

Chapter 4

The Reduction of the Continuous to the Discrete in the 19th and Early 20th Centuries

BOLZANO AND CAUCHY

The rapid development of mathematical analysis in the 18th century had not concealed the fact that its underlying concepts not only lacked rigorous definition, but were even (e.g. in the case of differentials and infinitesimals) of doubtful logical character. The lack of precision in the notion of continuous function—still vaguely understood as one which could be represented by a formula and whose associated curve could be smoothly drawn—had led to doubts concerning the validity of a number of procedures in which that concept figured. For example when Lagrange had formulated his method for “algebraizing” the calculus he had implicitly assumed that every continuous function could be expressed as an infinite series by means of Taylor’s theorem. Early in the 19th century this and other assumptions began to be questioned, thereby initiating an inquiry into what was meant by a function in general and by a continuous function in particular.

A pioneer in the matter of clarifying the concept of continuous function was the Bohemian priest, philosopher and mathematician **Bernard Bolzano** (1781-1848). In his *Reine analytischer Beweis* of 1817 he defines a (real-valued) function f to be continuous at a point x if the difference $f(x + \omega) - f(x)$ can be made smaller than any preselected quantity once we are permitted to take ω as small as we please. This is essentially the same as the definition of continuity in terms of the limit concept given a little later by Cauchy. Using this definition Bolzano goes on to show that the both the difference and the composition of any two continuous functions is continuous. Bolzano also formulated a definition of the derivative of a function free of the notion of infinitesimal, which later became standard:

I have no need then, of so restrictive a hypothesis in this matter as the one so often considered necessary, to wit: that the quantities to be calculated can become infinitely small. ... I ask one thing only: that these quantities, when they are variable, and not independently variable but dependently variable upon one or more other quantities, should possess a derivative, what Lagrange calls “une fonction dérivée”—if not for every value of their determining variable, then at least for all the values to which the process is to be validly applied. In other words: when x designates one of the independent variables, and $y = f(x)$ designates a variable dependent on it, then, if our calculation is to give a correct result for all values of x between $x = a$ and $x = b$, the mode of dependence of y upon x must be such that for all values of x between a and b the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(which arises from the division of the increase in y by the increase in x) can be brought as close as we wish to some constant, or to some quantity $f'(x)$ depending solely on x , by taking Δx sufficiently small; and subsequently, on our making Δx smaller still, either remains as close thereto or comes closer still.¹

¹ Bolzano (1950), p. 123.

Sometime before 1830 Bolzano invented a process for constructing continuous but nowhere differentiable functions¹. In this he anticipated Weierstrass's better known construction by some thirty years.

Like Galileo before him, Bolzano considered that actual infinity could come into existence through aggregation. In particular he held that a continuum was to be regarded as an infinite aggregate of points, and, like Ockham and Leibniz, he thought that density alone was sufficient for a set of points to make up a continuum:

If we try to form a clear idea of what we call a 'continuous extension' or 'continuum', we are forced to declare that a continuum is present when, and only when, we have an aggregate of simple entities (instants or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbour.

...one final question might be posed: how are we to interpret the assertion of those mathematicians who declare both that extension can never be generated by the mere accumulation of points however numerous, and also that it can never be resolved into simple points? Strictly speaking, we should on the one hand certainly teach that extension is never produced by a finite set of points, and produced by an infinite set only when, but always when, the...oft-mentioned condition is fulfilled—namely that each point of the set has, at each sufficiently small distance, a neighbour also belonging to the set; and on the other hand we should admit that not every partition of a spatial object works down to its simple parts: in no case one whose subsets are finite in number, and not even all that subdivide to infinity, say by successive halvings. Nevertheless, we must still insist that every continuum can be made up in the last analysis of points and points alone. Once the two [apparent opposites] are properly understood, they are perfectly consistent with one another. ²

Bolzano admitted the existence of infinitesimal quantities as the reciprocals of infinitely great quantities:

Now that the possibility of calculating with the infinitely great has been vindicated..., we assert the like for the infinitely small. For if N_0 is infinitely great, $\frac{1}{N_0}$ necessarily represents an infinitely small quantity, and we shall have no reason for denying objective reference to such an idea, at any rate in the general theory of quantity.³

Bolzano repudiated Euler's treatment of differentials as formal zeros in expressions such as dy/dx . He suggested instead that in determining the derivative of a function, increments Δx , Δy ,... be *finally* set to zero. He writes:

Once an equation between x and y is given it is usually a very easy and well-known matter to find [the] derivate of y . If for example

$$y^3 = ax^2 + a^3,$$

then we should have for ever Δx other than zero,

$$(y + \Delta y)^3 = a(x + \Delta x)^2 + a^3,$$

whence by the known rules

$$\frac{\Delta y}{\Delta x} = \frac{2ax + a\Delta x}{3y^2 + 3y\Delta y + \Delta y^2}$$

¹ *Ibid.*, p. 30.

² *Ibid.* pp. 129-131.

³ *Ibid.*, p. 109.

and the derived function of y , or in Lagrange's notation y' , would be discovered to be

$$\frac{2ax}{3y^2},$$

a function obtained from the expression for

$$\frac{\Delta y}{\Delta x}$$

by first suitably developing it, namely into a fraction whose numerator and denominators separate the terms multiplying Δx and Δy from those which do not, and then putting Δx and Δy equal to zero in the expression

$$\frac{2ax + a\Delta x}{3y^2 + 3y\Delta y + \Delta y^2}$$

thus arrived at.¹

Bolzano then remarks that it is perfectly in order to symbolize the derivative by $\frac{dy}{dx}$, provided that two things are understood, namely:

- (i) that all the Δx and Δy (or, if you like, the dx or dy written in their stead) which occur in the development of $\Delta y/\Delta x$ are to be regarded and treated as mere zeros; and
- (ii) that the symbol dy/dx shall not be regarded as the quotient of dy by dx , but expressly and exclusively as a symbol for the derivate of y with respect to x .²

Expressions involving differentials are construed by Bolzano as having a purely formal significance. In particular differentials themselves are never to be regarded as "the symbols of actual quantities, but always as equivalents to zero". An equation in which differentials figure is to be considered as

*nothing but a compound symbol so constituted that (i) if we carry out only such changes as algebra allows with the symbols of actual quantities (in this case, therefore, also divisions by dx and the like) and (ii) if we finally succeed in getting rid of the symbols dx , dy and so forth on both sides of the equation, then no false result will ever be the outcome.*³

Thus for Bolzano differentials have the status of "ideal elements", purely formal entities such as points and lines at infinity in projective geometry, or (as Bolzano himself mentions) imaginary numbers, whose use will never lead to false assertions concerning "real" quantities.

Given Bolzano's construal of infinitesimals as pure symbols, it was natural that he repudiated as chimerical the idea of actual infinitesimal geometric entities such as infinitesimal lines or areas:

Infinitely small distances have been assumed hardly less often than infinitely large ones, especially when it seemed necessary to treat as straight (or plane) those lines (or surfaces) of which no portion was both extended and really straight (or plane) on the plea for example of more easily determining their arcual length or the magnitude of their curvature.... People even went so far as to invent fictitious distances supposed

¹ *Ibid.*, p. 124. Bolzano's calculation here is formally equivalent to the procedure adopted in smooth infinitesimal analysis in which squares and higher powers of infinitesimals (but not the infinitesimals themselves) are set to zero: see Chapter 10 below.

² *Ibid.*, p. 125.

³ *Ibid.*, p. 127.

to be measured by infinitely small quantities of the second order, of the third order, or even of still higher orders.

Now if this process seldom led to false results, particularly in geometry, the only circumstance we can thank for it was...that in order to apply to determinable spatial extensions, variable quantities must be so constituted that, with the exception at most of isolated individual values, their first and second and all subsequent derivatives exist. For if they do exist, then what was being asserted of the 'infinitely small' lines and surfaces and volumes can, as a general rule, quite rightly be asserted of all lines and surfaces and volumes which, while always remaining finite, nevertheless can be taken as small as we please, or as we express it, infinitely decreased. The former assertions, mistakenly applied to 'infinitely small distances' were thus really true of the 'variable quantities'.

It can thus be understood, however, that such methods of describing the situation were bound to produce, and appear to prove, much that was paradoxical and even quite false. How scandalous it sounded, for example, when they said that every curve and surface was nothing but an assemblage of infinitely many straight lines and plane surfaces, which only needed to be considered in infinite multitude; and aggravated even this by adding the supposition of infinitely small lines and surfaces which were themselves curved...¹

In reality, however, infinitely small arcs are just as non-existent as infinitely small chords, and the statements which mathematicians make about their so-called 'infinitely small arcs and chords' are statements which they only prove for arcs and chords that we can take as small as we please.²

Although Bolzano anticipated the form that the rigorous formulation of the concepts of the calculus would assume, his work was largely ignored in his lifetime. The cornerstone for the rigorous development of the calculus was supplied by the ideas—essentially similar to Bolzano's—of the great French mathematician **Augustin-Louis Cauchy** (1789–1857). Cauchy's approach is presented in the three treatises *Cours d'analyse de l'École Polytechnique* (1821), *Résumé des leçons sur le calcul infinitésimal* (1823), and *Leçons sur le calcul différentiel* (1829). In Cauchy's work, as in Bolzano's, a central role is played by a purely arithmetical concept of limit freed of all geometric and temporal intuition. In the *Cours d'analyse* Cauchy defines the limit concept in the following way:

When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it as little as one wishes, this last is called the limit of the others.³

In the *Cours* Cauchy also formulates the condition for a sequence of real numbers to converge to a limit, and states his familiar criterion for convergence⁴, namely, that a sequence $\langle s_n \rangle$ convergent if and only if $s_{n+r} - s_n$ can be made less in absolute value than any preassigned quantity for all r and sufficiently large n . Cauchy proves that this is necessary for convergence, but as to sufficiency of the condition merely remarks "when the various conditions are fulfilled, the convergence of the series is assured."⁵ In making this latter assertion he is implicitly appealing to geometric intuition, since he makes no

¹ *Ibid.*, pp. 139-40.

² *Ibid.*, pp. 172-73.

³ Quoted in Kline (1972), p. 951.

⁴ This had been previously given by Bolzano.

⁵ Kline (1972), p. 963.

attempt to define real numbers, observing only that irrational numbers are to be regarded as the limits of sequences of rational numbers.

Cauchy chose to characterize the continuity of functions in terms of a rigorized notion of infinitesimal, which he defines in the *Cours d'analyse* as “a variable quantity [whose value] decreases indefinitely in such a way as to converge to the limit 0.” Here is his definition of continuity:

Let $f(x)$ be a function of the variable x , and suppose that, for each value of x intermediate between two given limits [bounds], this function constantly assumes a finite and unique value. If, beginning with a value of x contained between these two limits, one assigns to the variable x an infinitely small increment α , the function itself will take on as increment the difference $f(x + \alpha) - f(x)$, which will depend at the same time on the new variable α and on the value of x . This granted, the function $f(x)$ will be, between the two limits assigned to the variable x , a continuous function of the variable if, for each value of x intermediate between these two limits, the numerical value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely with that of α . In other words, the function $f(x)$ will remain continuous with respect to x between the given limits if, between these limits, an infinitely small increment of the variable always produces an infinitely small increment of the function itself.

We also say that the function $f(x)$ is a continuous function of x in the neighbourhood of a particular value assigned to the variable x , as long as it [the function] is continuous between these two limits of x , no matter how close together, which enclose the value in question.¹

Cauchy's definition of continuity of $f(x)$ in the neighbourhood of a value a amounts to the condition, in modern notation, that $\lim_{x \rightarrow a} f(x) = f(a)$.

In the *Résumé des leçons* Cauchy defines the derivative $f'(x)$ of a function $f(x)$ in a manner essentially identical to that of Bolzano. He then defines the differentials dy and dx in terms of the derivative by taking dx to be any finite quantity h and the corresponding differential dy of $y = f(x)$ to be $hf'(x)$. In defining the differentials dy and dx in such a way that their quotient is precisely $f'(x)$ Cauchy decisively reversed Leibniz's definition of the derivative of a function as the quotient of differentials.

The work of Cauchy (as well as that of Bolzano) represents a crucial stage in the renunciation by mathematicians—adumbrated in the work of d'Alembert—of (fixed) infinitesimals and the intuitive ideas of continuity and motion. Certain mathematicians of the day, such as Poisson and Cournot, who regarded the limit concept as no more than a circuitous substitute for the use of infinitesimally small magnitudes—which in any case (they claimed) had a real existence—felt that Cauchy's reforms had been carried too far². But traces of the traditional ideas did in fact remain in Cauchy's formulations, as evidenced by his use of such expressions as “variable quantities”, “infinitesimal quantities”, “approach indefinitely”, “as little as one wishes” and the like³.

¹ Quoted *ibid.*, pp. 951–2.

² Boyer (1959), p. 283.

³ Boyer (1959), p. 284. Fisher (1978) argues that here and there in his work Cauchy did “argue directly with infinitely small quantities treated as actual infinitesimals.”

RIEMANN

If analysts strove to eliminate the infinitesimal from the foundations of their discipline, the same cannot be said of the geometers, especially the differential geometers. Differential geometry¹ had first emerged in the 17th century through the injection of the methods of the calculus into coordinate geometry. While Euclidean or projective geometry is concerned with the *global* properties of geometric objects, the focus of differential geometry is the *local* or *infinitesimal* properties of such objects, that is, those arising in the immediate neighbourhoods of points, and which may vary from point to point. The language of differentials or infinitesimals is natural to differential geometry, and it was freely employed by those mathematicians, such as the Bernoullis, Clairaut, Euler, and Gauss, in their investigations into the subject².

The infinitesimal plays an important role in the revolutionary extension of differential geometry conceived by the great German mathematician **Bernhard Riemann** (1826–66)³. In 1854⁴ Riemann introduced the idea of an intrinsic geometry for an arbitrary “space” which he termed a *multiply extended manifold*. Riemann conceived of a manifold as being the domain over which varies what he terms a *multiply extended magnitude*. Such a magnitude M is called *n-fold extended*, and the associated manifold *n-dimensional*, if n quantities—called *coordinates*—need to be specified in order to fix the value of M . For example, the position of a rigid body is a 6-fold extended magnitude because three quantities are required to specify its location and another three to specify its orientation in space. Similarly, the fact that pure musical tones are determined by giving intensity and pitch show these to be 2-fold extended magnitudes. In both of these cases the associated manifold is *continuous* in so far as each magnitude is capable of varying continuously with no “gaps”. By contrast, Riemann termed *discrete* a manifold whose associated magnitude jumps discontinuously from one value to another, such as, for example, the number of leaves on the branches of a tree. Of discrete manifolds Riemann remarks:

*Concepts whose modes of determination form a discrete manifold are so numerous, that for things arbitrarily given there can always be found a concept...under which they are comprehended, and mathematicians have been able therefore in the doctrine of discrete quantities to set out without scruple from the postulate that given things are to be considered as being all of one kind. On the other hand there are in common life only such infrequent occasions to form concepts whose modes of determination form a continuous manifold, that the positions of objects of sense, and the colours, are probably the only simple notions whose modes of determination form a continuous manifold. More frequent occasion for the birth and development of these notions is first found in higher mathematics.*⁵

The size of parts of discrete manifolds can be compared, says Riemann, by straightforward counting, and the matter ends there. In the case of continuous

¹ The term “differential geometry” was introduced in 1894 by the Italian mathematician *Luigi Bianchi* (1856–1928).

² Cauchy too made important contributions to differential geometry, but he was much more circumspect than his fellows in the use of infinitesimals.

³ In the words of Hermann Weyl:

The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann’s geometry, and, indeed, the mainspring of all the eminent work of Riemann (1950, p. 92).

⁴ *On the Hypotheses which Lie at the Foundations of Geometry*, published 1868, translated in *A Source Book in Mathematics*, Smith (1959), pp. 411–25.

⁵ *Ibid.*, pp. 412–13.

manifolds, on the other hand, such comparisons must be made by *measurement*. Measurement, however, involves superposition, and so requires the positing of some magnitude—not a pure number—independent of its place in the manifold. Moreover, in a continuous manifold, as we pass from one element to another in a necessarily continuous manner, the series of intermediate terms passed through itself forms a one-dimensional manifold. If this whole manifold is now induced to pass over into another, each of its elements passes through a one-dimensional manifold, so generating a two-dimensional manifold. Iterating this procedure yields n -dimensional manifolds for an arbitrary integer n . Inversely, a manifold of n dimensions can be analyzed into one of one dimension and one of $n - 1$ dimensions. Repeating this process finally resolves the position of an element into n magnitudes.

Riemann thinks of a continuous manifold as a generalization of the three-dimensional space of experience, and refers to the coordinates of the associated continuous magnitudes as *points*. He was convinced that our acquaintance with physical space arises only *locally*, that is, through the experience of phenomena arising in our immediate neighbourhood. Thus it was natural for him to look to differential geometry to provide a suitable language in which to develop his conceptions. In particular, the *distance* between two points in a manifold is defined in the first instance only between points which are at *infinitesimal* distance from one another. This distance is calculated according to a natural generalization of the distance formula in Euclidean space. In n -dimensional Euclidean space, the distance ε between two points P and Q with coordinates (x_1, \dots, x_n) and $(x_1 + \varepsilon_1, \dots, x_n + \varepsilon_n)$ is given by

$$\varepsilon^2 = \varepsilon_1^2 + \dots + \varepsilon_n^2. \quad (1)$$

In an n -dimensional manifold, the distance between the points P and Q —assuming that the quantities ε_i are infinitesimally small—is given by Riemann as the following generalization of (1):

$$\varepsilon^2 = \sum g_{ij} \varepsilon_i \varepsilon_j,$$

where the g_{ij} are functions of the coordinates x_1, \dots, x_n , $g_{ij} = g_{ji}$ and the sum on the right side, taken over all i, j such that $1 \leq i, j \leq n$, is always positive. The array of functions g_{ij} is called the *metric* of the manifold. In allowing the g_{ij} to be functions of the coordinates Riemann allows for the possibility that the nature of the manifold or “space” may vary from point to point, just as the curvature of a surface may so vary.

Riemann concludes his discussion with the following words, the last sentence of which proved to be prophetic:

While in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in the case of a continuous manifold. Either then the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relations must be sought for outside that actuality, in colligating forces that operate on it. A decision on these questions can only be found by starting from the structure of phenomena that has been confirmed in experience hitherto...and by modifying the structure gradually under the compulsion of facts which it cannot explain...This path leads out into the domain of another science, into the realm of physics.¹

Riemann is saying, in other words, that if physical space is a continuous manifold, then its geometry cannot be derived *a priori*—as claimed, famously, by Kant—but can only be

¹ *Ibid.*, pp. 424–25.

determined by experience. In particular, and again in opposition to Kant, who held that the axioms of Euclidean geometry were necessarily and exactly true of our conception of space, these axioms may have no more than approximate truth.

The prophetic nature of Riemann's final sentence was realized in 1916 when his geometry—*Riemannian geometry*—was used as the basis for a landmark development in physics, Einstein's celebrated *General Theory of Relativity*. In Einstein's theory, the geometry of space is determined by the gravitational influence of the matter contained in it, thus perfectly realizing Riemann's contention that this geometry must come from "somewhere else", to wit, from physics.

WEIERSTRASS AND DEDEKIND

Meanwhile the German mathematician **Karl Weierstrass** (1815–97) was completing the banishment of spatiotemporal intuition, and the infinitesimal, from the foundations of analysis. To instill complete logical rigour Weierstrass proposed to establish mathematical analysis on the basis of number alone, to "arithmetize"¹ it—in effect, to replace the continuous by the discrete. "Arithmetization" may be seen as a form of mathematical atomism². In pursuit of this goal Weierstrass had first to formulate a rigorous "arithmetical" definition of real number. He did this by defining a (positive) real number to be a countable set of positive rational numbers for which the sum of any finite subset always remains below some preassigned bound, and then specifying the conditions under which two such "real numbers" are to be considered equal, or strictly less than one another.

Weierstrass was concerned to purge the foundations of analysis of all traces of the intuition of continuous motion—in a word, to replace the variable by the static. For Weierstrass a variable x was simply a symbol designating an arbitrary member of a given set of numbers, and a continuous variable one whose corresponding set S has the property that any interval around any member x of S contains members of S other than x .³ Weierstrass also formulated the familiar (ε, δ) definition of continuous function⁴: a function $f(x)$ is continuous at a if for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all x with $|x - a| < \delta$.⁵ He also discovered his famous *approximation theorem for continuous functions*: any continuous function defined on a closed interval of real numbers can be uniformly approximated to by a sequence of polynomials.

Following Weierstrass's efforts, another attack on the problem of formulating rigorous definitions of continuity and the real numbers was mounted by **Richard Dedekind** (1831–1916). Dedekind focussed attention on the question: exactly what is it that distinguishes a continuous domain from a discontinuous one? He seems to have

¹ According to Hobson (1907, p. 22), "the term 'arithmetization' is used to denote the movement which has resulted in placing analysis on a basis free from the idea of measurable quantity, the fractional, negative, and irrational numbers being so defined that they depend ultimately upon the conception of integral number."

² It would perhaps be too much of a conceptual stretch to identify arithmetization as a kind of neo-Pythagoreanism.

³ Boyer (1959), p. 286. This property—now known as *density in itself*—later came to be regarded as too weak to characterize a continuum; see the remarks on Cantor below.

⁴ The concept of function had by this time been greatly broadened: in 1837 Dirichlet suggested that a variable y should be regarded as a function of the independent variable x if a rule exists according to which, whenever a numerical value of x is given, a unique value of y is determined. (This idea was later to evolve into the set-theoretic definition of function as a set of ordered pairs.) Dirichlet's definition of function as a correspondence from which all traces of continuity had been purged, made necessary Weierstrass's independent definition of continuous function.

⁵ The notion of *uniform continuity* for functions was later introduced (in 1870) by Heine: a real valued function f is uniformly continuous if for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all x and y in the domain of f with $|x - y| < \delta$. In 1872 Heine proved the important theorem that any continuous real-valued function defined on a closed bounded interval of real numbers is uniformly continuous.

been the first to recognize that the property of density, possessed by the ordered set of rational numbers, is insufficient to guarantee continuity. In *Continuity and Irrational Numbers* (1872) he remarks that when the rational numbers are associated to points on a straight line, “there are infinitely many points [on the line] to which no rational number corresponds”¹ so that the rational numbers manifest “a gappiness, incompleteness, discontinuity”, in contrast with the straight line’s “absence of gaps, completeness, continuity.”² He goes on:

In what then does this continuity consist? Everything must depend on the answer to this question, and only through it shall we obtain a scientific basis for the investigations of all continuous domains. By vague remarks upon the unbroken connection in the smallest parts obviously nothing is gained; the problem is to indicate a precise characteristic of continuity that can serve as the basis for valid deductions. For a long time I pondered over this in vain, but finally I found what I was seeking. This discovery will, perhaps, be differently estimated by different people; but I believe the majority will find its content quite trivial. It consists of the following. In the preceding Section attention was called to the fact that every point p of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i.e., in the following principle:

‘If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.’³

Dedekind regards this principle as being essentially indemonstrable; he ascribes to it, rather, the status of an axiom “by which we attribute to the line its continuity, by which we think continuity into the line.”⁴ It is not, Dedekind stresses, necessary for *space* to be continuous in this sense, for “many of its properties would remain the same even if it were discontinuous.”⁵ And in any case, he goes on,

if we knew for certain that space were discontinuous there would be nothing to prevent us ... from filling up its gaps in thought and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be carried out in accordance with the above principle.⁶

The filling-up of gaps in the rational numbers through the “creation of new point-individuals” is the key idea underlying Dedekind’s construction of the domain of real numbers. He first defines a *cut* to be a partition (A_1, A_2) of the rational numbers such that every member of A_1 is less than every member of A_2 .⁷ After noting that each rational number corresponds, in an evident way, to a cut, he observes that infinitely many cuts fail to be engendered by rational numbers. The discontinuity or incompleteness of the domain of rational numbers consists precisely in this latter fact. That being the case, he continues,

¹ Ewald (1999), p. 770.

² *Ibid.*, p. 771.

³ *Ibid.*, p. 771.

⁴ *Ibid.*, p. 772.

⁵ *Ibid.*

⁶ *Ibid.*

⁷ *Ibid.*

*whenever we have a cut (A_1, A_2) produced by no rational number, we create a new number, an irrational number α , which we regard as completely defined by this cut (A_1, A_2) ; we shall say that the number α corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal if and only if they correspond to essentially different cuts.*¹

It is to be noted that Dedekind does not identify irrational numbers with cuts; rather, each irrational number is newly “created” by a mental act, and remains quite distinct from its associated cut.

Dedekind goes on to show how the domain of cuts, and thereby the associated domain of real numbers, can be ordered in such a way as to possess the property of *continuity*, viz.

*if the system \mathbb{R} of all real numbers divides into two classes A_1, A_2 such that every number a_1 of the class A_1 is less than every number a_2 of the class A_2 , then there exists one and only one number α by which this separation is produced.*²

Dedekind notes that this property of continuity is actually equivalent to two principles basic to the theory of limits; these he states as:

If a magnitude grows continually but not beyond all limits it approaches a limiting value

and

*if in the variation of a magnitude x we can, for every given positive magnitude δ , assign a corresponding interval within which x changes by less than δ , then x approaches a limiting value.*³

Dedekind’s definition of real numbers as cuts was to become basic to the set-theoretic analysis of the continuum.

CANTOR

The most visionary “arithmetizer” of all was **Georg Cantor** (1845–1918). Cantor’s analysis of the continuum in terms of infinite point sets led to his theory of transfinite numbers and to the eventual freeing of the concept of set from its geometric origins as a collection of points, so paving the way for the emergence of the concept of general abstract set central to today’s mathematics⁴.

¹ *Ibid.*, p. 773.

² *Ibid.*, p. 776.

³ *Ibid.*, p. 778.

⁴ The following observations in 1900 of A. Schoenflies, one of set-theory’s earliest contributors, are of interest in this connection. He saw the emergence of set theory, and the eventual discretization of mathematics, as primarily issuing from the effort to clarify the function concept, and only secondarily as the result of the struggle to tame the continuum:

The development of set theory had its source in the effort to produce clear analyses of two fundamental mathematical concepts, namely the concepts of argument and of function. Both concepts have undergone quite essential changes through the course of years. The concept of argument, specifically that of independent variable, originally coincided with [the] no further defined, naive concept of the geometric continuum; today it is common everywhere, to allow as domain of arguments any chosen value-set or point-set, which one can make up out of the continuum by rules defined in any way at all. Even more decisive is

At about the same time that Dedekind published his researches into the nature of the continuous, Cantor formulated his theory of the real numbers. This was presented in the first section of a paper of 1872 on trigonometric series¹. Like Weierstrass and Dedekind, Cantor aimed to formulate an adequate definition of the irrational numbers which avoided the presupposition of their prior existence, and he follows them in basing his definition on the rational numbers. Following Cauchy, he calls a sequence $a_1, a_2, \dots, a_n, \dots$ of rational numbers a *fundamental sequence* if there exists an integer N such that, for any positive rational ε , $|a_{n+m} - a_n| < \varepsilon$ for all m and all $n > N$. Any sequence $\langle a_n \rangle$ satisfying this condition is said to have a *definite limit* b . Dedekind had taken irrational numbers to be “mental objects” associated with cuts, so, analogously, Cantor regards these definite limits as nothing more than *formal symbols* associated with fundamental sequences². The domain B of such symbols may be considered an enlargement of the domain A of rational numbers, since each rational number r may be identified with the formal symbol associated with the fundamental sequence r, r, \dots, r, \dots . Order relations and arithmetical operations are then defined on B : for example, given three such symbols b, b', b'' associated with the fundamental sequences $\langle a_n \rangle, \langle a'_n \rangle, \langle a''_n \rangle$, the inequality $b < b'$ is taken to signify that, for some $\varepsilon > 0$ and N , $a_n - a'_n > \varepsilon$ for all $n > N$, while the equality $b + b' = b''$ is taken to express the relation $\lim(a_n + a'_n - a''_n) = 0$.

Having imposed an arithmetical structure on the domain B , Cantor is emboldened to refer to its elements as (real) *numbers*. Nevertheless, he still insists that these “numbers” have no existence except as representatives of fundamental sequences: in his theory

*the numbers (above all lacking general objectivity in themselves) appear only as components of theorems which have objectivity, for example, the theorem that the corresponding sequence has the number as limit.*³

Cantor next considers how real numbers are to be associated with points on the linear continuum. If a given point on the line lies at a distance from the origin O bearing a rational relation to the point at unit distance from that origin, then it can be represented by an element of A . Otherwise, it can be approached by a sequence $a_1, a_2, \dots, a_n, \dots$ of points each of which corresponds to an element of A . Moreover, the sequence $\langle a_n \rangle$ can be taken to be a fundamental sequence; Cantor writes:

*The distance of the point to be determined from the point O (the origin) is equal to b , where b is the number corresponding to the sequence.*⁴

In this way Cantor shows that each point on the line corresponds to a definite element of B . Conversely, each element of B should determine a definite point on the line. Realizing that the intuitive nature of the linear continuum precludes a rigorous proof of this property, Cantor simply assumes it as an axiom, just as Dedekind had done in regard to his principle of continuity:

the change which has befallen the notion of function. This change may be tied internally to Fourier's theorem, that a so-called arbitrary function can be represented by a trigonometric series; externally it finds expression in the definition which goes back to Dirichlet, which treats the general concept of function, to put it briefly, as equivalent to an arbitrary table ... It was left to Cantor to find the concepts which proved proper for a methodical investigation, and which made it possible to force infinite sets under the dominion of mathematical formulas and laws ... (Quoted in McLarty (1988), p. 83.)

¹ *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen*, *Mathematische Annalen* 5, 123–132.

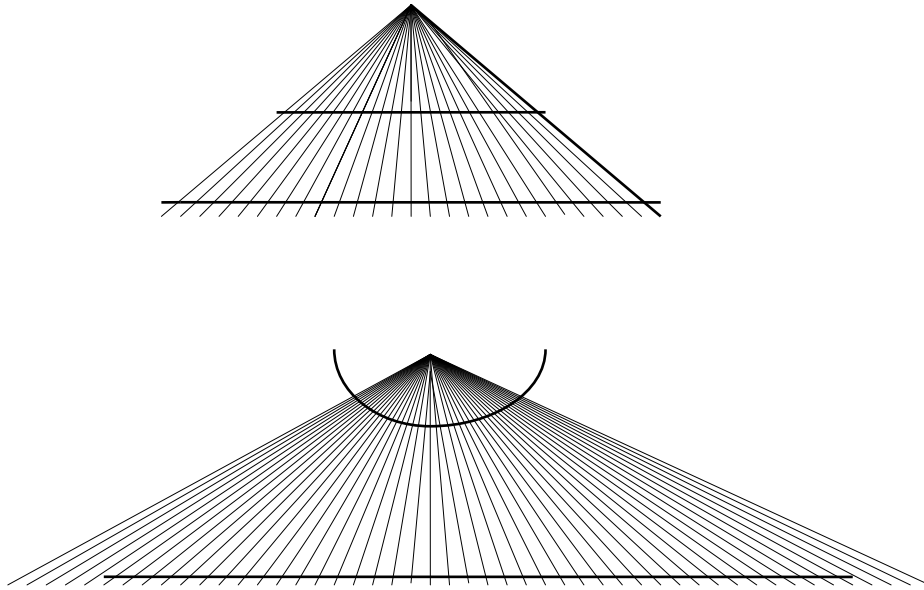
² Dauben (1979), p. 38.

³ Quoted *ibid.*, p. 39.

⁴ Quoted *ibid.*, p. 40.

Also conversely, to every number there corresponds a definite point of the line, whose coordinate is equal to that number.¹

For Cantor, who began as a number-theorist, and throughout his career cleaved to the discrete, it was numbers, rather than geometric points, that possessed objective significance. Indeed the isomorphism between the discrete numerical domain B and the linear continuum was regarded by Cantor essentially as a device for facilitating the manipulation of numbers.



Cantor's arithmetization of the continuum had another important consequence. It had long been recognized that the sets of points of any pair of line segments, even if one of them is infinite in length, can be placed in one-one correspondence—as in the diagrams above. This fact was taken to show that such sets of points have no well-defined “size”. But Cantor's identification of the set of points on a linear continuum with a domain of numbers enabled the *sizes* of point sets to be compared in a definite way, using the well-grounded idea of *one-one correspondence* between sets of numbers.

Thus in a letter to Dedekind written in November 1873 Cantor notes that the totality of natural numbers can be put into one-one correspondence with the totality of positive rational numbers, and, more generally, with the totality of finite sequences of natural numbers. It follows that these totalities have the same “size”; they are all *denumerable*. Cantor now raises the question of whether the natural numbers can be placed in one-one correspondence with the totality of all positive *real* numbers². He quickly answers his own question in the negative. In letters to Dedekind written during December 1873. Cantor shows that, for any sequence of real numbers, one can define numbers in every interval that are not in the sequence. It follows in particular that the whole set of real numbers is nondenumerable. Another important consequence concerns the existence of *transcendental numbers*, that is, numbers which are not

¹ Quoted *ibid.*, p. 40.

² Ewald (1999), p. 844.

algebraic in the sense of being the root of an algebraic equation with rational coefficients. In 1844 Liouville had established the transcendental nature of any number of the form

$$\frac{a_1}{10} + \frac{a_2}{10^{21}} + \frac{a_3}{10^{31}} + \dots$$

where the a_i are arbitrary integers from 0 to 9.¹ In his reply to Cantor's letter of November 1873, Dedekind had observed that the set of algebraic numbers is denumerable; it followed from the nondenumerability of the real numbers that *there must be many² transcendental numbers*.

By this time Cantor had come to regard nondenumerability as a necessary condition for the continuity of a point set, for in a paper of 1874 he asserts:

Moreover, the theorem...represents the reason why aggregates of real numbers which constitute a so-called continuum (say the totality of real numbers which are ≥ 0 and ≤ 1), cannot be uniquely correlated with the aggregate (v); thus I found the clear difference between a so-called continuum and an aggregate like the entirety of all real algebraic numbers.³

Cantor next became concerned with the question of whether the points of spaces of different dimensions—for instance a line and a plane—can be put into one-one correspondence. In a letter to Dedekind of January 1874 he remarks:

It still seems to me at the moment that the answer to this question is very difficult—although here too one is so impelled to say no that one would like to hold the proof to be almost superfluous.⁴

Nevertheless three years later Cantor, in a dramatic *volte-face*, established the existence of such correspondences between spaces of different dimensions. He showed, in fact, that (the points of) a space of any dimension whatsoever can be put into one-one correspondence with (the points of) a line. This result so startled him that, in a letter to Dedekind of June 1877 he was moved to exclaim *Je le vois, mais je ne le crois pas*.⁵

Cantor's discovery caused him to question the adequacy of the customary definition of the dimension of a continuum. For it had always been assumed that the determination of a point in a an n -dimensional continuous manifold requires n independent coordinates, but now Cantor had shown that, in principle at least, the job could be done with just a *single* coordinate. For Cantor this fact was sufficient to justify the claim that

... all philosophical or mathematical deductions that use that erroneous presupposition are inadmissible. Rather the difference that obtains between structures of different dimension-number must be sought in quite other terms than in the number of independent coordinates—the number that was hitherto held to be characteristic.⁶

¹ Kline (1972), p. 981.

² In fact nondenumerably many, although Cantor did not make this fact explicit until later.

³ Quoted in Dauben (1979), p. 53.

⁴ Ewald (1999), p. 850.

⁵ "I see it, but I don't believe it." *Ibid.*, p. 860.

⁶ *Ibid.*

In his reply to Cantor, Dedekind conceded the correctness of Cantor's result, but balked at Cantor's radical inferences therefrom. Dedekind maintained that the dimension-number of a continuous manifold was its "first and most important invariant"¹, and emphasized the issue of continuity:

For all authors have clearly made the tacit, completely natural presupposition that in a new determination of the points of a continuous manifold by new coordinates, these coordinates should also (in general) be continuous functions of the old coordinates, so that whatever appears as continuously connected under the first set of coordinates remains continuously connected under the second. ²

Dedekind also noted the extreme discontinuity of the correspondence Cantor had set up between higher dimensional spaces and the line:

*... it seems to me that in your present proof the initial correspondence between the points of the ρ -interval (whose coordinates are all irrational) and the points of the unit interval (also with irrational coordinates) is, in a certain sense (smallness of the alteration), as continuous as possible; but to fill up the gaps, you are compelled to admit, a frightful, dizzying discontinuity in the correspondence, which dissolves everything to atoms, so that every continuously connected part of the one domain appears in its image as thoroughly decomposed and discontinuous.*³

Dedekind avows his belief that no one-one correspondence between spaces of different dimensions can be continuous:

If it is possible to establish a reciprocal, one-to-one, and complete correspondence between the points of a continuous manifold A of a dimensions and the points of a continuous manifold B of b dimensions, then this correspondence itself, if a and b are unequal, is necessarily utterly discontinuous. ⁴

In his reply to Dedekind of July 1877 Cantor clarifies his remarks concerning the dimension of a continuous manifold:

...I unintentionally gave the appearance of wishing by my proof to oppose altogether the concept of a ρ -fold extended continuous manifold, whereas all my efforts have rather been intended to clarify it and to put it on the correct footing. When I said: "Now it seems to me that all philosophical and mathematical deductions which use that erroneous presupposition—" I meant by this presupposition not "the determinateness of the dimension-number" but rather the determinates of the independent coordinates, whose number is assumed by certain authors to be in all circumstances equal to the number of dimensions. But if one takes the concept of coordinate generally, with no presuppositions about the nature of the intermediate functions, then the number of independent, one-to-one, complete coordinates, as I showed, can be set to any number. ⁵

But he agrees with Dedekind that if "we require that the correspondence be continuous, then only structures with the same number of dimensions can be related to each other one-to-one."⁶ In that case, an invariant *can* be found in the number of independent

¹ *Ibid.*, p. 863.

² *Ibid.*

³ *Ibid.*, pp. 863-4.

⁴ *Ibid.*, p. 863.

⁵ *Ibid.*, p. 864.

⁶ *Ibid.*

coordinates, “which ought to lead to a definition of the dimension-number of a continuous structure.”¹ The problem is to correlate that dimension-number, a perfectly definite mathematical object, with something as elusive as an arbitrary continuous correspondence. Cantor writes:

However, I do not yet know how difficult this path (to the concept of dimension-number) will prove, because I do not know whether one is able to limit the concept of continuous correspondence in general. But everything in this direction seems to me to depend on the possibility of such a limiting.

*I believe I see a further difficulty in the fact that this path will probably fail if the structure ceases to be thoroughly continuous; but even in this case one wants to have something corresponding to the dimension-number—all the more so, given how difficult it is to prove that the manifolds that occur in nature are thoroughly continuous.*²

In rendering the continuous discrete, and thereby admitting *arbitrary* correspondences “of [a] frightful, dizzying discontinuity” between geometric objects “dissolved to atoms”, Cantor grasps at the same time that he has rendered the intuitive concept of spatial dimension a hostage to fortune³.

In 1878 Cantor published a fuller account⁴ of his ideas. Here Cantor explicitly introduces the concept of the *power*⁵ of a set of points: two sets are said to be of equal power if there existed a one-one correspondence between them. Cantor presents demonstrations of the denumerability of the rationals and the algebraic numbers, remarking that “the sequence of positive whole numbers constitutes...the least of all powers which occur among infinite aggregates.”⁶

The central theme of Cantor’s 1878 paper is the study of the powers of continuous n -dimensional spaces. He raises the issue of invariance of dimension and its connection with continuity:

*Apart from making the assumption, most are silent about how it follows from the course of this research that the correspondence between the elements of the space and the system of values x_1, x_2, \dots, x_n is a continuous one, so that any infinitely small change of the system x_1, x_2, \dots, x_n corresponds to an infinitely small change of the corresponding element, and conversely, to every infinitely small change of the element a similar change in the coordinates corresponds. It may be left undecided whether these assumptions are to be considered as sufficient, or whether they are to be extended by more specialized conditions in order to consider the intended conceptual construction of n -dimensional continuous spaces as one ensured against any contradictions, sound in itself.*⁷

¹ *Ibid.*

² *Ibid.*

³ At the same time Cantor’s recognition that the problem of defining dimension depends on the possibility of restricting the correspondences between the structures concerned is indicative of his great prescience as a mathematician. For the idea of taking as primary data *correspondences* between mathematical structures, as opposed to the structures themselves, was, through category theory, to prove seminal in the mathematics of the 20th century.

⁴ *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik 84, 242–58.

⁵ *Mächtigkeit*. Cantor mentions that the origin of this concept lies in the work of the Swiss geometer Jakob Steiner, who used it in his study of invariants of conic sections.

⁶ Quoted in Dauben (1979), p. 59.

⁷ *Ibid.*, p. 60.

The remarkable result obtained when one no longer insists on continuity in the correspondence between the spatial elements and the system of coordinates is described by Cantor in the following terms:

*As our research will show, it is even possible to determine uniquely and completely the elements of an n -dimensional continuous space by a single real coordinate t . If no assumptions are made about the kind of correspondence, it then follows that the number of independent, continuous, real coordinates which are used for the unique and complete determination of the elements of an n -dimensional continuous space can be brought to any arbitrary number, and thus is not to be regarded as a unique feature of the space.*¹

Cantor shows how this result can be deduced from the existence of a one-one correspondence between the set of reals and the set of irrationals, and then, by means of an involved argument, constructs such a correspondence.

Cantor seems to have become convinced by this time that the essential nature of a continuum was fully reflected in the properties of sets of points—a conviction which was later to give birth to abstract set theory. In particular a continuum's key properties, Cantor believed, resided in the range of powers of its subsets of points. Since the power of a continuum of any number of dimensions is the same as that of a linear continuum, the essential properties of arbitrary continua were thereby reduced to those of a line. In his investigations of the linear continuum Cantor had found its infinite subsets to possess just two powers, that of the natural numbers and that of the linear continuum itself. This led him to the conviction that these were the only possible powers of such subsets—a thesis later to be enunciated as the famous *continuum hypothesis*.

But the problem of establishing the invariance of dimension of spaces under continuous correspondences remained a pressing issue. Soon after the publication of Cantor's 1878 paper, a number of mathematicians, for example Lüroth, Thomae, Jürgens and Netto attempted proofs, but all of these suffered from shortcomings which did not escape notice². In 1879 Cantor himself published a proof which seems to have passed muster at the time, but which also contained flaws that were not detected for another 20 years³.

Satisfied that he had resolved the question of invariance of dimension, Cantor returned to his investigation of the properties of subsets of the linear continuum. The results of his labours are presented in six masterly papers published during 1879–84, *Über unendliche lineare Punktmannigfaltigkeiten* ("On infinite, linear point manifolds"). Remarkable in their richness of ideas, these papers contain the first accounts of Cantor's revolutionary theory of infinite sets and its application to the classification of subsets of the linear continuum. In the third and fifth of these are to be found Cantor's observations on the nature of the continuum.

In the third article, that of 1882, which is concerned with multidimensional spaces, Cantor applies his result on the nondenumerability of the continuum to prove the startling result that continuous motion is possible in discontinuous spaces. To be precise, he shows that, if M is any countable dense subset of the Euclidean plane \mathbb{R}^2 , (for example the set of points with both coordinates algebraic real numbers), then any pair of points of the discontinuous space $A = \mathbb{R}^2 - M$ can be joined by a continuous arc lying entirely within A ⁴. In fact Cantor claims even more:

¹ *Ibid.*, p. 60.

² Dauben (1979), pp. 70–72

³ *Ibid.*, pp. 72–76. The matter was only placed beyond doubt in 1911 when Brouwer showed definitively that the dimension of a Euclidean space is a topological invariant.

⁴ In modern terminology, spaces like A are *arcwise connected*.

*After all, with the same resources, it would be possible to connect the points...by a continuously running line given by a unique analytic rule and completely contained within the domain A.*¹

Cantor points out that the belief in the continuity of space is traditionally based on the evidence of continuous motion, but now it has been shown that continuous motion is possible even in discontinuous spaces. That being the case, the presumed continuity of space is no more than a hypothesis. Indeed, it cannot necessarily be assumed that physical space contains every point given by three real number coordinates. This assumption, he urges,

*...must be regarded as a free act of our constructive mental activity. The hypothesis of the continuity of space is therefore nothing but the assumption, arbitrary in itself, of the complete, one-to-one correspondence between the 3-dimensional purely arithmetic continuum (x, y, z) and the space underlying the world of phenomena.*²

These facts, so much at variance with received views, confirmed for Cantor once again that geometric intuition was a poor guide to the understanding of the continuum. For such understanding to be attained reliance must instead be placed on arithmetical analysis.

In the fifth paper in the series, the *Grundlagen* of 1883, is to be found a forthright declaration of Cantor's philosophical principles, which leads on to an extensive discussion of the concept of the continuum. Cantor distinguishes between the *intrasubjective* or *immanent* reality and *transsubjective* or *transient* reality of concepts or ideas. The first type of reality, he says, is ascribable to a concept which may be regarded

*as actual in so far as, on the basis of definitions, [it] is well distinguished from other parts of our thought, and stand[s] to them in determinate relationships, and thus modify the substance of our mind in a determinate way.*³

The second type, transient or transsubjective reality, is ascribable to a concept when it can, or must, be taken

*as an expression or copy of the events and relationships in the external world which confronts the intellect.*⁴

Cantor now presents the principal tenet of his philosophy, to wit, that the two sorts of reality he has identified invariably occur together,

*in the sense that a concept designated in the first respect as existent always also possess in certain, even infinitely many ways, a transient reality.*⁵

Cantor's thesis is tantamount to the principle that correct thinking is, in its essence, a reflection of the order of Nature. In a footnote Cantor places his thesis in the context of the history of philosophy. He claims that "it agrees essentially both with the principles of the Platonic system and with an essential tendency of the Spinozistic system" and that it can be found also in Leibniz's philosophy. But philosophy since that time has come to deviate from this cardinal principle:

¹ Quoted in Dauben (1979), p. 86.

² *Ibid.*

³ Ewald (1999), p. 895. Cantor refers specifically here to the concept of the natural numbers.

⁴ *Loc. cit.*

⁵ *Loc. cit.*

Only since the growth of modern empiricism, sensualism, and scepticism, as well as of the Kantian criticism that grows out of them, have people believed that the source of knowledge and certainty is to be found in the senses or in the so-called pure form of intuition of the world of appearances, and that they must confine themselves to these. But in my opinion these elements do not furnish us with any secure knowledge. For this can be obtained only from concepts and ideas that are stimulated by external experience, and are essentially formed by inner induction and deduction as something that, as it were, was already in us and is merely awakened and brought to consciousness.¹

This linkage between the immanent and transient reality of mathematical concepts—the fact that correct mathematical thinking reflects objective reality—has, in Cantor’s view, the important consequence that

mathematics, in the development of its ideas has only to take account of the immanent reality of its concepts and has absolutely no obligation to examine their transient reality.²

It follows that

mathematics in its development is entirely free³ and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established.⁴ In particular, in the introduction of new numbers it is only obliged to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to older numbers that they can in any given instance be precisely distinguished.⁵

For these reasons Cantor bestows his blessing on rational, irrational, and complex numbers, which “one must regard as being every bit as existent as the finite positive integers”; even Kummer’s introduction of “ideal” numbers into number theory meets with his approval. But not infinitesimal numbers, as we shall see.

Cantor begins his examination of the continuum with a tart summary of the controversies that have traditionally surrounded the notion:

The concept of the ‘continuum’ has not only played an important role everywhere in the development of the sciences but has also evoked the greatest differences of

¹ *Ibid.*, p. 918.

² *Ibid.*, p. 896.

³ So Cantor goes on to assert, in a famous pronouncement, “the *essence* of *mathematics* lies precisely in its *freedom*.” The only constraints on this freedom is the obligation for newly introduced concepts to be consistent and to stand in a well-determined relationship with concepts already present.

⁴ Here Cantor inserts a characteristic footnote, in which he puts forward a theory of the formation of concepts reminiscent of Plato’s doctrine of anamnesis:

The procedure in the correct formulation of concepts is in my opinion everywhere the same. One posits a thing with properties that at the outset is nothing other than a name or a sign A, and then in an orderly fashion gives it different, or even infinitely many, intelligible predicates whose meaning is known on the basis of ideas already at hand, and which may not contradict one another. In this way one determines the connection of A to the concepts that are already at hand, in particular to related concepts. If one has reached the end of this process, then one has met all the preconditions for awakening the concept A which slumbered inside us, and it comes into being accompanied by the intrasubjective reality which is all that can be demanded of a concept; to determine its transient meaning is then a matter for metaphysics. (Ewald 1999, p. 918)

⁵ Ewald (1999), p. 896.

opinion and event vehement quarrels. This lies perhaps in the fact that, because the exact and complete definition of the concept has not been bequeathed to the dissentients, the underlying idea has taken on different meanings; but it must also be (and this seems to me the most probable) that the idea of the continuum had not been thought out by the Greeks (who may have been the first to conceive it) with the clarity and completeness which would have been required to exclude the possibility of different opinions among their posterity. Thus we see that Leucippus, Democritus, and Aristotle consider the continuum as a composite which consists ex partibus sine fine divisilibus¹, but Epicurus and Lucretius construct it out of their atoms considered as finite things. Out of this a great quarrel arose among the philosophers, of whom some followed Aristotle, others Epicurus; still others, in order to remain aloof from this quarrel, declared with Thomas Aquinas that the continuum consisted neither of infinitely many nor of a finite number of parts, but of absolutely no parts. This last opinion seems to me to contain less an explanation of the facts than a tacit confession that one has not got to the bottom of the matter and prefers to get genteely out of its way. Here we see the medieval-scholastic origin of a point of view which we still find represented today, in which the continuum is thought to be an unanalysable concept, or, as others express themselves, a pure a priori intuition which is scarcely susceptible to a determination through concepts. Every arithmetical attempt at determination of this mysterium is looked on as a forbidden encroachment and repulsed with due vigour. Timid natures thereby get the impression that with the 'continuum' it is not a matter of a mathematically logical concept but rather of religious dogma.²

It is not Cantor's intention to "conjure up these controversial questions again". Rather, he is concerned to "develop the concept of the continuum as soberly and briefly as possible, and only with regard to the *mathematical* theory of sets". This opens the way, he believes, to the formulation of an exact concept of the continuum—nothing less than a demystification of the *mysterium*. Cantor points out that the idea of the continuum has heretofore merely been presupposed by mathematicians concerned with the analysis of continuous functions and the like, and has "not been subjected to any more thorough inspection."

Cantor next repudiates any use of temporal intuition in an exact determination of the continuum:

...I must explain that in my opinion to bring in the concept of time or the intuition of time in discussing the much more fundamental and more general concept of the continuum is not the correct way to proceed; time is in my opinion a representation, and its clear explanation presupposes the concept of continuity upon which it depends and without whose assistance it cannot be conceived either objectively (as a substance) or subjectively (as the form of an a priori intuition), but is nothing other than a helping and linking concept, through which one ascertains the relation between various different motions that occur in nature and are perceived by us. Such a thing as objective or absolute time never occurs in nature, and therefore time cannot be regarded as the measure of motion; far rather motion as the measure of time—were it not that time, even in the modest role of a subjective necessary a priori

¹ "from parts divisible without end". It may strike one as odd to find the atomists Leucippus and Democritus here bracketed with Aristotle as upholders of divisionism; presumably Cantor is implicitly distinguishing the formers' *material* atomism from their probable, or at least possible, assent to *theoretical* divisionism. Cf. Heath (1981), vol. I, p. 181, where the idea that Democritus may have upheld geometric indivisibles is dismissed on the grounds that "Democritus was too good a mathematician to have anything to do with such a theory."

² Ewald (1999), p. 903.

*form of intuition, has not been able to produce any fruitful, incontestable success, although since Kant the time for this has not been lacking.*¹

These strictures apply, *pari passu*, to spatial intuition:

*It is likewise my conviction that with the so-called form of intuition of space one cannot even begin to acquire knowledge of the continuum. For only with the help of a conceptually already completed continuum do space and the structure thought into it receive that content with which they can become the object, not merely of aesthetic contemplation or philosophical cleverness or imprecise comparisons, but of sober and exact mathematical investigations.*²

Cantor now embarks on the formulation of a precise arithmetical definition of a continuum. Making reference to the definition of real number he has already provided (i.e., in terms of fundamental sequences), he introduces the n -dimensional arithmetical space G_n as the set of all n -tuples of real numbers $(x_1 | x_2 | \dots | x_n)$, calling each such an *arithmetical point* of G_n . The distance between two such points is given by

$$\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + \dots + (x'_n - x_n)^2}.$$

Cantor defines an *arithmetical point-set* in G_n to be any “aggregate of points of the points of the space G_n that is given in a lawlike way”.

After remarking that he has previously shown that all spaces G_n have the same power as the set of real numbers in the interval $(0 \dots 1)$, and reiterating his conviction that any infinite point sets has either the power of the set of natural numbers or that of $(0 \dots 1)$ ³, Cantor turns to the definition of the general concept of a continuum within G_n . For this he employs the concept of *derivative* or *derived set* of a point set introduced in his 1872 paper on trigonometric series. Cantor had defined the derived set of a point set P to be the set of *limit points* of P , where a limit point of P is a point of P with infinitely many points of P arbitrarily close to it. A point set is called *perfect* if it coincides with its derived set⁴. Cantor observes that this condition does not suffice to characterize a continuum, since perfect sets can be constructed in the linear continuum which are dense in no interval, however small: as an example of such a set he offers the set⁵ consisting of all real numbers in $(0 \dots 1)$ whose ternary expansion does not contain a “1”.

Accordingly an additional condition is needed to define a continuum. Cantor supplies this by introducing the concept of a *connected* set. A point set T is connected in Cantor’s sense if for any pair of its points t, t' and any arbitrarily small number ε there is a finite sequence of points t_1, t_2, \dots, t_n of T for which the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \dots, \overline{t_nt'}$ are all less than ε . Cantor now observes:

...all the geometric point-continua known to us fall under this concept of connected point-set, as it easy to see; I believe that in these two predicates ‘perfect’ and ‘connected’ I have discovered the necessary and sufficient properties of a point-continuum. I therefore define a point-continuum inside G_n as a perfect-connected set. Here ‘perfect’ and ‘connected’ are not merely words but completely general predicates

¹ *Ibid.*, p. 904.

² *Ibid.*

³ This, *Cantor’s continuum hypothesis*, is actually stated in terms of the transfinite ordinal numbers introduced in previous sections of the *Grundlagen*.

⁴ In the terminology of general topology, a set is perfect if it is closed and has no isolated points.

⁵ This set later became known as the *Cantor ternary set* or the *Cantor discontinuum*.

*of the continuum; they have been conceptually characterized in the sharpest way by the foregoing definitions.*¹

Cantor points out the shortcomings of previous definitions of continuum such as those of Bolzano and Dedekind, and in a note dilates on the merits of his own definition:

Observe that this definition of a continuum is free from every reference to that which is called the dimension of a continuous structure; the definition includes also continua that are composed of connected pieces of different dimensions, such as lines, surfaces, solids, etc....I know very well that the word 'continuum' has previously not had a precise meaning in mathematics; so my definition will be judged by some as too narrow, by others as too broad. I trust that that I have succeeded in finding a proper mean between the two.

*In my opinion, a continuum can only be a perfect and connected structure. So, for example, a straight line segment lacking one or both of its end-points, or a disc whose boundary is excluded, are not complete continua; I call such point-sets semi-continua.*²

It will be seen that Cantor has advanced beyond his predecessors in formulating what is in essence a *topological* definition of continuum, one that, while still dependent on metric notions, does not involve an order relation³. It is interesting to compare Cantor's definition with the definition of continuum in modern general topology. In a well-known textbook⁴ on the subject we find a *continuum* defined as a compact connected subset of a topological space. Now within any *bounded* region of Euclidean space it can be shown that Cantor's continua coincide with continua in the sense of the modern definition. While Cantor lacked the definition of compactness, his requirement that continua be "complete" (which led to his rejecting as continua such noncompact sets as open intervals or discs) is not far away from the idea.

Cantor's analysis of infinite point sets had led him to introduce transfinite numbers⁵, and he had come to accept their objective existence as being beyond doubt. But throughout his mathematical career he maintained an unwavering, even dogmatic opposition to infinitesimals, attacking the efforts of mathematicians such as du Bois-Reymond and Veronese⁶ to formulate rigorous theories of actual infinitesimals. As far as as Cantor was concerned, the infinitesimal was beyond the realm of the possible; infiinitesimals were no more than "castles in the air, or rather just nonsense", to be classed "with circular squares and square circles".⁷ His abhorrence of infinitesimals went so deep as to move him to outright vilification, branding them as "Cholera-bacilli of mathematics."⁸

Cantor believed that the theory of transfinite numbers could be employed to explode the concept of infinitesimal once and for all. Cantor's specific aim was to refute all attempts at introducing infinitesimals through the abandoning of the *Archimedean*

¹ Ewald (1999), p. 906.

² *Ibid*, p. 919.

³ Cantor later turned to the problem of characterizing the linear continuum as an ordered set. His solution was published in 1895 in the *Mathematische Annalen* (Dauben, Chapter 8.) For a modern presentation, see §3 of Ch. 6 of Kuratowski-Mostowski (1968)..

⁴ Hocking and Young (1961). See also Chapter 6 below.

⁵ For an account, see Hallett (1984)

⁶ Other proponents of infinitesimals of the time include Johannes Thomae and Otto Stolz: see Fisher (1981). For du Bois-Reymond and Veronese see Chapter 5 below.

⁷ Fisher (1981), p. 118.

⁸ Dauben (1979) , p. 233.

principle—i.e. the assertion that for any positive real numbers $a < b$, there is a sufficiently large natural number n for which $na > b$. (Domains in which this principle fails to hold are called nonarchimedean). In a paper of 1887, Cantor attempted to demonstrate that the Archimedean property was a necessary consequence of the “concept of linear quantity” and “certain theorems of transfinite number theory”, so that the linear continuum could contain no infinitesimals. He concludes that “the so-called Archimedean axiom is not an axiom at all, but a theorem which follows with logical necessity from the concept of linear quantity.”¹ Cantor’s argument relied on the claim that the product of a positive infinitesimal, should such exist, with one of his transfinite numbers could never be finite². But since a proof of this claim was not supplied, Cantor’s alleged demonstration that infinitesimals are impossible must be regarded as inconclusive.³

Cantor’s rejection of infinitesimals stemmed from his conviction that his own theory of transfinite ordinal and cardinal numbers exhausted the realm of the numerable, so that no further generalization of the concept of number, in particular any which embraced infinitesimals, was admissible.⁴ For Cantor, transfinite numbers were grounded in *transient reality*, while infinitesimals and similar chimeras could not be accorded such a status.⁵ Recall Cantor’s assertion:

¹ Fisher (1981), p. 118.

² In this connection Fraenkel observes (1976, p. 123) that Cantor’s cardinals and ordinals themselves constitute nonarchimedean domains for the simple reason that, when α is finite and β is transfinite, then $n\alpha < \beta$ for any n . While admitting the legitimacy of the sort of nonarchimedean domains put forward by mathematicians such as Veronese and du Bois-Reymond, Fraenkel distinguishes between Cantor’s from the other nonarchimedean domains through the fact that in the former any cardinal or ordinal can be reached by the repeated addition of unity (if necessary, transfinitely), while in the latter such a procedure is not even definable. Fraenkel concludes: “This contrast explains in what sense other non-Archimedean domains contain ‘relatively infinite’ magnitudes while the transfiniteness of cardinals and ordinals is an ‘absolute’ one.”

It is worth quoting Fraenkel’s full endorsement of Cantor’s rejection of infinitesimal numbers:

Cantor, when undertaking a “continuation of the series of real integers beyond thye infinite” and showing the usefulness of this generalization of the process of counting, refused to consider infinitely small magnitude beyond the “potential infinite” of analysis based on the concept of limit. The ‘infinitesimals’ of analysis, as is well known, refer to an infinite process and not to a constant positive value which, if greater than zero, could nit be infinitely small...

Opposing this attitude, some schools of philosophers... and later sporadic mathematicians proposed resuming the vague attempts of most 17th and 18th century mathematicians to base calculus on infinitely small magnitudes, the so-called infinitesimals or differentials. After the introduction of transfinite numbers by Cantor such attitudes pretended to be justified by set theory because there ought to be reciprocals (inverse ratios) to the transfinite numbers, namely the ostensible infinitesimals of various degrees representing the ratios of finite to transfinite numbers.

These views have been thoroughly rejected by Cantor and by the mathematical world in general. The reason for this uniformity was not dogmatism, which is a rare feature in mathematics and then almost invariably fought off; nobody has pleaded more ardently than Cantor himself that liberty of thought was the essence of mathematics and that prejudices had a very short life. The argument was not even that the admission of infinitesimals was self-contradictory, but just that it was sterile and useless... This uselessness contrasts strikingly with the success of the transfinite numbers regarding both their applications and their task of generalizing finite counting and ordering. (Fraenkel 1976, pp. 121-2.)

But it has to be said (as remarked by Fisher 1981) that concerning infinitesimals Cantor did display a dogmatic attitude and did argue, in effect, that the admission of infinitesimals was self-contradictory. While Cantor’s intolerant attitude towards the infinitesimal is not, strictly speaking, inconsistent with the “freedom” of mathematics in his sense, it does seem to reflect his deep-seated conviction (reported in Dauben 1979, pp. 288–91) that his transfinite set theory was the product of “divine inspiration”, so that anything in conflict with it must be anathematized.

³ Fisher (1981), p. 118. Even so, Cantor’s argument (in amended form) seems to have convinced a number of influential mathematicians, including Peano and Russell, of the untenability of infinitesimals.

⁴ Dauben (1979), p. 235.

⁵ *Ibid.*, p. 236.

In particular, in the introduction of new numbers it is only obliged to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to older numbers that they can in any given instance be precisely distinguished.

So Cantor could not grant the infinitesimal an immanent reality which was compatible with “older” numbers—among which he of course included his transfinite numbers—for had he done so he would perforce have had, in accordance with his own principles, to grant the infinitesimal transient reality. This seems to be the reason for Cantor’s determination to demonstrate the inconsistency of the infinitesimal with his concept of transfinite number.¹

RUSSELL

Bertrand Russell (1872–1970) began his philosophical career as a Hegelian, but he soon abandoned Hegel in favour of the logical approach to philosophy espoused by mathematicians such as Cantor, Frege and Peano. In 1903 Russell published his great work *The Principles of Mathematics*. In writing this work Russell’s principal objective was to demonstrate the *logicist* thesis that “pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles”. In particular the concept of *continuity* comes under close scrutiny. The work’s Part V—a kind of paean to Weierstrass and Cantor—is devoted to an analysis of the idea of continuity, and its relation to the infinite and the infinitesimal.

Before getting down to a full analysis of these topics, which “[have] been generally considered the fundamental problem[s] of mathematical philosophy”, Russell remarks that, thanks to the labours of modern mathematicians, the whole problem has been completely transformed:

Since the time of Newton and Leibniz, the nature of infinity and continuity had been sought in discussions of the so-called Infinitesimal Calculus. But it has been shown that this Calculus is not, as a matter of fact, in any way concerned with the infinitesimal, and that a large and most important branch of mathematics is logically prior to it. It was formerly supposed—and herein lay the real strength of Kant’s mathematical philosophy—that continuity had an essential reference to space and time, and that the calculus (as the word fluxion suggests) in some way presupposed motion or at least change. In this way, the philosophy of space and time was prior to that of continuity, the Transcendental Aesthetic preceded the Transcendental Dialectic, and the antinomies (at least the mathematical ones) were essentially spatio-temporal. All this has been changed by modern mathematics. What is called the arithmetization of mathematics has shown that all the problems presented, in this respect, by space and time, are already present in pure arithmetic.²

While the theory of infinity has two forms, “cardinal and ordinal, of which the former springs from the logical theory of number”, the new theory of the continuous that Russell champions so enthusiastically is “purely ordinal”. Indeed, he goes on,

we shall find it possible to give a general definition of continuity, in which no appeal is made to the mass of unanalyzed prejudice which Kantians call “intuition”; and ...

¹ A related point is made by Fraenkel (1976), p. 123.

²Russell (1964), p. 259. Earlier (in Ch. XXIII) Russell presents an argument to show that, while continuity, infinity and the infinitesimal have been traditionally associated with the category of Quantity, in fact they are more properly regarded as being ordinal and arithmetical in nature.

*we shall find that no other continuity is involved in space and time. And we shall find that, by a strict adherence to the doctrine of limits, it is possible to dispense entirely with the infinitesimal, even in the definition of continuity and the foundations of the calculus.*¹

Russell's presentation of the theory of real numbers in the *Principles* begins with characteristic *brio*:

*The philosopher may be surprised, after all that has already been said concerning numbers, to find that he is only now to learn about real numbers; and his surprise will be turned to horror when he learns that real is opposed to rational. But he will be relieved to learn that real numbers are not numbers at all, but something quite different.*²

Real numbers, according to Russell, are nothing more than certain *classes* of rational numbers. By way of illustration, "the class of rationals less than $\frac{1}{2}$ is a real number, associated with, but obviously not identical with, the rational number $\frac{1}{2}$." Russell remarks of this theory:

*[It] is not, so far as I know, explicitly advocated by any other author, though Peano suggests it and Cantor comes very near to it.*³

Russell proposes that real numbers are to be what he calls *segments* of rational numbers, where

*a segment of rationals may be defined as a class of rationals which is not null, nor yet coextensive with the rationals themselves ... and which is identical with the class of rational less than a (variable) term of itself, i.e. with the class of rationals x such that there is a rational y of the said class such that x is less than y .*⁴

That is, a subset S of the set \mathbb{Q} of rational numbers is a segment provided that $\emptyset \neq S \neq \mathbb{Q}$ and, for any $x \in \mathbb{Q}$, $x \in S$ if and only if $\exists y \in S. x < y$. It is curious that Russell does not mention Dedekind in connection with this definition (although Dedekind's construction of the irrational numbers is outlined a few pages further on). For the definition of real numbers in terms of Russell's "segments" is, from a purely formal standpoint, the same as that given by Dedekind in terms of his "cuts"⁵. But Russell seems to have regarded the *content* of his definition as differing in a significant way from that of Dedekind. In Russell's view, Dedekind merely *postulates* the existence of an irrational number corresponding to each of his "cuts". Dedekind does not actually construct or otherwise prove the existence of such numbers, in particular he does not claim that the cut *itself* is the number. In Russell's eyes, his own definition had the merit of *explicitly identifying* the object (number) in question as a class without resorting to abstraction or "intuition." It is interesting to note that this point, so important to the philosopher

¹ *Ibid.*, p. 260.

² *Ibid.*, p. 270.

³ *Ibid.*

⁴ *Ibid.*, p. 271.

⁵Indeed, in Hobson (1957), we find Russell's definition of real number (introduced, with due acknowledgment, as Russell's "form" of the definition) included in the section of the book entitled "The Dedekind Theory of Irrational Numbers" (*loc. cit.*, pp. 23-27). Hobson refers to Russell's "segments" as "lower segments"; and in fact Russell's real numbers are the same as those lower segments of Dedekind cuts as possess no largest member.

Russell, was not ascribed so great a significance by practicing mathematicians, who saw Russell's definition as no more than a "less abstract" version of Dedekind's¹.

After reviewing Cantor's definition of real numbers, Russell next proceeds to a discussion of Cantor's definitions of continuity. He begins with a light-hearted dig at Hegel:

*The notion of continuity has been treated by philosophers, as a rule, as though it were incapable of analysis. They have said many things about it, including the Hegelian dictum that everything discrete is continuous and vice-versa. This remark, as being an exemplification of Hegel's usual habit of combining opposites, has been tamely repeated by all his followers. But as to what they meant by continuity and discreteness, they have preserved a discreet and continuous silence; only one thing was evident, that whatever they did mean could not be relevant to mathematics, or to the philosophy of space and time.*²

Russell contrasts the "continuity" of the rational numbers—the fact that between any two there is another³—with "that other kind of continuity, which was seen to belong to space." This latter form of continuity, Russell says,

*was treated, as Cantor remarks, as a kind of religious dogma, and was exempted from that conceptual analysis which is requisite to its comprehension. Indeed it was often held to show, especially by philosophers, that any subject-matter possessing it was not validly analyzable into elements. Cantor has shown that this view is mistaken, by a precise definition of the kind of continuity which must belong to space. This definition, if it is to be explanatory of space, must, as he rightly says, be effected without any appeal to space.*⁴

In his *Introduction to Mathematical Philosophy* of 1919 Russell enlarges on the contrast between the "arithmetical" characterization of continuity and the intuitive notion:

The definitions of continuity ... of Dedekind and Cantor do not correspond very closely to the vague idea which is associated with the notion in the mind of the man in the street or of the philosopher. They conceive continuity rather as absence of separateness, the sort of general obliteration of distinctions which characterises a thick fog. A fog gives an impression of vastness without definite multiplicity or division. It is this sort of thing that the metaphysician means by "continuity", declaring it, very truly, to be characteristic of his mental life and that of children and animals.

The general idea of vaguely indicated by the word "continuity" when so employed, or by the word "flux", is one which is certainly quite different from that which we have been defining [i.e., the arithmetical one]. Take, for example, the series of real numbers. Each is what it is, quite definitely and uncompromisingly; it does not pass over by imperceptible degrees into another; it is a hard, separate unit, and its distance from every other unit is finite, though it can be made less than any given finite amount assigned in advance. The question of the relation between the kind of continuity existing among the real numbers and the kind exhibited, e.g., by what we see at a given time, is a difficult and intricate one. It is not to be maintained that the two kinds are simply identical, but it may, I think, be very well maintained that the

¹Hobson (1957), p. 24.

² Russell (1964), p. 287.

³ Russell calls this property "compactness"; it is now usually referred to as "density".

⁴ Russell (1964), p. 288.

*mathematical conception ... gives the abstract logical scheme to which it must be possible to bring empirical material by suitable manipulation, if that material is to be called "continuous" in any precisely definable sense.*¹

Returning now to the *Principles*, Russell considers continuity to be a purely ordinal notion, and accordingly Cantor's later definition of continuity in purely order-theoretic terms is superior, in Russell's eyes, to the earlier one which involves metric considerations². Following a brief examination of the theory of infinite cardinals and ordinals³, Russell turns to the calculus and the infinitesimal. It is Russell's contention that, despite its traditional denomination as the "Infinitesimal" Calculus, "there is no allusion to, or implication of, the infinitesimal in any part of this branch of mathematics." Russell's unbounded enthusiasm for the actual infinite was accompanied by some hostility to the infinitesimal, although it ran less deep than Cantor's. Russell begins his discussion of the calculus with a withering account of Leibniz's muddled use of infinitesimals⁴:

*The philosophical theory of the Calculus has been, ever since the subject was first invented, in a somewhat disgraceful condition. Leibniz himself—who, one would have supposed, should have been competent to give a correct account of his own invention—had ideas, upon this topic, which can only be described as extremely crude. He appears to have held that, if metaphysical subtleties are left aside, the calculus is only approximate, but is justified practically by the fact that the errors to which it gives rise are less than those of observation. When he was thinking of Dynamics, his belief in the actual infinitesimal hindered him from discovering that the calculus rests on the doctrine of limits, and made him regard his dx and dy as neither zero, nor finite, nor mathematical fictions, but as really representing the units to which, in his philosophy, infinite division was supposed to lead. And in his mathematical expositions of the subject, he avoided giving careful proofs, contenting himself with the enumeration of rules. At other times, it is true, he definitely rejects infinitesimals as philosophically valid; but he failed to show how, without the use of infinitesimals, the results obtained by means of the Calculus could be exact, and not approximate.*⁵

Newton, however, fares rather better in Russell's account (as he did in Hegel's); Russell continues:

In this respect, Newton is preferable to Leibniz: his Lemmas give the true foundation of the Calculus in the doctrine of limits, and, assuming the continuity of space and time in Cantor's sense, they give valid proofs of its rules so far as spatio-temporal magnitudes are concerned. But Newton was, of course, entirely ignorant of the fact that his Lemmas depend upon the modern theory of continuity; moreover, the appeal

¹ Russell (1995) p. 105.

² With the later subsumption of both order and metric under general topology, Cantor's two definitions of continuity, when referred to ordered and metric topological spaces, respectively, become equivalent.

³ This too receives a sparkling introduction which is worth quoting:

The mathematical theory of infinity may almost be said to begin with Cantor. The Infinitesimal Calculus, though it cannot wholly dispense with infinity, has as few dealings with it as possible, and contrives to hide it away before facing the world. Cantor has abandoned this cowardly policy, and has brought the skeleton out of its cupboard. He has been emboldened in this course by denying that it is a skeleton. Indeed, like many other skeletons, it was wholly dependent on its cupboard, and vanished in the light of day. (Ibid., p. 304).

⁴ While Russell's disdain for the infinitesimal may have tempered somewhat his estimation of Leibniz, he never wavered in his regard for the latter as "one of the supreme intellects of all time" (1945, p. 581).

⁵ Russell (1964), p. 325.

*to time and change, which appears in the word fluxion, and to space, which appears in the Lemmas, was wholly unnecessary, and served merely to hide the fact that no definition of continuity had been given.*¹

But Russell is not quite finished with his censure of Leibniz and the infinitesimalists, for he goes on immediately:

*Whether Leibniz avoided this error, seems highly doubtful: it is at any rate certain that, in his first published account of the Calculus, he defined the differential coefficient by means of the tangent to a curve. And by his emphasis on the infinitesimal, he gave a wrong direction to speculation as to the Calculus, which misled all mathematicians before Weierstrass (with the exception, perhaps, of De Morgan), and all philosophers down to the present day.*²

Russell proceeds to show, in painstaking detail, that the definitions of continuity, differentiability and integrability of a function in terms of limits involve no reference to the infinitesimal whatsoever, so establishing his claim that the Calculus can be washed entirely free of the notion. He remarks:

*Until recent times, it was universally believed that continuity, the derivative, and the definite integral all involved actual infinitesimals, i.e., that even if the definitions of these notions could be formally freed from explicit mention of the infinitesimal, yet, where the definitions applied, the actual infinitesimal must always be found. This belief is now generally abandoned. The definitions which have been given in previous chapters do not in any way imply the infinitesimal, and this notion appears to have become mathematically useless.*³

But what of the concept of the infinitesimal on its own account? It is to this issue that Russell next turns his attention.

Russell begins by observing that the infinitesimal has, in general, lacked precise definition:

*It has been regarded as a number or magnitude which, though not zero, is less than any finite number or magnitude. It has been the dx or dy of the Calculus, the time during which a ball thrown vertically upwards is at rest during the highest point of its course, the distance between a point on a line and the next point, etc., etc.*⁴

None of these amount to anything like a precise definition in view of the facts, first, that the differential is not a quantity, nor $\frac{dy}{dx}$ a fraction; secondly, a properly developed theory of motion shows that there is no time during which a ball is at rest at its highest point; and lastly, the idea of the distance between consecutive points “presupposes that there are consecutive points—a view which there is every reason to deny.”

Russell suggests that the sole precise definition of infinitesimal makes of it a purely relative notion, “correlative to something arbitrarily assumed to be finite.” This relative notion is obtained by denying the Archimedean principle that any pair of numbers or comparable magnitudes P , Q are relatively finite in the sense that, if P be the lesser, then there is a finite integer n such that $nP > Q$. In that case P may be defined to be infinitesimal with respect to Q , and Q infinite with respect to P , if $nP < Q$ for any integer

¹ *Ibid.*, p. 325-6.

² *Ibid.*, p. 326.

³ *Ibid.*, p. 331.

⁴ *Ibid.*

n. As far as magnitudes are concerned, Russell says, the only way of defining the infinitesimal and, indeed, the infinite, is through the use (or denial) of the Archimedean principle, for

*Of a magnitude not numerically measurable, there is nothing to be said except that it is greater than some of its kind, and less than others; but from such propositions infinity cannot be obtained. Even if there be a magnitude greater than all others of its kind, there is no reason for regarding it as infinite. Finitude and infinity are essentially numerical notions, and it is only by relation to numbers that these terms can be applied to other entities.*¹

Russell admits that instances of the (relative) infinitesimal can be found, even some of significance. One example he offers arises in connection with his introduction, earlier on in the *Principles*, of the concept of *magnitude of divisibility*. Russell proposes this concept as a way of overcoming one difficulty posed by the Cantor's reduction of the continuous to the discrete, namely the consequence that any two continua, however different they may be in metric size, are always the same size numerically in the sense of being composed of the same (transfinite) number of points (or "terms"). As Russell says, "there must be ... some other respect in which [say] two periods of twelve hours are equal, while a period of one hour and another of twenty-three hours are unequal."² Russell's suggestion is then to introduce, corresponding to each aggregate, a magnitude called its *magnitude of divisibility*. Roughly speaking, an aggregate's magnitude of divisibility represents the "number" of parts into which it can be divided with respect to a given determination of the meaning of "part". For example, if the aggregates in question are finite sets, and "parts" are singletons, then the corresponding magnitudes of divisibility are natural numbers; if the aggregates are infinite sets and "parts" are again singletons, then the corresponding magnitudes of divisibility are transfinite cardinals. On the other hand if the aggregates are intervals on a line and the "parts" are intervals of unit length, then the corresponding magnitudes of divisibility are the nonnegative real numbers.

Now if divisibility can be regarded as a magnitude in the sense above, then "it is plain", says Russell, "that the divisibility of any whole containing a finite number of simple parts is infinitesimal as compared with one containing an infinite number. The number of parts being taken as the measure, every infinite whole will be greater than *n* times every finite whole, whatever finite number *n* may be."³

Russell offers a number of further examples of infinitesimals in the same spirit: any line is infinitesimal with respect to an area, an area with respect to a volume, and a bounded volume with respect to the whole of space. On the other hand, the real numbers have been provided with an unequivocal definition as segments of rationals, and this fact "renders the non-existence of infinitesimals [among the real numbers] demonstrable." Russell's conclusion is

*if it were possible, in any sense to speak of infinitesimal numbers, it would have to be in some radically new sense.*⁴

Strangely, Russell (whose first published work was devoted to the foundations of geometry) fails to mention the geometer Veronese's attempts at introducing infinitesimals. But he touches on du Bois-Reymond's orders of infinity and infinitesimality of functions. This Russell does with reluctance, given that "on this

¹ *Ibid.*, p. 332.

² *Ibid.*, p. 151.

³ *Ibid.*, pp. 332-3. This example is essentially the same as that mentioned above by Fraenkel.

⁴ *Loc. cit.*, p. 335.

question the greatest authorities are divided.” But in the end Russell sides with Cantor in deciding that “these infinitesimals are mathematical fictions.”

In sum, Russell concludes,

*the infinitesimal... is a restricted and mathematically very unimportant conception, of which continuity and infinity are alike independent.*¹

So much for the infinitesimal as a mathematical concept. Russell next turns to the philosophical import of the notion. Again, a playful introduction:

*We have concluded our summary review of what mathematics has to say concerning the continuous, the infinite, and the infinitesimal. And here, if no previous philosophers had treated of these topics, we might leave the discussion, and apply our doctrines to space and time. For I hold the paradoxical opinion that what can be mathematically demonstrated is true. As, however, almost all philosophers disagree with this opinion, and as many have written elaborate arguments in favour of views different from those above expounded, it will be necessary to examine controversially the principal types of opposing theories and to defend, as far as possible, the points in which I differ from standard writers.*²

As his source for these “opposing theories”, Russell focuses on Hermann Cohen’s 1883 neo-Kantian work, *Das Prinzip der Infinitesimal-Methode und Seine Geschichte* (“The Principle of the Method of Infinitesimals and its History”) ³. In this work Cohen develops the view that the infinitesimal is essentially intensive magnitude, possessing the capacity of acting as a kind of generating element of the real as it is presented to the mind. For Cohen, mathematical infinitesimals are entities which, while real, “cannot be directly intuited as insular or discrete elements of being”⁴. As such, the infinitesimal mirrors “the relation of thinking and intuition which is to characterize all of modern science.” Cohen argues that, “far from being limited only to mathematical or scientific knowledge, the same process at the heart of the infinitesimal lies at the heart of all forms of perception.”⁵ As an index of change, the infinitesimal “allows a proper understanding of change in the world.”⁶ Cohen ascribed equal importance to the idea of continuity, which, he says, “is the general basis of consciousness.” The wider context in which the mind necessarily places each particular presented to it is necessarily a “continuous plenum.”⁷

¹ *Loc. cit.*, p. 337.

² *Loc. cit.*, p. 338.

³ For an illuminating discussion of this work and its historical context see Moynahan (2003).

⁴ *Op. cit.*, p. 44.

⁵ *Ibid.*, p. 45.

⁶ *Ibid.*

⁷ In this connection it is worth mentioning that no less a figure than Leo Tolstoy ascribed similar importance to the continuous and the infinitesimal. In Book 3, Part 3 of *War and Peace* we read:

To elicit the laws of history we must leave aside kings, ministers, and generals, and select for study the homogeneous, infinitesimal elements which influence the masses. No one can say how far it is possible for a man to advance in this way to an understanding of the laws of history; but it is obvious that this is the only path to that end...

It is impossible for the human intellect to grasp the idea of absolute continuity of motion. Laws of motion only become comprehensible to man when he can examine arbitrarily selected units of that motion. But at the same time it is this arbitrary division of continuous motion into discontinuous units which gives rise to a large proportion of human error.

A new branch of mathematics [i.e., the calculus], having attained the art of reckoning with infinitesimals, can now yield solutions in ... complex problems of motion which before seemed insoluble. ... This new branch of mathematics ... by admitting the conception, when dealing with problems of motion, of the

It happens that Cohen's work was the subject of a review by Frege in 1885¹. Here is how Frege sums up the basic idea of Cohen's treatise:

Cohen brings reality into a peculiar connection with the differential by going back, it seems, to the anticipations of perception whose principle according to Kant is this: 'In all appearances, the real that is an object of sensation has intensive magnitude, that is, a degree.' Now the differential is an intensive magnitude. If, e.g., x is a distance on the straight line, then dx , its differential or infinitesimal increment, is not to be thought of as an extensive magnitude or as itself a distance; this would lead to contradictions, and it is precisely because mathematicians wanted to let the differential pass throughout as an extensive magnitude that they got entangled in the well-known difficulties. These difficulties can be removed, not by logic, but by the critique of knowledge, which is the term the author uses for 'theory of knowledge', because it shows that an infinitesimal number is an intensive magnitude which, as such, has a power of realization: 'It does not merely represent the unit of reality; but also realizes as such; it confers reality upon Being in Quality'. [Again], 'If the differential constitutes reality as a constitutive condition of thought, then the integral designates the real as object.' The dx is therefore to be conceived of as, say, an intensive magnitude concentrated at the end point of x , comparable to an electrical charge, or as a power to increase the distance, like for example the last bud on a bough in which we can recognize a striving for growth. These pictures occurred to me as I was reading the book; they are not to be attributed to the author himself. ...

*The contrast between extensive and intensive goes back to the contrast between intuition and thought, since the quality which corresponds to the intensive magnitude is a category of thought. The extensive magnitude of intuition is thus opposed to the intensive one of sensation. The two sources of knowledge must always be combined if the knowledge is to be objective. Reality, as a means of thought, is able to come in where intuition alone fails, for the latter has the character of ideality. If the infinitesimal is to be fit to do justice to the requirements of reality, it must be withdrawn from intuition, provided that reality is to mean a condition of experience on the part of thought. Accordingly, continuity is also separated from intuition and assigned to thought.*²

infinitely small and thus conforming to the chief condition of motion (absolute continuity), corrects the inevitable error which the human intellect cannot but make if it considers separate units of motion instead of continuous motion.

In the investigation of the laws of historical movement precisely the same principle operates. The march of humanity, springing as it does from an infinite multitude of individual wills, is continuous. The discovery of the laws of this continuous movement is the aim of history.

Only by assuming an infinitesimally small unit of observation—a differential of history (that is, the common tendencies of men)—and arriving at the art of integration (finding the sum of the infinitesimals) can we hope to discover the laws of history.

¹ An English translation of Frege's review may be found in Frege (1984), pp. 108-12.

² *Loc. cit.*, pp. 109-10. In 1906 Cohen's student Ernst Cassirer summed up Cohen's outlook more sympathetically:

The basic idea of Cohen's work can be stated quite briefly: if we want to achieve a true scientific grounding of logic, we should not begin from any sort of completed existence. What naive intuition takes as its obvious and secure possession, this is for logic a real problem; what it assumes as directly 'given', this is what must be critically analyzed and taken apart in its crucial conditions for thought. We should not begin with any objective Being, no matter of what sort and no matter [in] what relation we place ourselves to it: for every "being" is in the first place a product and a result which the operation of thought and its systematic unity has as a presupposition. A foundational conceptual setting of this sort, an intellectual tradition in which we can first speak of "reality" in a scientific sense, is found by Cohen in the idea of the infinitesimal as it is detailed and fixed in modern mathematics. (Quoted in Moynahan 2003, p. 42.)

Frege, not surprisingly, is very critical of all this, remarking that “I do not find that the infinitesimal has an intimate connection with reality.” Above all, he says, “What I miss everywhere is proofs.”

Russell, for his part, is equally critical of Cohen’s claims for the infinitesimal. A first objection is that Cohen unquestioningly treats differentials as actual entities, a view which Russell regards as having been exploded by the theory of limits:

But when we turn to such works as Cohen’s, we find the dx and dy treated as separate entities, as the intensively real elements of which the continuum is composed The view that the Calculus requires infinitesimals is apparently not thought open to question; at any rate, no arguments whatever are brought up to support it. This view is certainly assumed as self-evident by most philosophers who discuss the Calculus. ¹

But Russell’s principal objection is that approaches to the problem of the infinitesimal such as Cohen’s identify the problem as possessing an epistemological, rather than a purely logical, character, and so “depends upon the pure intuitions as well as the categories.” Russell rejects “this Kantian opinion [which] is wholly opposed to the philosophy which underlies the present work.” Cohen’s claim that intensive magnitude is infinitesimal extensive magnitude is also rejected, on the grounds that

the latter must always be smaller than finite extensive magnitudes, and must therefore be of the same kind with them; while intensive magnitudes seem never in any sense smaller than any extensive magnitudes.²

Russell quotes Cohen’s summary of his own theory:

That I may be able to posit an element in and for itself, is the desideratum, to which corresponds the instrument of thought reality. This instrument of thought must first be set up, in order to be able to enter into that combination with intuition, with the consciousness of being given, which is completed in the principle of intensive magnitude. This presupposition of intensive reality is latent in all principles, and must be made independent. This presupposition is the meaning of reality and the secret of the concept of the differential. ³

Russell rejects most of this, but remarks that,

What we can agree to, and what, I believe, confusedly underlies the above statement, is, that every continuum must consist of elements or terms; but these, as we have seen, will not fulfill the function of the dx or dy which occur in old-fashioned accounts of the Calculus. Nor can we agree that “this finite” (i.e. that which is the object of physical science [to quote Cohen] “can be thought of as a sum of those infinitesimal intensive realities, as a definite integral.” The definite integral is not a sum of elements of a continuum, although there are such elements: for example, the length of a curve, as obtained by integration, is not the sum of its points, but strictly and only the limit of the lengths of inscribed polygons... There is no such thing as an infinitesimal stretch; if there were, it would not be an element of a continuum. ⁴

In sum, says Russell,

¹ Russell (1964), p. 339.

²*Ibid.*, p. 344.

³*Ibid.*

⁴*Ibid.*, p.345.

*infinitesimals as explaining continuity must be regarded as unnecessary, erroneous, and self-contradictory.*¹

Having dismissed the philosophical claims of the infinitesimal, Russell finally turns his attention to the philosophical difficulties posed by the continuum. It is made clear that attention is to be confined to the *arithmetical*—that is to Cantor’s—continuum, and that the continuum as it is presented to “intuition” is to be excluded from consideration. Following this decree Russell remarks:

*It has always been held to be an open question whether the continuum is composed of elements; and even when it has been allowed to contain elements, it has been often alleged to be not composed of these. This latter view was maintained by even by so stout a supporter of elements in everything as Leibniz. But all these views are only possible in regard to such continual as space and time. The arithmetical continuum is an object selected by definition, and known to embodied in at least one instance, namely the segments of the rational numbers. I shall [later] maintain ... that spaces afford other instances of the arithmetical continuum. The chief reason for the elaborate and paradoxical theories of space and time and their continuity, which have been constructed by philosophers, has been the supposed contradictions in a continuum composed of elements. The thesis of the present chapter is, that Cantor’s continuum is free from contradictions. This thesis, as is evident, must be firmly established, before we can allow that spatial-temporal continuity is of Cantor’s kind. In this argument I shall assume as proved ... that the continuity to be discussed does not involve the admission of actual infinitesimals.*²

Russell demonstrates the freedom from contradiction of Cantor’s continuum through a new analysis of Zeno’s paradoxes. He translates each paradox into arithmetical language and then shows that the resulting assertions are not paradoxical at all. He introduces his demonstration with another dramatic paragraph:

*In this capricious world, nothing is more capricious than posthumous fame. One of the most notable victims of posterity’s lack of judgment is the Eleatic Zeno. Having invented four arguments, all immeasurably subtle and profound, the grossness of subsequent philosophers pronounced him to be a mere ingenious juggler, and his arguments to be one and all sophisms. After two thousand years of continual refutation, these sophisms were reinstated, and made the foundation of a mathematical renaissance, by a German professor, who probably never dreamed of a connection between himself and Zeno. Weierstrass, by strictly banishing all infinitesimals, has at last shown that we live in an unchanging world, and that the arrow, at every moment of its flight, is truly at rest. The only point where Zeno probably erred was in inferring (if he did infer) that, because there is no change, therefore the world must be in the same state at one time as at another. This consequence by no means follows, and in this point the German professor is more constructive than the ingenious Greek. Weierstrass, being able to embody his opinions in mathematics, where familiarity with truth eliminates the vulgar prejudices of common sense, has been able to give to his propositions the respectable air of platitudes; and if his result is less delightful to the lover of reason than Zeno’s bold defiance, it is at any rate more calculated to appease the mass of academic mankind.*³

¹*Ibid.* Russell would have been greatly surprised to learn that Cohen’s conception of infinitesimals as intensive magnitudes can in fact be given a precise mathematical sense. See Chapter 10 below.

²*Ibid.*, p. 347.

³*Ibid.*, pp. 347-8.

Here we consider Russell's resolutions of two of the paradoxes, that of Dichotomy and that of the Arrow. The Dichotomy is stated by Russell as "There is no motion, for what moves must reach the middle of its course before it reaches the end." That is to say, Russell continues, "whatever motion we assume to have taken place, this presupposes another motion, and so on *ad infinitum*. Hence there is an endless regress in the mere idea of assigned motion." To state this argument in arithmetical form, Russell considers the class, or set, of real numbers between 0 and 1. This class, he says, is an infinite whole,

*whose parts are logically prior to it: for it has parts, and it cannot subsist if any of the parts are lacking. Thus the numbers from 0 to 1 presuppose those from 0 to $\frac{1}{2}$, these presuppose those from 0 to $\frac{1}{4}$, and so on.*¹

So it would seem to follow that

*there is an infinite regress in the notion of any infinite whole; but without infinite wholes, real numbers cannot be defined, and arithmetical continuity, which applies to an infinite series, breaks down.*²

Russell refutes this argument by observing that a class of real numbers, being given *intensionally* as the class of terms satisfying a given predicate (rather than *extensionally* by enumeration of its members) is not logically posterior to its parts. In particular, the class of all real numbers between 0 and 1 forms

*a definite class, whose meaning is known as soon as we know what is meant by real number, 0, 1, and between.*³

It follows that "the particular members of the class, and the smaller classes contained in it, are not logically prior to the class." The infinite regress, which "consists merely in the fact that every segment of real numbers has parts which are again segments" is rendered harmless by observing that these parts are not logically prior to it. Russell concludes that "the solution of the difficulty lies in the theory of denoting and the intensional definition of a class." So much for the Dichotomy.

Of Zeno's arrow puzzle—"If everything is in rest or in motion in a space equal to itself, and if what moves is always in the instant, the arrow in its flight is immovable"—, Russell remarks:

*This has usually been thought so monstrous a paradox as scarcely to deserve serious discussion. To my mind, I must confess, it seems a very plain statement of a very elementary fact, and its neglect has, I think, caused the quagmire in which the philosophy of change has long been immersed.*⁴

Russell's dissolution of the paradox is to divest it of all reference to change, so revealing it to be

*a very important and very widely applicable platitude, namely: "Every possible value of a variable is a constant."*⁵

¹ *Ibid.*, p. 348.

² *Ibid.*

³ *Ibid.*, p. 350.

⁴ *Ibid.*, p. 350.

⁵ *Ibid.*, p. 351.

Russell's claim here is that the variable position of the arrow is in essence a variable in the mathematical sense; and since a mathematical variable is (according to Russell) just a symbol denoting an arbitrary constant, the arrow's flight is, in the eyes of pure logic, just a conjunction of assertions of the form "the arrow was at a certain place at a certain time". Each of these correlated places and times is a constant, and the arrow is at rest at all of them. "This simple logical fact", says Russell, "seems to constitute the essence of Zeno's contention that the arrow is always at rest."

Russell contends that in addition to its purely logical character, Zeno's argument says something fundamental about continua. In the case of motion, it is the denial that there is such a thing as a *state* of motion; once "change" is eradicated, motion is in fact nothing more than the occupation of different places at different times. In the case of a continuous variable, the thrust of Zeno's argument may be taken to be the denial of the existence of actual infinitesimals. "For," says Russell,

*infinitesimals are an attempt to extend to the values of a variable the variability which belongs to it alone. When once it is firmly realized that all the values of a variable are constants, it becomes easy to see, by taking any two such values, that their difference is always finite, and hence there are no infinitesimal differences. If x be a variable which may take all values between 0 and 1, then, taking any two of these values, we see that their difference is finite, although x is a continuous variable... This static conception of the variable is due to the mathematicians, and its absence in Zeno's day led him to suppose that continuous change was impossible without a state of change, which involves infinitesimals and the contradiction of a body being where it is not.*¹

From these analyses Russell infers that

Zeno's arguments, though they prove a very great deal, do not prove that the continuum, as we have become acquainted with it, contains any contradiction whatever. Since his day the attacks on the continuum, have not, so far as I know, been conducted with any new or more powerful weapons.

Russell again raises his hat to Cantor:

*The notion to which Cantor gives the name of continuum may, of course, be called by any other name in or out of the dictionary, and it is open to every one to assert that he himself means something quite different by the continuum. But these verbal questions are purely frivolous. Cantor's merit lies, not in meaning what other people mean, but in telling us what he means himself—an almost unique merit, where continuity is concerned. He has defined, accurately and generally, a purely ordinal notion, free, as we now see, from contradictions, and sufficient for all Analysis, Geometry, and Dynamics. ... The salient points in the definition of the continuum are (1) the connection with the doctrine of limits, (2) the denial of infinitesimal segments. These two points being borne in mind, the whole philosophy of the subject becomes illuminated.*²

One final conclusion is drawn:

The denial of infinitesimal segments resolves an antinomy which has long been an open scandal, I mean the antinomy that the continuum both does and does not consist of elements. We see now that both may be said, though in different senses. Every continuum is a series consisting of terms, and the terms, if not indivisible, at

¹ *Ibid.*, p. 351–2.

² *Ibid.*, p. 353.

any rate are not divisible into new terms of the continuum. In this sense there are elements. But if we take consecutive terms together with their asymmetrical relations as constituting what may be called ... an ordinal element, then, in this sense, our continuum has no elements. If we take a stretch to be essentially serial, so that it must consist of at least two terms, then there are no elementary stretches; and if our continuum be one in which there is distance, then like wise there are no elementary distances. But in neither of these cases is there the slightest logical ground for elements. The demand for consecutive terms springs ... from an illegitimate use of mathematical induction. And as regards distance, small distances are no simpler than large ones, but all ... are alike simple. And large distances do not presuppose small ones Thus the infinite regress from greater to smaller distances or stretches is of the harmless kind, and the lack of elements need not cause any logical inconvenience. Hence the antinomy is resolved, and the continuum, so far at least as I am able to discover, is wholly free from contradictions.¹

As we have seen, Russell's analysis of the continuum rests chiefly on denying the existence of infinitesimals. He correctly identifies infinitesimals as "an attempt to extend to the values of a variable the variability which belongs to it alone", and he is right in his assertion that if the values of a variable must always be constants, infinitesimals as "variable values" cannot exist. But he could not have foreseen that, 60 years or so after writing this passage (yet still, remarkably, within his own lifetime), developments in mathematics would enable variation to be reincorporated into the subject in such a way as to allow the admission of the infinitesimal in essentially the sense that he repudiates with such *élan*. Another development that Russell could not have anticipated is that the rigorous reintroduction of the infinitesimal in this sense would not require abandoning the law of noncontradiction, as he seems, with some justice, to have thought: the requisite logical adjustment turned out to be the dropping of the law of excluded middle². Russell would also have been greatly surprised—perhaps even dismayed—to learn that the conception of infinitesimals as intensive magnitudes can in fact be given a precise mathematical sense³.

HOBSON'S CHOICE

Published in 1907, *The Theory of Functions of a Real Variable*, by the prominent English mathematician **E. W. Hobson** (1856-1933), was the first systematic exposition in English of the new analysis. In this work, which was to prove very influential, Hobson makes a number of interesting observations concerning the process by which the continuum of intuition had, in the course of the 19th century, come to be replaced by the arithmetical continuum:

Before the development of analysis was made to rest upon a purely arithmetical basis, it was usually considered that the field of operations was the continuum given by our intuition of extensive magnitude especially of spatial or temporal magnitude, and of the motion of bodies through space.

The intuitive idea of continuous motion implies that, in order that a body may pass from one position A to another position B, it must pass through every intermediate position in its path. An attempt to answer the question, what is meant by every

¹ *Ibid.*, pp. 353-4.

² See Chapter 9 below.

³ See Chapter 9 below.

intermediate position, reveals the essential difficulties of this question, and gives rise to a demand for an exact theoretical treatment of continuous magnitude.

The implication in the idea of continuous magnitude shews that, between A and B, other positions A', B' exist, which the body must occupy at definite times; that between A', B', other such positions exist, and so on. The intuitive notion of the continuum, and that of continuous motion, negate the idea that such a process of subdivision can be conceived of as having a definite termination. The view is prevalent that the intuitional notions of continuity and of continuous motion are fundamental and sui generis; and that they are incapable of being exhaustively described by a scheme of specification of positions. Nevertheless, the aspect of the continuum as a field of possible positions is the one which is accessible to Arithmetic Analysis, and with which alone Mathematical Analysis is concerned. That property of the intuitional continuum, which may be described as unlimited divisibility, is the only one that is immediately available for use in Mathematical thought,; and this property is not sufficient for the purposes in view, until it has been supplemented by a system of axioms and definitions which shall suffice to provide a complete and exact description of the possible positions of points and other geometrical objects which can be determined in space. Such a scheme constitutes an abstract theory of spatial magnitude .¹

Hobson defends the arithmetic continuum as the necessary outcome of an exact theory of measurable quantity:

The term arithmetic continuum is used to denote the aggregate of real numbers, because it is held that the system of numbers of this aggregate is adequate for the complete analytical representation of what is known as continuous magnitude. The theory of the arithmetic continuum has been criticized on the ground that it is an attempt to find the continuum within the domain of number, whereas number is essentially discrete. Such an objection presupposes the existence of some independent conception of the continuum, with which that of the aggregate of real numbers can be compared. At the time when the theory of the arithmetic continuum was developed, the only conception of the continuum which was extant was that of the continuum as given by intuition; but this, as we shall shew, is too vague a conception to be fitted for an object of exact mathematical thought, until its character as a pure intuitional datum has been clarified by exact definitions and axioms. The discussions connected with arithmetization have led to the construction of abstract theories of measurable quantity; and these all involve the use of some system of arithmetic, as providing the necessary language for the description of the relations of magnitudes and quantities. It would thus appear to be highly probable that, whatever abstract conception of the intuitional continuum of quantity and magnitude may be developed, a parallel conception of the arithmetic continuum, though not necessarily identical with the one which we have discussed, will be required. To any such scheme of numbers, the same objection might be raised as has been referred to above; but if the objection were a valid one, the complete representation of continuous magnitudes would, under any theory of such magnitudes, be impossible. It is clear that only in connection with an exact abstract theory of magnitude that any question as to the adequacy of the continuum of real numbers for the measurement of magnitudes can arise. For actual measurement of physical, or of spatial, or temporal magnitudes, the rational numbers are sufficient; such measurement being essentially

¹ Hobson (1957), pp. 54-5.

*of an approximate character only, the degree of error depending on the accuracy of the instruments employed.*¹

He admits the possibility of constructing arithmetic continua with essentially different properties, perhaps even containing infinitesimals:

*The disputable idea that the theory here explained [i.e. that of Dedekind] necessarily implies that a continuum is to be regarded as a set of points, which are elements not possessing magnitude, has frequently been a stumbling-block in the way of the acceptance of the view of the spatial continuum which has been indicated above. It has been held that, if space is to be regarded as made up of elements, these elements must themselves possess spatial character; and this view has given rise to various theories of infinitesimals or of indivisibles, as components of spatial magnitudes. The most modern and complete theory of this kind has been developed by Veronese² and is based on a denial of the Principle of Archimedes ...*³

But in the end archimedean systems are to be preferred on the grounds of simplicity:

*The validity of Veronese's system has been criticized by Cantor and others on the ground that the definitions contained in it, relating to equality and inequality, lead to contradiction; it is however unnecessary for our purpose to enter into the controversy on this point. The straight line of geometry is an ideal object of which any properties whatever may be postulated, provided that they satisfy the conditions, (1), that they form a valid scheme, i.e. one that does not lead to contradiction, and (2), that the object defined is such that it is not in contradiction with empirical straightness and linearity. There is no a priori objection to the existence of two or more such adequate conceptual systems, each self-consistent even if they be incompatible with one another; but of such rival schemes the simplest will naturally be chosen for actual use. Assuming then the possibility of setting up a valid non-Archimedean system for the straight line, still the simpler system, in which the system of Archimedes is assumed, is to be preferred, because it gives a simpler conception of the nature of the straight line, and is adequate for the purposes for which it was devised.*⁴

As Hobson's work shows, by the beginning of the 20th century, mathematical analysis had come to be placed on a set-theoretic foundation, supplanting the older methods of analysis based on infinitesimals and the intuitive continuum. In geometry, by contrast, the process of set-theorization was considerably less rapid. The geometer Sophus Lie, for example, was "untouched by it in the 1890s"⁵. Hermann Weyl's and Tullio Levi-Civita's work in the 1920s in mathematics and physics avoids the use of set-theoretic methods, making extensive use of infinitesimals, even though both "believed that ϵ - δ style foundations were better in principle."⁶ Nor is there much trace of set theory in the work of the geometer Élie Cartan, who was active in the 1930s. In fact set theory did not come to dominate geometry until the mid-20th century.

¹ Hobson (1957), pp. 53-4.

² See Chapter 5 below.

³ Hobson (1957), pp. 57-8.

⁴ *Ibid.*, p. 58.

⁵ McLarty (1988), p. 87.

⁶ *Ibid.*