all the heavenly bodies, and fills the whole space of the heavens... This opinion, however, that attraction is essential to all matter, is subject to so many other inconveniences, that it is hardly possible to allow it a place in a rational philosophy. It is certainly much safer to proceed on the idea, that what is called attraction is a power contained in the subtile matter which fills the whole space of the heavens; though we cannot tell how.*[3]

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[1] Bell (1965), Vol I, p. 152

[2] Euler (1843), Vol 1, pp. 83-84.

[3] *Ibid.*, pp. 254-5.

In a sequence of later letters Euler mounts a determined attack against both material atomism and monadism. He argues that it is an inherent property of extension, established on geometric grounds, to be infinitely divisible; this being granted, it follows that bodies too, as instances of the extended, must be likewise. But, says Euler, certain philosophers deny this conclusion, insisting that the divisibility of bodies

> *extends only to a certain point, and that you may come at length to particles so minute that, having no magnitude, they are no longer divisible. These ultimate particles, which enter into the composition of bodies, they denominate* simple beings *and* monads.[1]

Indeed,

> *There was a time when the dispute respecting monads employed such general attention, and was conducted with so much warmth, that it forced its way into company of every description, that of the guard-room not excepted. There was scarcely a lady at court who did not take a decided part in favour of monads or against them. In a word, all conversation was engrossed by monads—no other subject could find admission.*[2]

The partisans of monads are

> *obliged to affirm that bodies are not extended, but have only an appearance of extension. They imagine that by this they have subverted the argument adduced in support of the divisibility in infinitum. But if body is not extended, I should be glad to know from whence we derived the idea of extension; for if body is not extended, nothing in the world is, since spirits are still less so. Our idea of extension, therefore, would be altogether imaginary and chimerical.*[3]

The monadists' case ultimately rests, says Euler, on the *principle of sufficient reason*, but applied in a question-begging way:

> *Bodies, say they, must have their sufficient reason somewhere; but if they were divisible to infinity, such reason could not take place; and hence they conclude, with an air altogether philosophical,* that as every thing must have its sufficient reason, it is absolutely necessary that all bodies should be composed of monads—*which was to be demonstrated.... It were greatly to be wished that a reasoning so slight could elucidate to us questions of this importance; but I frankly confess I understand nothing of the matter.*[4]

Euler argues that the monadists' seemingly reasonable basic premise to the effect that every compound being is made up of simple beings, is in fact fallacious since it leads to contradictions. He writes:

> *In effect, they* [the monadists] *admit that bodies are extended; from this point [they] set out to establish the proposition that they are compound beings; and having hence deduced that bodies are compounded of simple beings, they are obliged to allow that simple beings are incapable of producing real extension, and consequently that the extension of bodies is mere illusion.*

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[1] Euler (1843), Vol. II, p. 39.
[2] *Ibid.*, p. 39-40.
[3] *Ibid.*, p. 41.
[4] *Ibid.*, pp. 50–51.

> *An argument whose conclusion is a direct contradiction of the premises is singularly strange: the reasoning sets out with advancing that bodies are extended; for if they were not, how could it be known that they are compound beings—and then comes to the conclusion that they are not so. Never was a fallacious argument, in my opinion, more completely refuted than this has been. The question was,* Why are bodies extended? *And, after a little turning and winding, it is answered,* Because they are not so. *Were I to be asked, Why has a triangle three sides? and I should reply that it is a mere illusion—would such a reply be deemed satisfactory?*

Euler rejected the concept of infinitesimal in its sense as a quantity less than any assignable magnitude and yet unequal to 0, arguing:

> *There is no doubt that every quantity can be diminished to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals 0. It is in harmony also with that definition of infinitely small things, by which the things are said to be less than any assignable quantity; it certainly would have to be nothing; for unless it is equal to 0, an equal quantity can be assigned to it, which is contrary to the hypothesis.*[1]

In that case differentials must be zeros, and $\mathrm{d}y/\mathrm{d}x$ the quotient 0/0. Since for any number α, $\alpha \cdot 0 = 0$, Euler maintained that the quotient 0/0 could represent any number whatsoever[2]. For Euler *qua* formalist the calculus was essentially a procedure for determining the value of the expression 0/0 in the manifold situations it arises as the ratio of evanescent increments.

But in the mathematical analysis of natural phenomena, Euler, along with a number of his contemporaries, did employ what amount to infinitesimals in the form of minute, but more or less concrete "elements" of continua. This is illustrated in his work on fluid flow, where his virtuosity in employing the principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts[3] can be seen in full flower. In his fundamental papers on the subject[4] Euler derives the *equation of continuity* for a fluid free of viscosity but of varying density flowing smoothly in space. At any point $O = (x, y, z)$ in the fluid and at any time t, the fluid's density ρ and the components u, v, w of the fluid's velocity are given as functions of x, y, z, t. Euler considers the elementary volume element **E**—an infinitesimal parallelepiped—with origin O and edges OA, OB, BC of infinitesimal lengths $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ (see figure 1). Fluid flow during the infinitesimal time $\mathrm{d}t$ transforms the volume element **E** into the infinitesimal parallelepiped **E′** with vertices O', A', B', C'. Euler calculates the lengths of the sides $O'A'$, $O'B'$, $B'C'$ to be respectively

$$\mathrm{d}x\left(1+\mathrm{d}t\frac{\partial u}{\partial x}\right),\quad \mathrm{d}y\left(1+\mathrm{d}t\frac{\partial v}{\partial y}\right),\quad \mathrm{d}z\left(1+\mathrm{d}t\frac{\partial w}{\partial z}\right),$$

ignoring infinitesimal terms of higher order than the second. The volume of **E′** is then

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[1] Quoted from Euler's *Institutiones* of 1755 in Kline (1972), p 429.

[2] Or, to put it another way, (real) numbers are just the ratios of infinitesimals: this is a reigning principle of smooth infinitesimal analysis, see Chapter 10 below.

[3] Weyl (1950), p. 92.

[4] Euler, *Principia motus fluidorum* and *Principes généraux du mouvement des fluides,* 1755. Summarized in Dugas (1988), pp. 301-304.

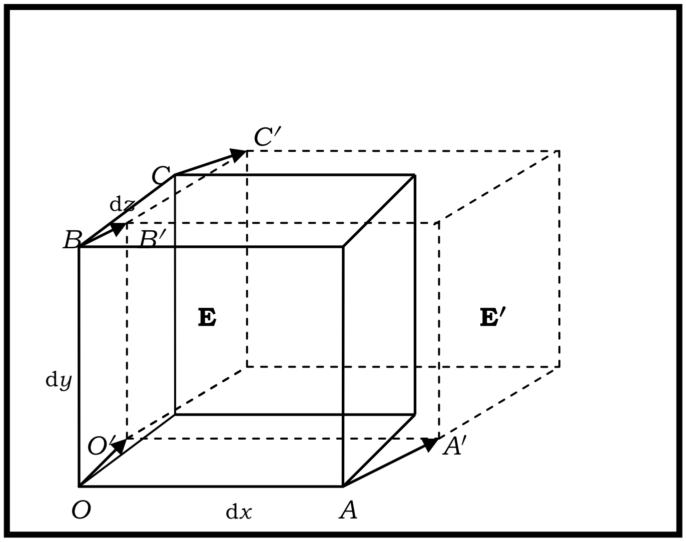

Figure 1

the product of these quantities, which, ignoring terms in dt of second and higher order is seen to be

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z\left(1+\mathrm{d}t\frac{\partial u}{\partial x}+\mathrm{d}t\frac{\partial v}{\partial y}+\mathrm{d}t\frac{\partial w}{\partial z}\right).$$

Since the coordinates of O' are (x+udt, y+vdt, z+wdt), the fluid density ρ' there at time t + dt is

$$\rho+\mathrm{d}t\frac{\partial\rho}{\partial t}+u\mathrm{d}t\frac{\partial\rho}{\partial x}+v\mathrm{d}t\frac{\partial\rho}{\partial y}+w\mathrm{d}t\frac{\partial\rho}{\partial z}.$$

Euler now invokes the principle of conservation of mass to assert that the masses of the fluid in **E** and **E′** are the same, so that,

> *as the density is reciprocally proportional to the volume, the quantity* ρ' *will be related to* ρ *as* dxdydz *is related to*
>
> $$\mathrm{d}x\mathrm{d}y\mathrm{d}z\left(1+\mathrm{d}t\frac{\partial u}{\partial x}+\mathrm{d}t\frac{\partial v}{\partial y}+\mathrm{d}t\frac{\partial w}{\partial z}\right);$$
>
> *whence, by carrying out the division, the very remarkable condition which relates from the continuity of the fluid,*
>
> $$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}=0.$$
>
> *This may be written more simply as*

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

and, for an incompressible fluid, it reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

It will be seen from this calculation that Euler treats the volume element **E** not as an atom or monad in the strict sense—as part of a continuum it must of necessity be divisible—but as being of sufficient minuteness to preserve its rectilinear *shape* under infinitesimal flow, yet allowing its *volume* to undergo infinitesimal change. This idea was to become fundamental in continuum mechanics.

Euler also made important contributions to the development of the function concept. As has been pointed out, the notion of functionality or functional relation arose in connection with continuous variation; indeed the term "function" itself, introduced by Leibniz in a manuscript of 1673[1], was used by him to denote a variable length related in a specified way to a variable point on a curve. In 1718 the scope of the concept was greatly enlarged when John Bernoulli defined a "function of a variable magnitude" as a quantity made up in any way of this variable magnitude and constants.[2] In 1730 Bernoulli introduced the distinction between algebraic and transcendental functions[3], where by the latter he meant integrals of algebraic functions. In 1734 Euler introduced the characteristic function-argument notation $f(x)$. His *Introductio in analysin infinitorum* of 1748 gives unprecedented prominence to the function concept; there he extends Bernoulli's definition of function still further by defining it to be any *analytical* expression defined from variable quantities and constants, where the term "analytical" includes polynomials, power series, and logarithmic and trigonometric expressions. For Euler a *continuous* function meant a function "unbroken" in the sense of being specified by a single analytic formula; hence a *discontinuous* function meant one "broken" in the sense of requiring different analytic expressions in different domains of the independent variable. In Boyer's words, for Euler "functionality became a matter of formal representation, rather than conceptual recognition of a relation."[4] This is true; nevertheless Euler's formalistic treatment of function freed the concept from its geometric origins and paved the way for the general concept of function which appeared in the middle of the nineteenth century.

FROM D'ALEMBERT TO CARNOT

While Euler treated infinitesimals as formal zeros, that is, as fixed quantities, his contemporary **Jean le Rond d'Alembert** (1717–83) took a different view of the matter. Following Newton's lead, he conceived of infinitesimals or differentials in terms of the limit concept, which he formulated by the assertion that one varying quantity is the limit of another if the second can approach the other more closely than by any given quantity.[5] D'Alembert firmly rejected the idea of infinitesimals as fixed quantities, asserting

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[1] Kline (1972), p. 340.

[2] Art. *Function,* Encyclopedia Britannica, Eleventh Edition, 1910-11.

[3] *Ibid.*

[4] Boyer(1959), p. 243.

[5] *Ibid.*, p. 247.

> *A quantity is something or nothing: if it is something, it has not yet vanished; if it is nothing, it has literally vanished. The supposition that there is an intermediate state between the two is a chimera.*[1]

D'Alembert saw the idea of limit as supplying the methodological root of the differential calculus:

> *The differentiation of equations consists simply in finding the limits of the ratios of finite differences of two variables included in the equation.*[2]

For d'Alembert the language of infinitesimals or differentials was just a convenient shorthand for avoiding the cumbrousness of expression required by the use of the limit concept.

Although d'Alembert anticipated the doctrine of limits which would later come to provide the rigorous foundation for analysis long sought by mathematicians, the majority of his contemporaries regarded his formulation of the limit concept as being no less vague, and much less convenient, than the concept of infinitesimal it was intended to supplant. Indeed, of 28 publications on the calculus appearing from 1754 to 1784, just 6 employ the limit concept[3]. The use of the latter was not to become routine until the very end of the 19th century.

In 1742 **Colin Maclaurin** (1698–1746) published his *Treatise of Fluxions*. In this work Maclaurin sets out to demonstrate the essential validity of Newton's fluxional theory by rigorous derivation "after the manner of the ancients, from a few unexceptionable principles"[4]. His approach involved discarding Leibnizian infinitesimals and differentials, but retaining the fundamental notions of Newton's fluxional theory, in particular instantaneous velocity.

Maclaurin gives a masterly account of the process by which the rigorous procedures of Archimedes and other Greek mathematicians came gradually to be replaced by arguments, convenient but of doubtful logical validity, involving infinities and infinitesimals. He writes:

> *But when the principles and strict methods of the ancients...were so far abandoned, it was difficult for the Geometricians to determine where they should stop. After they had indulged themselves in admitting quantities, of various kinds, that were not assignable, in supposing such things to be done as could not possibly be effected (against the constant practice of the ancients), and had involved themselves in the mazes of infinity; it was not easy for them to avoid perplexity, and sometimes error, or to fix bounds to these liberties once they were introduced. Curves were not only considered as polygons of an infinite number of infinitely little sides, and their differences deduced from the different angles that were supposed to be formed from these sides; but infinites and infinitesimals were admitted of infinite orders...*[5]

From geometry, Maclaurin observes, the infinites and infinitesimals passed into "philosophy", i.e. natural science,

> *carrying with them the obscurity and perplexity that cannot fail to accompany them. An actual division, as well as a divisibility of matter* in infinitum, *is admitted by some. Fluids are imagined consisting of infinitely small particles, which are composed of others infinitely less; and this subdivision is supposed to be continued without*

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[1] Quoted in Boyer (1959), pp. 248
[2] Quoted *ibid.*, pp. 247–8
[3] *Ibid.*,p. 250.
[4] Maclaurin, *Treatise of Fluxions,* Preface, in Ewald (1999) *From Kant to Hilbert,* p. 93
[5] *Ibid,* pp. 107- 8

> *end. Vortices are proposed, for solving the phaenomena of nature, of indefinite or infinite degrees, in imitation of the infinitesimals in geometry...*[1]

Maclaurin notes that the introduction of infinitesimals into the description of natural phenomena carries with it the constraint that natural processes always occur continuously, so ruling out physical atomism:

> *Nature is confined in her operations to operate by infinitely small steps. Bodies of a perfect hardness are rejected, and the old doctrine of atoms treated as imaginary, because in their actions and collisions they might pass at once from motion to rest, or from rest to motion, in violation of this law. Thus the doctrine of infinites is interwoven with our speculations in geometry and nature.*[2]

Despite the clarity and originality of Maclaurin's treatise, its style of presentation *in modo geometrico* is in truth a backward step. For, as the Continental mathematicians recognized, the analysis of continuous variation had transcended the modes of reasoning in classical geometry and could no longer be adequately accommodated there. Historians of mathematics have noted that the wide influence of Maclaurin's treatise on British mathematicians led them to adopt the mathematical style of Archimedes and to ignore the more fertile methods of analysis emerging on the Continent. As a result, British mathematicians lagged behind their Continental counterparts for nearly a century.

The last efforts of the 18th century mathematicians to demystify infinitesimals and banish the persistent doubts concerning the soundness of the calculus both appeared in France in 1797. These were the *Théorie des Fonctions Analytiques* by the great Franco-Italian mathematician **Joseph-Louis Lagrange** (1736-1813) and the *Reflexions sur la Metaphysique du Calcul Infinitesimal* by the mathematician and "organisateur de la victoire" of the French Revolution, **Lazare Carnot** (1753-1823).

The *Théorie des Fonctions Analytiques* embodies an "algebraic" approach to the calculus. Lagrange had long been sceptical concerning infinitesimals, but was at the same time less than enamoured of the idea of a limit, considering it metaphysically suspect. Nor did the method of fluxions appeal to him, as it involved the extraneous concept of motion. The treatment of differentials as formal zeros he also regarded as dubious.[3] In seeking a method for obtaining the results of the calculus which avoided these pitfalls, he came up with an idea based on the *Taylor expansion*[4] of a function:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots \ .$$

Here the coefficients $f'(x)$, $f''(x)$, ... are the first, second, ... *derivatives* of *f.* These derivatives had originally been determined through the use of fluxions or infinitesimals, but Lagrange proposed to avoid these by *defining* the successive derivatives of a function to be the coefficients—which Lagrange called the *derived functions*[5] of the given given function—in its Taylor expansion. In this way the differential calculus was to be purged of all metaphysical difficulties, in fact becoming no more than a straightforward method for finding the derived functions of a given function.

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[1] *Ibid.,* p. 108.

[2] *Ibid.*

[3] Boyer (1959), p. 251.

[4] First introduced in 1715 by the English mathematician **Brook Taylor** (1685-1731).

[5] Hence the term *derivative.* The notation $f'(x)$ was also introduced by Lagrange.

Carnot's aim, by contrast, was not to banish the infinitesimal but to divest the concept of all trace of vagueness or obscurity. He attempts to achieve this by conceiving of infinitesimals as variable quantities:

> *We will call every quantity, which is considered as continually decreasing (so that it may be made as small as we please, without being at the same time obliged to make those quantities vary the ratio of which it is our object to determine), an Infinitely small Quantity.*[1]
>
> *...You ask me what Infinitesimal quantities mean? I declare to you that I never by that expression mean metaphysical and abstract existences, as this abridged name seems to imply; but real, arbitrary quantities, capable of becoming as small as I wish, without being compelled at the same time to make those quantities vary whose ratio it was my intention to discover.* [2]

But despite their reality, the inherent variability of infinitesimal quantities necessitates that they be discharged at the conclusion of a calculation:

> *...You ask me if my calculation is perfectly exact and rigorous? I reply in the affirmative, as soon as I have arrived at eliminating from it the Infinitesimal quantities spoken of above, and have reduced it so as to contain ordinary Algebraic quantities alone.* [3]

But Carnot did not follow d'Alembert in regarding the use of infinitesimals as a convenient shorthand for an underlying use of a limit concept. Rather, Carnot suggests that when infinitesimals are taken as real quantities the efficacy of the calculus is then explained by the compensation of errors, a view defended earlier by Berkeley[4] . He contrasts this with the explanation, resting on the Law of Continuity, associated with Euler's view that infinitesimals are no more than zeros:

> *We may then regard the Infinitesimal Analysis in two points of view: by considering the infinitely small quantities either as real quantities or as absolutely zero. In the former case, Infinitesimal Analysis is nothing more than a calculation of compensation of errors: and in the second it is the art of comparing vanishing quantities together and with others, in order to deduce from these comparisons the ratios, whatever they may be, which exist between the proposed quantities. These quantities, as equal to zero, ought to be overlooked in the calculation when they are found in addition with any real quantity; or when they are subtracted from them: but nevertheless they have...ratios very interesting to discover, and such as are determined by the law of continuity, to which the system of auxiliary quantities is subject in its changes. Now in order to readily apprehend this law of continuity, we may easily observe, that we are obliged to consider the quantities in question, at some distance from the term when they vanish altogether, to forestall them from presenting the indefinite ratio of* 0 *to* 0*: but this distance is arbitrary, and has no other object than to enable us to judge the more easily of the ratios which exist between vanishing quantities: these are the ratios which we have in view, whilst we regard the infinitely small quantities as absolutely zero, and not those which have not yet arrived at the term of their annihilation. These last, which have been* called *infinitely small, are never intended themselves to enter into the calculation, regarded under the point of view of which we are at present speaking, but are only employed to assist the imagination, and to point*

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[1] Carnot (1832), p. 14.
[2] *Ibid.,* p. 33
[3] *Ibid.,* p. 34.
[4] See below.

> *out the law of continuity which determines any ratio whatever of the vanishing quantities to which they correspond.* [1]

Carnot's work continued to be influential well into the 19th century before it was swept away by the limit concept.

2. The Philosophers

BERKELEY

Infinitesimals, differentials, evanescent quantities and the like coursed through the veins of the calculus throughout the 18th century. Although nebulous—even logically suspect—these concepts provided, *faute de mieux,* the tools for deriving the great wealth of results the calculus had made possible. And while, with the notable exception of Euler, many 18th century mathematicians were ill-at-ease with the infinitesimal, they would not risk killing the goose laying such a wealth of golden mathematical eggs. Accordingly they refrained, in the main, from destructive criticism of the ideas underlying the calculus. Philosophers, however, were not fettered by such constraints.

The philosopher **George Berkeley** (1685-1753), noted both for his subjective idealist doctrine of *esse est percipi* and his denial of general ideas, was a persistent critic of the presuppositions underlying the mathematical practice of his day. His most celebrated broadsides were directed at the calculus, but in fact his conflict with the mathematicians went deeper. For his denial of the existence of abstract ideas of any kind went in direct opposition with the abstractionist account of mathematical concepts held by the majority of mathematicians and philosophers of the day. The central tenet of this doctrine, which goes back to Aristotle, is that the mind creates mathematical concepts by *abstraction*, that is, by the mental suppression of extraneous features of perceived objects so as to focus on properties singled out for attention. Berkeley rejected this, asserting that mathematics as a science is ultimately concerned with objects of sense, its admitted generality stemming from the capacity of percepts to serve as signs for all percepts of a similar form[2].

Berkeley's empiricist philosophy had initially led him to claim that geometry, correctly conceived, can make reference only to actually perceived lines and figures. Thus in his early and unpublished *Philosophical Commentaries* he rejects infinite divisibility and proposes jettisoning classical geometry in favour of a new account of geometry formulated in terms of perceptible minima[3]. By the time he came to write the *Principles of Human Knowledge,* (published in 1710), his radicalism *vis-a-vis* geometry had softened somewhat, but he again attacks infinite divisibility on the grounds that, since to exist is to be perceived, only perceived extension exists, and it is manifest that this is not infinitely divisible:

> *Every particular finite extension which may possibly be an object of our thought is an* idea *existing only in the mind, and consequently each part thereof must be perceived. If, therefore, I cannot perceive innumerable parts in any finite extension that I consider, it is certain that they are not contained in it; but it is evident that I cannot distinguish innumerable parts in any particular line, surface or solid, which I either perceive by sense, or figure to myself in my mind: wherefore I conclude that they are not contained in it. Nothing can be plainer to me than that the extensions I have in view are no other than my own ideas; and it is no less plain that I cannot resolve any*

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[1] Carnot (1832), pp. 101-2.

[2] Jesseph (1993), p. 37.

[3] *Ibid.*, p. 57. The doctrine of perceptible minima is, of course, subject to the same objections that had been raised against previous attempts at analyzing extension in terms of atoms.

> *one of my ideas into an infinite number of other ideas; that is, they are not infinitely divisible.*[1]

It will be seen that Berkeley, like Epicurus, reads the thesis of infinite divisibility as the assertion that an extended magnitude must contain an actual infinity of parts[2].

Not surprisingly, Berkeley pours scorn on those who adhere to the concept of infinitesimal:

> *Of late the speculation about infinites have run so high, and grown to such strange notions, as have occasioned no scruples and disputes among the geometers of the present age. Some there are of great note who, not content with holding that finite lines may be divided into an infinite number of parts, do yet farther maintain that each of those infinitesimals is itself subdivisible into an infinity of other parts or infinitesimals of a second order, and so on* ad infinitum. *These, I say, assert there are infinitesimals of infinitesimals of infinitesimals, etc., without ever coming to an end! so that according to them an inch does not barely contain an infinite number of parts, but an infinity of an infinity of an infinity* ad infinitum *of parts. Others there be who hold all orders of infinitesimals below the first to be nothing at all; thinking it with good reason absurd to imagine there is any positive quantity or part of extension which, though multiplied infinitely, can never equal the smallest given extension. And yet on the other hand it seems no less absurd to think the square, cube, or other power of a positive real root, should itself be nothing at all; which they who hold infinitesimals of the first order, denying all of the subsequent orders, are obliged to maintain.*[3]

Berkeley maintains that the use of infinitesimals in deriving mathematical results is illusory, and that they can be eliminated:

> *If it be said that several theorems undoubtedly true are discovered by methods in which infinitesimals are made use of, which could never have been if their existence included a contradiction in it; I answer that upon a thorough examination it will not be found that in any instance it is necessary make use of or conceive infinitesimal parts of finite lines, or even quantities less than the* minimum sensible; *nay, it will be evident this is never done, it being impossible. And whatever mathematicians may think of fluxions or the differential calculus and the like, a little reflexion will shew them, that in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense. They may, indeed, call these little and almost insensible quantities infinitesimals or infinitesimals of infinitesimals, if they please: but at bottom this is all, they being in truth finite, nor does the solution of problems require the supposing any other.*[4]

In his 1721 treatise *De Motu* Berkeley adopts a more tolerant attitude towards infinitesimals, regarding them as useful fictions in somewhat the same way as did Leibniz:

> *Just as a curve can be considered as consisting of an infinity of right lines, even if in truth it does not consist of them but because this hypothesis is useful in geometry, in*

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[1] Berkeley (1960), §124.

[2] According to Jesseph (1993, p. 67), Berkeley was largely unaware of the tradition of "mathematical atomism" in ancient and medieval philosophy.

[3] Berkeley (1960), §130. It is of interest here to note that the final sentence of this quotation is an explicit rejection of the concept of nilpotent infinitesimal which had been defended by Nieuwentijdt against Leibniz 250 years later, that concept was to be revived in smooth infinitesimal analysis. See Chapter 10 below.

[4] *Ibid.,* §132.

> *the same way circular motion can be regarded as traced and arising from an infinity of rectilinear directions, which supposition is useful in the mechanical philosophy.*[1]

In *The Analyst* of 1734 Berkeley launched his most sustained and sophisticated critique of infinitesimals and the whole metaphysics of the calculus. Addressed *To an Infidel Mathematician*[2], the tract was written with the avowed purpose of defending theology against the scepticism shared by many of the mathematicians and scientists of the day. Berkeley's defense of religion amounts to the claim that the reasoning of mathematicians in respect of the calculus is no less flawed than that of theologians in respect of the mysteries of the divine:

> *...he who can digest a second or third fluxion, a second or third difference*[3]*, need not, methinks, be squeamish about any point in divinity*[4].

Berkeley's arguments are directed chiefly against the Newtonian fluxional calculus. Typical of his objections is that in attempting to avoid infinitesimals by the employment of such devices as evanescent quantities and prime and ultimate ratios Newton has in fact violated the law of noncontradiction by first subjecting a quantity to an increment and then setting the increment to 0, that is, denying that an increment had ever been present. As for fluxions and evanescent increments themselves, Berkeley has this to say:

> *And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?*[5]

Nor did the Leibnizian method of differentials escape Berkeley's strictures:

> *Instead of flowing quantities and their fluxions,* [the Leibnizians] *consider the variable finite quantities as increasing or diminishing by the continual addition or subduction of infinitely small quantities. Instead of the velocities wherewith increments are generated, they consider the increments or decrements themselves, which they call differences, and which are supposed to be infinitely small. The difference of a line is an infinitely little line; of a plane an infinitely little plain. They suppose finite quantities to consist of parts infinitely little, and curves to be polygons, whereof the sides are infinitely little, which by the angles they make with each other determine the curvity of the line. Now to conceive a quantity infinitely small, that is, infinitely less than any sensible or imaginable quantity, or than any the least finite magnitude is, I confess, above my capacity...* [6]

Berkeley asserts that he is not challenging the conclusions drawn by the analysts, but only the methods by which these conclusions are drawn. His own view is that the methods of the calculus actually work through the introduction of what he calls "contrary errors". In finding the tangent to a curve by means of differentials, for example, increments are first introduced; but these determine the secant, not the tangent. This error is then expunged by ignoring higher differentials, and so

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[1] Berkeley, *De Motu,* §61. In Ewald (1999).

[2] Likely the astronomer Edmund Halley (1656-1742).

[3] By "difference" Berkeley means "differential".

[4] Berkeley, *Analyst,* §7. In Ewald (1999).

[5] *Ibid.,* §35.

[6] *Ibid.,* §§5, 6.

> *by virtue of a twofold mistake you arrive, though not at science, yet at truth.*[1]

This explanation of the validity of the results of the calculus, which became known as the principle of *compensation of errors,* was endorsed by a number of 18th century mathematicians, including Euler, Lagrange and, as we have observed, Carnot.

Berkeley's criticisms of the methodology of the calculus, while pointed, were by no means wholly destructive. Indeed, *The Analyst* has been identified as marking "a turning point in the history of mathematical thought in Great Britain."[2] By exposing with such severity the logical inadequacies of the foundations of the calculus, Berkeley provoked an avalanche of responses from mathematicians anxious to clarify the concepts underlying the calculus and so to place its results and methods beyond reasonable doubt.

HUME

The views of the radical empiricist **David Hume** (1711-76) are similar in certain respects to those of Epicurus[3]. For Hume, as for Berkeley, to comprehend or to have an idea of a thing is to have a mental picture of it[4], a picture not inferior in immediacy to a sense impression. This requirement automatically places beyond comprehension both the infinite and the infinitesimal. Accordingly Hume rejects infinite divisibility, even in thought, on the grounds that anything infinitely divisible must contain an infinity of parts, while the mind is incapable of grasping an infinity in its actuality. Thus Hume writes in Part II of his *Treatise of Human Nature:*

> *'Tis universally allow'd, that the capacity of the mind is limited, and can never attain a full and adequate conception of infinity: And tho' it were not allow'd, 'twould be sufficiently evident from the plainest observation and evidence. 'Tis also obvious, that whatever is capable of being divided* in infinitum, *must consist of an infinite number of parts, and that 'tis impossible to set any bounds to the number of parts, without setting bounds at the same time to the division. It requires scarce any induction to conclude from hence, that the* idea, *which we form of any finite quality, is not infinitely divisible, but that by proper distinctions and separations we may run up this idea to inferior ones, which will be perfectly simple and indivisible. In rejecting the infinite capacity of the mind, we suppose it may arrive at an end in the division of its ideas; nor are there any possible means of evading the evidence of this conclusion*[5].

Hume makes the interesting observation that, even though one may grasp the concept of an arbitrarily small numerical fraction *applied* to an idea, the idea *itself* is not indefinitely divisible:

> *When you tell me of the thousandth or the ten thousandth part of a grain of sand, I have a distinct idea of their different proportions; but the images, which I form in my mind to represent the things themselves, are nothing different from each other, nor inferior to that image, by which I represent the grain of sand itself, which is suppos'd so vastly to exceed them. What consists of parts is distinguishable into them, and what is distinguishable is separable. But whatever we may imagine of the thing, the*

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[1] *Ibid.,* §22.
[2] Cajori (1919), p. 89.
[3] Furley (1967), Ch. 10.
[4] *Ibid.*, p. 137.
[5] Hume (1962), II, 1.

> *idea of a grain of sand is not distinguishable, nor separable into twenty, much less a thousand, ten thousand, or an infinite number of different ideas.*[1]

Hume concludes that the boundedness of divisibility in imagination leads inevitably to indivisible minima, that is, to atoms:

> *'Tis therefore certain, that the imagination reaches a* minimum, *and may raise up to itself an idea, of which it cannot conceive any sub-division, and which cannot be distinguished without a total annihilation.*[2]

And what holds of the imagination holds equally of the senses:

> *'Tis the same case with the impressions of the senses as with the ideas of the imagination. Put a spot of ink on paper, fix your eye upon that spot, and retire to such a distance, that at last you lose sight of it: 'tis plain, that the moment before it vanish'd the image or impression was perfectly indivisible.*[3]

But while sense perception may present us with apparent minima which reason tells us must be composed of a multitude of parts, Hume urges that the atoms of the imagination are genuinely indivisible, and not so merely as a result of mental limitation:

> *We may hence discover the error of the common opinion, that the capacity of the mind is limited...and that 'tis impossible for the imagination to form an adequate idea, of what goes beyond a certain degree of minuteness... . Nothing can be more minute, than some ideas, which we form in the fancy; and images, which appear to the senses; since there are ideas and images perfectly simple and indivisible. The only defect of our senses is, that they give us disproportion'd images of things, and represent as minute and uncompounded what is really great and composed of a vast number of parts. This mistake we are not sensible of: but taking the impressions of those minute objects, which appear to the senses, to be equal or nearly equal to the objects, and finding by reason, that there are other objects vastly more minute, we too hastily conclude, that these are inferior to any idea of our imagination or impression of the senses.*[4]

Hume next contends that what holds of ideas holds equally of the objects represented by them, so that, since ideas are not infinitely divisible, neither are objects:

> *Wherever ideas are adequate representations of objects, the relations, contradictions and agreements of the ideas are all applicable to the objects... . But our ideas are adequate representations of the most minute parts of extension; and thro' whatever divisions and subdivisions we may suppose these parts to be arriv'd at, they can never become inferior to some ideas we can form. The plain consequence is, that whatever* appears *impossible and contradictory upon the comparison of these ideas, must be* really *impossible and contradictory, without any farther excuse or evasion.*
>
> *Everything capable of being infinitely divided contains an infinite number of parts; other wise the division would be stopt short by the indivisible parts, which we should immediately arrive at. If therefore any finite extension be infinitely divisible, it can be no contradiction to suppose, that a finite extension contains an infinite number of parts; And* vice versa, *if it be a contradiction to suppose, that a finite extension*

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[1] *Ibid.* But presumably the idea of a grain of sand is separable into the ideas "grain" and "sand"

[2] *Ibid.*.

[3] *Ibid.*.

[4] *Ibid.*

> *contains an infinite number of parts, no finite extension can be infinitely divisible. But that this latter supposition is absurd, I easily convince myself by the consideration of my clear ideas. I first take the least idea I can form of a part of extension, and being certain that there is nothing more minute than this idea, I conclude, that whatever I discover by its means must be a real quality of extension. I then repeat this idea once, twice, thrice, &c, and find the compound idea of extension, arising from its repetition, always to augment, and become double, triple, quadruple, &c, till at last it swells up to a considerable bulk, greater or smaller, in proportion as I repeat greater or less the same idea. When I stop in the addition of parts, the idea of extension ceases to augment; and were I to carry on this addition* in infinitum, *I clearly perceive, that the idea of extension must also become infinite. Upon the whole I conclude, that the idea of an infinite number of parts is individually the same idea with that of infinite extension; that no finite extension is capable of containing an infinite number of parts; and consequently that no finite extension is infinitely divisible.*[1]

In a nutshell, Hume's argument is the following. No matter how small an actual extension may be, it can always be represented as an idea in the mind. Now suppose that an actual extension were divisible into infinitely many parts. Each of these parts can be represented as an idea, and so each is no smaller than the minimal idea of extension conceivable. But then the my idea of the given extension, as the sum of (the ideas of) its parts, would have to be as least as large as the sum of infinitely many minimal ideas of extension, and so itself infinite. Since the finite mind cannot contain an infinite idea, a contradiction results.

Hume's atomism, to which he refers as "the doctrine of indivisible points", was thus both conceptual and physical. Hume says that in order to be grasped these points or atoms of extension must possess sensible qualities such as colour or solidity:

> *The idea of space is convey'd to the mind by two senses, the sight and touch; nor does anything ever appear extended, that is not either visible or tangible. That compound impression, which represents extension, consists of several lesser impressions, that are indivisible to the eye or feeling, and may be call'd impressions of atoms or corpuscles endow'd with colour and solidity..*[2]..

According to Hume the doctrine of sensible points or atoms provides a way of avoiding the infinite divisibility of extension which mathematicians have always considered ineluctable:

> *It has often been maintained in the schools, that extension must be divisible,* in infinitum, *because the system of mathematical points is absurd; and that system is absurd, because a mathematical point is a non-entity, and consequently can never by its conjunction with others form a real existence. This wou'd be perfectly decisive, were there no medium betwixt the infinite divisibility of matter, and the non-entity of mathematical points. But there is evidently a medium, viz. the bestowing of colour or solidity on these points; and the absurdity of both the extremes is a demonstration of the truth and reality of this medium.* [3]

The exact nature of Hume's sensible, or indivisible points remains somewhat unclear. Should they be taken as extended entities like Epicurus's minima or as extensionless mathematical points? Not the former, since Hume denies them "real extension"; but also not the latter since they can be compounded to form extended magnitudes. Whatever his "points" may be, Hume admits that they are "entirely

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[1] *Ibid., II, 2,*

[2] *Ibid., II, 3*

[3] *Ibid., II, 4.*

useless" as providing a standard of comparison of the sizes of extended magnitudes, since it is impossible to compute the number of points these contain.

It has been suggested[1] that Hume later came round to the view that sensible points could be taken as minimal parts of extension, a position he decisively rejected in the *Treatise.* This aspect of Hume's doctrine remains puzzling and controversial.

KANT

The opposition between continuity and discreteness plays a significant role in the philosophical thought of **Immanuel Kant** (1724–1804). His mature philosophy, *transcendental idealism*, rests on the division of reality into two realms. The first, the *phenomenal* realm, consists of appearances or objects of possible experience, configured by the forms of sensibility and the epistemic categories. The second, the *noumenal* realm, consists of "entities of the understanding to which no objects of experience can ever correspond"[2], that is, things-in-themselves.

Regarded as magnitudes, appearances are spatiotemporally extended and continuous, that is infinitely, or at least limitlessly, divisible. Space and time constitute the underlying order of phenomena, so are ultimately phenomenal themselves, and hence also continuous:

> *The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity. Space and time are* quanta continua*, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is again a space or a time. Space therefore consists solely of spaces, time solely of times. Points and instants are only limits, that is, mere positions which limit space and time. But positions always presuppose the intuitions which they limit or are intended to limit; and out of mere positions, viewed as constituents capable of being given prior to space and time, neither space nor time can be constructed. Such magnitudes may also be called* flowing*, since the synthesis of productive imagination involved in their production is a progression in time, and the continuity of time is ordinarily designated by the term flowing or flowing away.* [3]

As objects of knowledge, appearances are continuous *extensive* magnitudes, but as objects of sensation or perception they are, according to Kant, *intensive* magnitudes. By an intensive magnitude Kant means a magnitude possessing a *degree* and so capable of being apprehended by the senses: for example brightness or temperature. Intensive magnitudes are entirely free of the intuitions of space or time, and "can only be presented as unities". But, like extensive magnitudes, they are continuous:

> *Every sensation... is capable of diminution, so that it can decrease and gradually vanish. Between reality in the field of appearance and negation there is therefore a continuity of many possible intermediate sensations, the difference between any two of which is always smaller than the difference between the given sensation and zero or complete negation.*[4]

Moreover, appearances are always presented to the senses as intensive magnitudes:

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[1] Furley (1967) p. 142.

[2] Körner (1955), p. 94.

[3] Kant (1964), p. 204.

[4] *Ibid.*, p. 203.

> *…the real in the field of appearance always has a magnitude. But since its apprehension by means of mere sensation always takes place in an instant and not through successive synthesis of different sensations, and therefore does not proceed the parts to the whole, the magnitude is to be met only in the apprehension. The real has therefore magnitude, but not extensive magnitude. ... Every reality in the field of appearance has therefore intensive magnitude.* [1]

Kant regards as "remarkable" the facts that

> *of magnitudes in general we can know* a priori *only a single* quality, *namely, that of continuity, and ... in all quality (the real in appearances) we can know nothing save [in regard to] their intensive* quantity, *namely that they have degree. Everything else has to be left to experience.*[2]

As for the concept of a thing-in-itself, it signifies "only the thought of something in general, in which I abstract from everything that belongs to the form of sensible intuition."[3] Kant seems to have regarded things-in-themselves as discrete entities in the sense of not being divisible to infinity, and hence, like Leibniz's real entities, as being compounded from simples. This may be inferred from his assertion in the *Metaphysical Foundations of Natural Science* of 1786 that a thing-in-itself "must in advance already contain within itself all the parts in their entirety into which it can be divided."[4] So were a thing-in-itself to be infinitely divisible, one could infer that it consists of an infinite multitude of parts. This, however, is impossible "because there is a contradiction involved in thinking of an infinite number as complete, inasmuch as the concept of an infinite number already implies that it can never be wholly complete."[5]

While Kant never deviated from the claim that space and time are divisible without limit, his opinion on the divisibility of matter underwent alteration. In the *Physical Monadology* of 1756, for example, he attempts to establish the compatibility of the *indivisibility* of physical monads or atoms with the infinite divisibility of space itself. Kant's argument is essentially that while substances must be compounded from simple parts and so cannot be infinitely divisible, this does not apply to space because it is not itself a substance but no more than a well-founded phenomenon, a "certain appearance of the external relation of substances".[6] He begins by arguing that bodies must be compounded from monads, or simple parts:

> *Bodies consist of parts, each of which separately has an enduring existence. Since, however, the composition of such parts is nothing but a relation, and hence a determination which is itself contingent, and which can be denied without abrogating the existence of the things having this relation, it is plain that all composition of a body can be abolished, though all the parts which were formerly combined together nonetheless continue to exist. When all composition is abolished, moreover, the parts which are left are not compound at all; and thus they are completely free from plurality of substances, and, consequently, they are simple. All bodies, whatever, therefore, consist of absolutely simple fundamental parts, that is to say, monads.*[7]

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[1] *Ibid., p.* 203.

[2] *Ibid.,* p. 208.

[3] *Ibid.,* p. 270.

[4] Kant (1970), p. 53. Cf. *Critique of Pure Reason, Observation on the Second Antinomy:*

> *Though it may be true that when a whole, made up of substances, is thought by the pure understanding alone, we must, prior to all composition, of it, have the simple...*

[5] *Ibid.,* p. 53.

[6] Kant's thus echoes Leibniz - with the exception that Leibniz's monads were not physical.

[7] *Physical Monadology,* Proposition II, in Kant (1992).

The infinite divisibility of space is then established by means of an argument similar to that in the *Port-Royal Logic,* with the consequence that space does not consist of simple parts. In a later commentary Kant remarks that from the fact that bodies are composed of monads and yet the space they occupy is infinitely divisible it would be wrong to infer that physical monads are "infinitely small particles of a body":

> *For it is abundantly plain that space, which is entirely free from substantiality and which is the appearance of the external relations of unitary monads, will not be exhausted by division continued to infinity. However, in the case of any compound whatever, where composition is nothing but an accident and in which there are substantial subjects of composition, it would be absurd if it admitted infinite division. For if a compound were to admit infinite division, it would follow that all the fundamental parts whatever of a body would be so constituted that, whether they were combined with a thousand, or ten thousand, or millions of millions—in a word, no matter how many—they would not constitute particles of matter. This would certainly and obviously deprive a compound of all substantiality; it cannot, therefore, apply to the bodies of nature.*[1]

That is, if bodies were infinitely divisible, their fundamental parts, as indivisibles, must perforce be unextended and could not then be recombined to form an extended object. It follows that each body consists of a determinate number of simple elements. Kant goes on to argue that while each such physical monad is not only situated in space, and actually fills the space it occupies, it does not follow from the admitted divisibility of that space that the monad is likewise divisible. In a commentary to this Kant remarks:

> *The line or surface which divides a small space into two parts certainly indicates that one part of the space exists outside the other. But since space is not a substance but a certain appearance of the external relation of substances, it follows that the possibility of dividing the relation of one and the same substance into two parts is not incompatible with the simplicity of, or if you prefer, the unity of that substance. For what exists on each side of the dividing line is not something which can be so separated from the substance that it preserves an existence of its own, apart from the substance itself and in separation from it, which would, of course, be necessary for real division which destroys simplicity. What exists on each side of the dividing line is an action which is exercised on both sides of one and the same substance: in other words, it is a relation, in which the existence of a certain plurality does not amount to tearing the substance itself into parts.*[2]

It is through its indivisibility or simplicity that a physical monad is distinguished from the space it happens to occupy. But this cannot of itself explain the fact that it occupies that *particular part* of space. The explanation, says Kant, is to be found in the relations of the monad with the substances external to it, and so ultimately in the monad's intherent *impenetrability.* This "prevents the monads immediately present to it on each side from drawing closer to each other" and so limits "the degree of proximity by which they are able to approach it". The monad thus "fills the space by the sphere of its activity". Kant identifies this activity as a *force.* The force has two components, one repulsive, preventing penetration by other monads; the other attractive, ensuring that the monad has a determinate form.

As with Leibnizian monads, it is therefore the *activity* of Kant's monads which prevents them from "collapsing" into mere geometrical points. And Kant follows Leibniz in describing this activity as a *force.* But Kant's monads, in contradistinction with Leibniz's, are material entities, so that, while Leibniz saw the "force" of his monads as

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[1] *Physical Monadology,* Scholium to Prop. IV.

[2] *Ibid.,* Scholium to Prop. V.

being akin to the impulses of the soul, Kant endowed his monads with the physical forces of attraction and repulsion.

In a startling *volte-face*, Kant came to repudiate his earlier view that matter is not divisible to infinity[1]. In the *Metaphysical Foundations of Natural Science* of 1786 he now insists that, in a space filled with matter, "every part of the space contains repulsive force to counteract on all sides the remaining parts", so that "every part of a space filled by matter is separable from the remaining parts, insofar as they are material substance." From the mathematical divisibility of the space to infinity there now follows the infinite divisibility of matter "into parts each of which is in turn material substance."

The upshot is that, like space and time, matter too must be assigned to the realm of appearance:

> *...space is no property appertaining to anything outside of our senses, but is only the subjective form of our sensibility. Under this form objects of our external senses appear to us, but we do not know them as they are constituted in themselves. We call this appearance matter.* [2]

But, again, the infinite divisibility of matter (or of space, time, or indeed any appearance) does not imply that it consists in *actuality* of infinitely many parts; the infinity involved is Aristotle's potential infinite:

> *That matter consists of infinitely many parts can indeed be thought by reason, even though this thought cannot be constructed and rendered intuitable. For with regard to what is actual only by being given in representation, there is not more given than is met with in the representation, i.e., as far as the progression of the representation reaches. Therefore, one can only say of appearances, whose division goes on to infinity, that there are as many parts of the appearance as we give, i.e., as far as we want to divide. For the parts insofar as they belong to the existence of an appearance exist only in thought, that is, in the division itself. The division indeed goes on to infinity, but it is never given as infinite; and hence it does not follow that the divisible contains within itself an infinite number of parts in themselves, that are outside of our representation, merely because the division goes on to infinity. For it is not the division of the thing but only the division of its representation that can be infinitely continued. Any division of the object (which is itself unknown) can never be completed and hence can never be entirely given. Therefore, any division of the representation proves no actual infinite multitude to be in the object (since such a multitude would be an express contradiction).* [3]

In the *Critique of Pure Reason* (1781) Kant brings a new subtlety (and, it must be said, tortuousity) to the analysis of the opposition between continuity and discreteness. This may be seen in the second of the celebrated Antinomies in that work, which concerns the question of the mereological composition of matter, or extended substance. Is it (*a*) discrete, that is, consists of simple or indivisible parts, or (*b*) continuous, that is, contains parts within parts *ad infinitum*? Although (*a*), which Kant calls the *Thesis* and (*b*) the *Antithesis* would seem to contradict one another, Kant offers proofs of both assertions. The resulting contradiction may be resolved, he asserts, by observing that while the antinomy "relates to the division of appearances"[4], the arguments for (*a*) and (*b*) implicitly treat matter or substance as things-in-themselves:

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[1] I am grateful to my colleague Lorne Falkenstein for pointing this out to me.

[2] Kant (1970), p. 55.

[3] *Ibid.,* p.54.

[4] Kant (1977), p. 83.

> *For these* [i.e., appearances] *are mere representations; and the parts exist merely in their representation, consequently in the division (i.e., in a possible experience where they are given) and the division reaches only so far as such experience reaches. To assume that an appearance, e.g., that of a body, contains in itself before all experience all the parts which any experience can ever reach is to impute to a mere appearance, which can exist only in experience, an existence previous to experience. In other words, it would mean that mere representations exist before they can be found in our faculty of our representations. Such an assumption is self-contradictory, as also every solution of our misunderstood problem, whether we maintain that bodies in themselves consist of an infinite number of parts or of a finite number of simple parts.* [1]

Kant concludes that both Thesis and Antithesis "presuppose an inadmissible condition" and accordingly "both fall to the ground, inasmuch as the condition, under which alone either of them can be maintained, itself falls."

Kant identifies the inadmissible condition as the implicit taking of matter as a thing-in-itself, which in turn leads to the mistake of taking the division of matter into parts to subsist independently of the act of dividing. In that case, the Thesis implies that the sequence of divisions is finite; the Antithesis, that it is infinite. These cannot be both be true of the *completed* (or at least completable) sequence of divisions which would result from taking matter or substance as a thing-in-itself.[2] Now since the truth of both assertions has been shown to follow from that assumption, *it* must be false, that is, matter and extended substance are appearances only. And for appearances, Kant maintains, divisions into parts are not completable in experience, with the result that such divisions can be considered neither finite nor infinite:

> *We must therefore say that the number of parts in a given appearance is in itself neither finite nor infinite. For an appearance is not something existing in itself, and its parts given in and through the regress of the decomposing synthesis, a regress which is never given in its absolute completeness, either as finite or infinite.*[3]

It follows that, for appearances, both Thesis and Antithesis are *false.*

Later in the *Critique* Kant enlarges on the issue of divisibility:

> *If we divide a whole which is given in intuition, we proceed from something conditioned to the conditions of its possibility. The division of the parts...is a regress in the series of these conditions. The absolute totality of this series would be given only if the regress could reach* simple *parts. But if all the parts in a continuously progressing decomposition are themselves again divisible, the division, that is, the regress from the conditioned to its conditions*, proceeds in infinitum*. For the conditions (the parts) are themselves contained in the conditioned, and since this is given complete in an intuition that is enclosed between limits, the parts are all one and all given together with the conditioned. The regress may not, therefore be entitled merely a regress* in indefinitum. *... Nevertheless we are not entitled to say of a whole which is divisible to infinity that* it is made up of infinitely many parts. *For although all parts are contained in the intuition of the whole,* the whole division *is not so contained, but consists only in the continuous decomposition, that is, the regress itself, whereby the series first becomes actual. Since this regress is infinite, all the members or parts at which it arrives are contained in the whole, viewed as an* aggregate. *But the whole* series of the division *is not so contained, for it is a*

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[1] *Ibid.*

[2] As already observed, Kant would probably maintain the truth of the Thesis in that event.

[3] Kant (1964), p. 448.

> *successive infinite and never* whole, *and cannot, therefore, exhibit an infinite multiplicity, or any combination of an infinite multiplicity in a whole.*[1]

What Kant seems to be saying here is that, while each part generated by a sequence of divisions of an intuited whole is given with the whole, the sequence's incompletability prevents *it* from forming a whole; *a fortiori* no such sequence can be claimed to be actually infinite.

Kant draws similar conclusions in respect of his First Antinomy, which concerns the opposition between the boundedness and unboundedness of space and time. Jonathan Bennett, in his thought-provoking paper, *The Age and Size of the World*[2], sums up the conclusion of the First Antinomy as follows:

> *Although Kant denies that the world can be infinitely old or large, he thinks that it cannot be finitely large or old either.*

In explicating this assertion, Bennett concludes that what Kant means by "the world is not finite in size" is "no finite amount of world includes all the world there is", or "every finite quantity of world excludes some world". Bennett submits that this last statement "seems to Kant to be a weaker statement than the statement that there is an infinite amount of world." More generally, Bennett suggests that

> *Kant is one of those who think that*
>
> *Every finite set of F's excludes at least one F,* (1)
>
> *though it contradicts the statement that there are only finitely many F's, is nevertheless weaker than*
>
> *There is an infinite number of F's* (2).

Bennett implies that Kant is simply mistaken here, that in fact (1) and (2) are equivalent. But is this right? Let bring to bear some contemporary mathematical ideas on the matter.

Call a set A *finite* if for some natural number n, all the members of A can be enumerated as a list $a_0, \ldots, a_n$; *potentially infinite* if it is not finite, that is, if, for any natural number n, and any list of n members of A, there is always a member of A outside the list; *actually infinite* if there is a list $a_0, \ldots, a_n, \ldots$(one for each natural number n) of *distinct* members of A; and *Kantian* if it is potentially, but not actually infinite, that is, if it is neither finite nor actually infinite[3]. Now *it is possible for a set to be Kantian*, just as Kant (according to Bennett) thought the actual world was.

For suppose that we are given a potentially infinite set A, and we attempt to show that it is actually infinite by arguing as follows. We start by picking a member a_0 of A; since A is potentially infinite, there must be a member of A different from a_0; pick such a member and call it a_1. Now again by the fact that A is potentially infinite, there is a member of A different from a_0, a_1—pick such and call it a_2. In this way we generate a list $a_0, a_1, a_2, \ldots$ of distinct members of A; so, we are tempted to conclude, A is actually infinite. But clearly the cogency of this argument hinges on our presumed ability to "pick", for each n, an element of A distinct from $a_0, \ldots, a_n$—an ability enshrined in the

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[1] *Ibid.*, p. 459.

[2] Bennett (1971).

[3] In the usual set-theoretic terminology, my term "potentially infinite" corresponds to "infinite"; "actually infinite" to "transfinite" or "Dedekind infinite"; and "Kantian" to "infinite Dedekind finite".

set-theoretic principle known as the *axiom of choice.*[1] Now the axiom of choice is, as Gödel showed in 1938[2], a perfectly consistent mathematical assumption. But, as Paul Cohen showed in 1964[3], its denial is equally consistent. In fact, it can be denied in such a way as to prevent the argument just presented from going through, that is, to allow the presence of potentially infinite sets which are not at the same time actually infinite. That is, the existence of Kantian sets is consistent with the axioms of set theory (and classical logic) *as long as the axiom of choice is not assumed.*

The axiom of choice may be regarded as a principle ensuring that the universe of sets is "static" in the sense that families of sets "sit still" long enough to enable elements to be extracted from them. Accordingly the existence of "Kantian" sets is compatible with classical set theory, as long as it has not been rendered "overstatic" through the imposition of the axiom of choice.

Another, more direct way of obtaining a Kantian set is to allow our "sets" to undergo *explicit variation*, to wit, variation over discrete time (that is, over the natural numbers). For consider the following universe of discourse[4] $\mathcal{U}$. Its objects are all sequences of maps between sets:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots A_n \xrightarrow{f_n} A_{n+1} \longrightarrow \cdots$$

Such an object may be thought of as a set $\boldsymbol{A}$ "varying over (discrete) time": A_n is its "value" at time n. Now consider the temporally varying set

$$\boldsymbol{K} = (\{0\} \longrightarrow \{0,1\} \longrightarrow \{0,1,2\} \cdots \longrightarrow \{0,1,\ldots,n\} \longrightarrow \cdots)$$

in which all the arrows are identity maps. In $\mathcal{U}$, $\boldsymbol{K}$ "grows" indefinitely and hence potentially infinite. On the other hand at each specific time it is finite and so is not actually infinite. In short, in the universe of "sets through time", $\boldsymbol{K}$ is a Kantian set.

It would seem then that Kant's conclusion in the First Antinomy that space and time are neither finite nor infinite—that is, potentially infinite—*is* coherent (or at least consistent) after all. This applies equally to the Second Antinomy, at least in respect of Kant's contention that ongoing sequences of divisions of appearances cannot be assumed to terminate, and are accordingly neither finite nor infinite. But these assertions only make sense in a conceptual framework conceived as undergoing some form of variation[5].

HEGEL

The concepts of continuity and discreteness, albeit in a unorthodox and esoteric form, play an important role in the philosophy of **G. W. F. Hegel** (1770–1831). Hegel saw continuity and discreteness as being locked in an indissoluble dialectical relationship—a "unity of opposites". Continuity and discreteness are the "moments", that is, the

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[1] A straightforward version of the axiom of choice is the following: for any collection $\mathcal{A}$ of nonempty sets no two of which have common elements there is a set having exactly one element in common with each member of $\mathcal{A}$.

[2] Gödel (1940).

[3] Cohen (1966).

[4] *Cognoscenti* will recognize $\mathcal{U}$ as the topos of sets varying over the natural numbers. But within this universe not only the axiom of choice but also the law of excluded middle has to be abandoned, a course that Kant would likely have found most unpalatable.

[5] See Chapter 9 below.

defining or constituting attributes, of the category of Quantity; the latter is itself a "simple unity of Discreteness and Continuity".[1]

In Hegel's conception continuity is, as it was for Parmenides, first and foremost a form of unity or identity; in the *Science of Logic* (1812–16) continuity is characterized as

> *simple and self-identical self-relation, interrupted by no limit or exclusion; not however, an immediate unity, but a unity of the Ones which are for themselves. The externality of plurality is still here contained, but as something undifferentiated and uninterrupted. In continuity, plurality is posited as it is in itself; each of the many is what the others are, each is equal to the other, and hence plurality is simple and undifferentiated equality.*[2]

Continuity thus still entails plurality, but as "something undifferentiated and uninterrupted." In continuity, Hegel says

> *plurality is posited as it is itself; each of the many is what the others are, each is equal to the other, and hence [the] plurality is simple and undifferentiated equality.* [3]

He remarks that imagination "easily changes Continuity into Combination, that is, into an external relation of the Ones to one another"[4] . But on the other hand, "continuity is not external but peculiar to [the One], and founded in its essence". Atomism, says Hegel, "remains entangled" in this "externality of continuity". By contrast, mathematics

> *rejects a metaphysic which should be content to allow time to consist of points of time, space in general (or as a first step, the line) of points in space, the plane, of lines, and the whole of space, of planes; it allows no validity to such discontinuous Ones. And although it determines, for instance, the magnitude of a plane as consisting of the sum of an infinity of lines, yet this discreteness is taken only as a momentary image; and the infinite plurality of lines implies, since the space which they are meant to constitute is after all limited, that their discreteness has already been transcended.*[5]

It is to this fact, Hegel says, that we must attribute "the conflict or Antinomy of the infinite divisibility of Space, of Time, of Matter, and so on.": here he is referring to Kant's second Antinomy in the *Critique of Pure Reason.* For Hegel,

> *This antinomy consists solely in the necessity of asserting Discreteness as much as Continuity. The one-sided assertion of Discreteness gives an infinite or absolute division (and thus something indivisible) for principle; and the one-sided assertion of Continuity, infinite divisibility.* [6]

Hegel finds Kant's analysis of the antinomy wanting in that an absolute separation is made, inadmissibly, between continuity and discreteness. He writes:

> *Looked at from the point of view of mere discreteness, substance, matter, space and time, and so on, are absolutely divided, and the One is their principle. From the point of view of continuity, this One is merely suspended: division remains divisibility, the possibility of dividing remains possibility, without ever actually reaching the atom. Now ... still continuity contains the moment of the atom, since continuity exists simply*

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[1] Hegel (1961), p. 204.
[2] *Ibid.,* p. 200.
[3] *Ibid.*
[4] *Ibid.*
[5] *Ibid.,* pp. 202-3.
[6] *Ibid.,* p. 204.

> *as the possibility of division; just as accomplished division, or discreteness, cancels all distinctions between the Ones—for each simple One is what every other is, —and for that very reason contains their equality and therefore their continuity. Each of the two opposed sides contains the other in itself, and neither can be thought of without the other; and thus it follows that, taken alone, neither determination has truth, but only their unity. This is the true dialectic consideration of them, and the true result.* [1]

Hegel next embarks on a discussion of continuous and discrete magnitude. Having observed that Quantity, which embodies both continuity and discreteness, continuous magnitude is identified as Quantity "posited only in one of its determinations, namely, continuity."[2] Now "continuity is one of the moments of Quantity which requires the other moment, discreteness, to complete it", so that

> *continuity is only coherent and homogeneous unity as unity of discrete elements, and posited thus, it is no longer mere moment but complete Quantity: this is Continuous Magnitude.* [3]

Quantity is identified by Hegel as "externality in itself", and Continuous Magnitude is "this externality as propagating itself without negation, as a context which remains at one with itself." By contrast Discrete Magnitude is "this externality as non-continuous or interrupted". If continuity is identity, then discreteness is distinguishability. But while Discrete Magnitude is multiplicity, but this multiplicity is not that of "the multitude of atoms and the void", for

> *Discrete Magnitude is Quantity; and for that very reason* [its] *discreteness is continuous. The continuity in discreteness consists in the fact that the Ones are equal to one another, or have the same unity. Discrete magnitude, then, is the externality of much One posited as the same, and not of the many Ones in general; it is posited as the Many of one unity.*[4]

For Leibniz, the Many in One was manifested in continuous extension, but Hegel saw it in discrete magnitude.

There is an interesting attempt by Bertrand Russell, in *The Principles of Mathematics,* to elucidate the Hegelian conception of continuity and discreteness:

> *The word* continuity *has borne among philosophers, especially since the time of Hegel, a meaning totally unlike that given to it by Cantor. Thus Hegel says: "Quantity, as we saw, has two sources: the exclusive unit, and the identification or equalization of these units. When we look, therefore, at its immediate relation to self, or at the characteristic of selfsameness made explicit by abstraction, quantity is* Continuous *magnitude; but when we look at the One implied in it, it is* Discrete *magnitude." When we remember that quantity and magnitude, in Hegel, both mean "cardinal number", we may conjecture that this assertion amounts to the following: "Many terms, considered as having a cardinal number, must all be members of one class; in so far as they are merely an instance of the class-concept, they are indistinguishable from one another, and in this aspect the whole that they compose is called* continuous; *but in order to their manyness, they must be* different *instances of the class-concept, and in this respect the whole that they compose is called* discrete.[5]

If this is right, then Hegel's conception of *discrete* magnitude may be seen as corresponding to Cantor's famous definition of *set*:

> *By a "set" we mean any collection M into a whole of definite distinct objects m... of our perception or thought.* [6]

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[1] *Ibid.,* p. 211.
[2] *Ibid.,* p. 213.
[3] *Ibid.,* p. 214.
[4] *Ibid.,* p. 214.
[5] Russell (1964), p. 346.
[6] Quoted in Dauben (1979), p. 170.

And Hegel's conception of *continuous* magnitude would further correspond to Cantor's notion of *power* or *cardinal number:*

> *By the* power *or* cardinal number *of a set M (which consists of distinct, conceptually separate elements m, m′, ... and is to this extent determined and limited), I understand the general concept or generic character (universal) which one obtains by abstracting from the elements of the set, as well as from all connections which the elements may have (be it between themselves or other objects), but in particular from the order in which they occur, and by reflecting only upon that which is common to all sets which are equivalent to M.* [1]

It is delightfully dialectical that Hegel seems to have identified as continuous what Cantor held to be discrete.

The *Science of Logic* contains an extensive discussion of the ideas underlying the calculus. Like Berkeley, d'Alembert and Lagrange, Hegel was critical of the use mathematicians had made of infinitesimals and differentials. But far from rejecting the infinitesimal, Hegel was concerned to assign it a proper location within his philosophical scheme, whose reigning principle was the division of reality into the triad of Being, Nothing, and Becoming. For Cavalieri infinitesimals possessed Being, and for Euler they were Nothing, but for Hegel they fell under the category of Becoming. He writes:

> *In an equation where x and y are posited primarily as determined through a ratio of powers, x and y as such are still meant to denote Quanta*[2]: *now this meaning is entirely lost in the so-called infinitesimal differences. dx and dy are no longer Quanta and are not supposed to signify such; they have a significance only in their relation, a meaning merely as moments. They no longer are Something (Something being taken as Quantum), nor are they finite differences; but they are also not Nothing or the indeterminate nil. Apart from their relation they are pure nil; but they are meant to be taken as moments of the relation, as determinations of the differential coefficient* $\frac{dx}{dy}$.[3]
>
> *Now when the mathematics of the infinite* [i.e., the infinitesimal] *still maintained that these quantitative determinations were vanishing magnitudes, that is, magnitudes which no longer are any Quantum but also not nothing, it seemed abundantly clear that that such an intermediate state, as it was called, between Being and Nothing did not exist. ... The unity of Being and Nothing is indeed not a* state; *for a state would be a determination of Being and Nothing such as might have been reached by these moments only contingently, as it were through disease or external influence, and through erroneous thinking; but, on the contrary, this mean and unity, this vanishing and, equally, Becoming is, in fact, their only truth.*[4]

In Hegel's subsequent review of how the infinitesimal has been conceived by mathematicians of the past, those who regarded infinitesimals as fixed quantities receive short shrift, while those who saw infinitesimals in terms of the limit concept (which in Hegel's eyes fell under the appropriate category of Becoming) are praised. Thus, for example, Newton is praised for his explanation of fluxions (in the *Principia,*

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[1] Quoted *ibid.,* p. 221.

[2] By *Quantum* Hegel means determinate Quantity, that is, Quantity of a definite size.

[3] Hegel (1961), p. 269.

[4] *Ibid.,* pp. 269–70.

see ….) not in terms of indivisibles, but in terms of "vanishing *divisibilia*", and, further, "not [in terms of] sums and ratios of determinate parts, but [in terms of] the limits (*limites*) of the sums and ratios."[1] Newton's conception of generative or variable magnitudes also receives Hegel's endorsement. He quotes Newton to the effect that

> *"[Any finite magnitude] is considered as variable in its incessant motion and flow of increase and decrease, and so its momentary augmentation [I give] the name of* Moments. *These, however, must not be taken as particles of determinate magnitude (particulae finitae). They are not moments themselves, but magnitudes produced by moments; the generative principles or beginnings of finite magnitudes must here be understood.*

And then comments:

> *—An internal distinction is here made in Quantum; it is taken first as product or Determinate Being, and next in its Becoming, as its beginning and principle, that is , as it is in its concept or (what is the same thing) in its qualitative determination. In the latter the qualitative differences, the infinitesimal incrementa or decrementa, are moments only; and it is only in what has been generated that we have Quantum or that which has passed over into the indifference of determinate existence and into externality.* [2]

The use of fixed infinitesimals, on the other hand, Hegel deplores:

> *The idea of infinitely small quantities (latent also in increment and decrement) is far inferior to the mode of conception* [just] *indicated. The idea supposes them to be of such a nature that they may be neglected in relation to finite magnitudes; and not only that, but also their higher orders relative to the lower order, and the products of several relative to one.—With Leibniz this demand to neglect (which previous inventors of methods referring to this kind of magnitude also bring into play) becomes more strikingly prominent. It is this chiefly which gives an appearance of inexactitude and express incorrectness, the price of convenience, to this calculus in the course of its operation.*[3]

Nor does Euler's view of infinitesimals as formal zeros fare much better:

> *In this regard Euler's idea especially must be cited. On the basis of Newton's general definition, he insists that the differential calculus considers the ratios of incrementa of a magnitude, while the infinitesimal difference as such is to be regarded wholly as nil.—It will be clear from the above how this is to be understood: the infinitesimal difference is nil only quantitatively, it is not a qualitative nil, but, as nil of quantum, it is pure moment of a ratio only. There is no magnitudinal difference; but for that reason it is, in a manner, wrong to express as incrementa or decrementa as as differences those moments which are called infinitely small magnitudes. ... the difficulty is self-evident when it is said that for themselves the incrementa are each nil, and that only their ratios are being considered; for a nil is altogether without determinateness. Thus this image, although it reaches the negative aspect of Quantum and expressly asserts it, yet does not simultaneously seize this negative in its positive meaning of qualitative determinations of quantity, which, if torn away from the ratio and treated as Quanta, would each be but a nil.*[4]

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[1] *Ibid.,* p. 271.
[2] *Ibid.,* p. 274.
[3] *Ibid.*
[4] *Ibid.,* p. 275–6.

Hegel goes on to discuss some of the methods that mathematicians have employed to resolve the conceptual difficulties caused by the use of infinitesimals. He pays particular attention to Lagrange's attempt to eliminate infinitesimals from the calculus through the use of Taylor expansions. Hegel considers that the Taylor expansion of a function "must not only be regarded as a sum, but as qualitative moments of a conceptual whole."[1] That being the case, he says, the basic calculus procedure of omitting from the Taylor series terms with higher powers has "a significance wholly different from that which belongs to their omission on the ground of relative smallness"[2] He continues:

> *And here the general assertion can be made that the whole difficulty of the principle would be removed, if the qualitative meaning of the principle were indicated and the operation made dependent on it, in place of the formalism which identifies the determination of the differential with the problem which gives it its name, the distinction generally between a function and its variation after its variable magnitude has received an increment. In this sense it is clear that the first term of the series resulting from the development of* $(x + dx)^n$ *quite exhausts the differentia of* x^n*. Thus the neglect of the other terms is not due to their relative smallness;—and no inexactitude, no mistake or error is here assumed which is supposed to be compensated or rectified by another error .... A ratio, and not a sum, is here in question; and, therefore, with the first term the differential is fully found...* [3].

Hegel (correctly) regards the differential coefficient $\frac{dy}{dx}$ as the limit of a *ratio* in which the term *dy*, in particular, is not to be "taken as difference or increment in the sense that it is only the numerical difference of the Quantum from the Quantum of another ordinate."[4] If differentials *are*, incorrectly, construed in this way, the matter is "obscured", for then:

> *Limit has not here the meaning of ratio: it counts only as the ultimate value which another and similar magnitude steadily approaches in such a manner, that the difference between them may be as small as desire, and the ultimate relation a relation of equality. Thus the infinitesimal difference is the ghost of a difference between one Quantum and another, and, when it is thus imagined, the qualitative nature, according to which dx is related essentially as a determination of ratio not to x but to dy, is in the background. ...—In this kind of determination, geometers are chiefly at pains to make intelligible the approximation of a magnitude to its limit, clinging to this aspect of the difference between Quantum and Quantum, where it is no difference and yet still is a difference. But in any case Approximation is a category which in itself means and makes intelligible nothing; dx has already passed through approximation, it is neither near nor is it nearer; and "infinitely near" itself means the negation of nearness and approximation.*[5]

Hegel observes that this incorrect construal of differentials amounts to considering "the incrementa or infinitesimal differences ... only from the side of the Quantum that vanishes in them, and as the limit of this: they are, then, taken as unrelated moments."[6]. And from this, he says, "the inadmissible idea would follow, that it would be permissible to equate in the ultimate ratio abscissae and ordinates, or else sines and

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[1] *Ibid.,* p. 280.

[2] *Ibid.,* p. 280-1. Hegel regards as highly dubious the procedure of omitting terms in a sum because of their "relative smallness".

[3] *Ibid.,* p. 281–2.

[4] *Ibid.,* p. 287.

[5] *Ibid.*

[6] *Ibid.*

cosines, tangents, and versed sines,—anything, in fact." That is to say, the illegitimate equating of entities of differing *types.* Interestingly, Hegel does not see this inadmissible procedure at work when infinitesimal portions of curves are taken to be straight lines. He writes:

> *This idea seems to be operating when a curve is treated as a tangent; for the curve too is incommensurable with the straight line, and its element of a different quality from the element of the straight line. And it seems even more irrational and less permissible than the confusion of abscissae and ordinates, versed sine and cosine, and so forth, when* (quadrata rotundis) *a part—though infinitely small—of a curve is taken for part of a tangent and thus is treated as a straight line.—However this treatment differs essentially from the confusion we have just denounced; and this is its justification,—in the triangle*[1]*, which has for its sides the element of a curve and the elements of its abscissae and ordinates, the relation is the same as though this element of the curve were the element of a straight line—the tangent: the angles, which constitute the essential relation (that is, that which remains in these elements after abstraction made from the finite magnitudes belonging to them), are the same.*[2]

He concludes:

> *We can also express ourselves in this matter as follows:— straight lines, as being infinitely small, have passed over into curves, and the ratio which subsists between them in their infinity is a ratio of curves. The straight line according to its definition is the shortest distance between two points, and therefore its difference from the curve is based upon the determination of amount, upon the smaller number of distinguishable steps on this route,—and this is a quantitative determination. But when it is taken as intensive magnitude* [3]*, or infinite moment, or element, this determination vanishes with it, and with it vanishes the difference from the curve, which is based solely on a difference in Quantum.—Taken as infinitesimal, therefore, straight line and curve have no quantitative relation, and hence (on the basis of the accepted definition) no qualitative difference relatively to each other: the latter now passes into the former.* [4]

Hegel is often regarded a philosopher who did not take mathematics very seriously. The fact that he devoted a substantial portion of *The Science of Logic* to the infinitesimal calculus speaks to the contrary[5].

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[1] I.e., the differential triangle.

[2] *Ibid.,* p. 287–8.

[3] Hegel distinguishes between *extensive* and *intensive* magnitude. When a magnitude is regarded as a multiplicity, it is extensive; regarded as a unity, it is intensive.

[4] *Ibid.,* p. 288.

[5] Hegel's disciple Karl Marx was also preoccupied by the infinitesimal calculus. See Marx (1983).