

Chapter 5

Dissenting Voices: Divergent Conceptions of the Continuum in the 19th and Early 20th Centuries

Despite the great success of Weierstrass, Dedekind and Cantor in constructing the continuum from arithmetical materials, a number of thinkers of the late 19th and early 20th centuries remained opposed, in varying degrees, to the idea of explicating the continuum concept entirely in discrete terms. These include the mathematicians *du Bois-Reymond*, *Veronese*, *Poincaré*, *Brouwer* and *Weyl*, and the philosophers *Brentano* and *Peirce*.

DU BOIS-REYMOND

Paul du Bois-Reymond (1831–1889), against whose theory of infinities and infinitesimals Cantor fought so hard, was a prominent mathematician of the later 19th century who made significant contributions to real analysis, differential equations, mathematical physics and the foundations of mathematics. While accepting many of the methods of the Dedekind-Cantor school, and indeed embracing the idea of the actual infinite, he rejected its associated philosophy of the continuum on the grounds that it was committed to the reduction of the continuous to the discrete. So in 1882 he writes:

*The conception of space as static and unchanging can never generate the notion of a sharply defined, uniform line from a series of points however dense, for, after all, points are devoid of size, and hence no matter how dense a series of points may be, it can never become an interval, which must always be regarded as the sum of intervals between points.*¹

Du Bois-Reymond took a somewhat mystical view of the continuum, asserting that its true nature, being beyond the limits of human cognition, would forever elude the understanding of mathematicians. Nevertheless this did not prevent him from developing his own theory of the mathematical continuum, a *continuum of functions*, during the 1870s and 80s. This was introduced in an article of 1870-1 as the *calculus of infinities*. Here du Bois-Reymond considers “functions ordered according to the limit of their quotients”². The orderings of functions, in du Bois-Reymond’s notation,

$$f(x) \prec \varphi(x), f(x) \sim \varphi(x), f(x) \succ \varphi(x),$$

are defined respectively by

$$\lim_{x \rightarrow \infty} f(x)/\varphi(x) = \infty, \lim_{x \rightarrow \infty} f(x)/\varphi(x) \text{ is finite and } \neq 0, \lim_{x \rightarrow \infty} f(x)/\varphi(x) = 0.$$

Thus, for example, $e^x \succ x \succ \log(x)$ and $x^p \succ x$ for any $p > 1$, while $cx^r \sim x^r$ for any c and r . When $f(x) \prec \varphi(x)$, $f(x)$ is said to have an “infinity greater than $\varphi(x)$ ”; when $f(x) \succ \varphi(x)$, $f(x)$ may be thought of (although du Bois-Reymond does not say this explicitly) as being infinitesimal in comparison with $\varphi(x)$. Du Bois-Reymond considers sequences of

¹ Quoted in Ehrlich (1994), p. x

² Fisher(1981), p. 102.

functions linearly ordered under $<$ or $>$. Such “scales of infinity”¹ can be caused to become arbitrarily complex by the continued interpolation of new such sequences between terms. Du Bois-Reymond draws an analogy with the ordered set of real numbers:

*Just as between two functions two functions as close with respect to their infinities as one may want, one can imagine an infinity of others forming a kind of passage from the first function to the second, one can compare the sequence F [a scale of infinity] to the sequence of real numbers, in which one can also pass from one number to a number very little different from it by an infinity of other ones.*²

While du Bois-Reymond uses the term “infinities” in connection with his classification of functions, he does not at this point speak of infinite numbers or actual infinities. But in an article of 1875 he drops his reservations on the matter, and boldly begins by asserting:

*I decided to publish this continuation of my research on functions becoming infinite in German after I overcame my aversion to using the word ‘infinite (unendlich)’ as a substantive, like the French their ‘infini’. I even flatter myself that, by this ‘infinite (unendlich)’, I have enriched our mathematical vocabulary in a noteworthy way.*³

He goes on to say:

*In earlier articles I have distinguished the different infinities of functions by their different magnitudes so that they form a domain of quantities (the infinitary) with the stipulation that the infinity of $\phi(x)$ is to be regarded as larger than that of $\psi(x)$ or equal to it...according as the quotient $\phi(x)/\psi(x)$ is infinite or finite. Thus in the infinitary domain of quantities the quotient enters in place of the difference in the ordinary domain of numbers. Between the two domains there are many analogies... I can add further that the most complete symmetry exists between functions becoming zero and becoming infinity, in such a way that everywhere the positive numbers correspond in the most striking way to becoming infinity, the negative numbers to becoming zero, zero to remaining finite. Instead of numbers as fixed signs in the domain of numbers, one has in the infinitary domain of quantities an unlimited number of simple functions; the exponential functions, the powers, the logarithmic functions, that likewise form fixed points of comparisons, and between whose arbitrarily close infinities a limitless number of infinities different from each other can be inserted.*⁴

In a paper of 1877 du Bois-Reymond compares his system of “infinities” and that of “ordinary” numbers. He introduces the concept of “numerical continuity”, an idea which he suggests underlies the introduction of irrational numbers. To illustrate the idea, du Bois-Reymond offers as a metaphor the distribution of the stars on a great circle in the sky.⁵ The readily identified brighter stars he compares to rational numbers with small numerators and denominators. Use of telescopes reveals the presence of new stars in any region, however small, but patches of darkness are always found between them. And then

our imagination, or speculation, peoples this as it were asymptotically uniform nothingness which always remains, with matter whose radiation or our observation

¹ The term is Hardy’s: see Hardy (1910).

² Quoted in Fisher (1981), p. 104.

³ Quoted *ibid.*, p. 105.

⁴ *Ibid.*, p. 106.

⁵ *Ibid.*, pp. 107-8.

*can no longer make accessible. In our thought, we may believe there is no end, and we admit no empty spot in the sky.*¹

This is analogous to the generation of rational and irrational numbers:

*Thus through more precise consideration the rational numbers always approach more closely to one another, yet in our minds gaps are always left between them, which mathematical speculation then fills with the irrationals.*²

According to du Bois-Reymond this is essentially the way in which “numerical continuity” has arisen. He sees mathematical intuition as assigning equal authenticity to geometric and numerical quantity, but the attainment of complete equality between the two can only be attained through the use of the limit concept in introducing the irrationals. And the insertion of the irrationals between the rationals is an extension of the primitive concept of number to an equally primitive, but more comprehensive, concept of continuous quantity. That being the case, the comparison between numerical and geometric quantities may conceal further subtleties.³

One such subtlety is brought to light in connection with continuous families of curves. When these are allowed to increase with growing rapidity their approximative behaviour is quite different from that associated with ordinary spatial continuity. Du Bois-Reymond writes:

*If we think of two different quickly increasing functions, then all the transitions from one to the other are spatially conceivable and present in our minds. We cannot conceive anywhere a gap between two curves increasing to infinity or in the neighbourhood of one such curve, which could not be filled with curves; on the contrary, each curve is accompanied by curves which proceed arbitrarily close to it, to infinity.*⁴

Now, unlike the points on a line segment, the curves which run between two such curves do not form a “simple infinity”, that is, they do not depend on just a single parameter. Du Bois-Reymond shows that this infinity is “unlimited” in the sense that it is not n -fold for any finite n .⁵ He continues:

*...just as in the ordinary domain of quantities we can only express quantities numerically exactly by means of rational numbers, since the other numbers are not actual numbers but only limits of such numbers⁶: so we can only express infinities with well-defined functions, of which we only have at our disposal up to now those belonging to the family of logarithms, powers, exponential functions.*⁷

Du Bois-Reymond next notes the difference between the approximative behaviour of real numbers and that of “infinities” associated with functions⁸. While one can approximate a number, say $\frac{1}{2}$, by many sequences in such a way that any number, however close to $\frac{1}{2}$, will fall between two members of any such sequence, the situation is quite different for the functions associated with infinities. For example, consider the sequence of functions

¹ Quoted *ibid.*, p. 107

² Quoted *ibid.*, p. 107.

³ *ibid.*, p. 108.

⁴ Quoted *ibid.*, p. 108.

⁵ *Ibid.*, p. 108.

⁶ This view of irrational numbers is evidently in direct opposition to Cantor's.

⁷ Quoted *ibid.*, pp. 108-9.

⁸ *Ibid.*, p.109.

$$x^{1/2}, x^{2/3}, \dots, x^{p/p+1}, \dots$$

The exponents of the members of this sequence approach 1, but it is not hard to establish the existence of functions whose infinities fall between *all* of the infinities of the members of this sequence and the function x to which the sequence converges in an appropriate sense. For example, $x^{\frac{\log \log x}{\log \log x + 1}}$ is readily shown to be such a function.

Du Bois-Reymond next proceeds to demonstrate the generality of this phenomenon:

*One cannot approximate a given infinity $\lambda(x)$ with any sequence of functions $\varphi_p(x)$, $p = 1, 2, \dots$ in such a way that one could not always specify a function $\psi(x)$ which satisfies for arbitrarily large values of p $\lambda(x) \geq \psi(x) \geq \varphi_p(x)$.*¹

He continues:

*Now the fact that we can with no conceivable sequence of functions approach without limit a given infinity, certainly has something strange about it. For it would be ... completely counter to our intuition to suppose that there is necessarily a gap, for example around the line $y = x$. We can always fill this gap in our thoughts with curves which accompany the line $y = x$ to infinity.*²

However, du Bois-Reymond finds here

*no irreconcilable conflict of the results of different forms of thought, but only one of the ideas of a perhaps not very familiar but still not inaccessible spatial behaviour.*³

For du Bois-Reymond this only indicates the presence of “a gap between in the analogy between ordinary and infinitary quantities”, the manifestation of “a behaviour peculiar to the infinitary domain”.⁴

In his book *Die allgemeine Functionentheorie* of 1882 du Bois-Reymond presents his views on the nature and existence of infinitesimals. He begins by stating that in the analysis of “continuous mathematical quantities”, one begins with a “geometric quantity” and tries to relate other quantities to it.⁵ So the finite decimals are assigned correlates on a segment, that is, “points”. This correlation between finite decimals and points is then extended to infinite decimals by a limit process. But the totality of such points can never form a complete segment, since

*points are just dimensionless, and therefore an arbitrarily dense sequence of points can never become a distance.*⁶

Here we see once again a rejection of the idea that the continuous is reducible to the discrete. Consequently, a geometric segment must contain something other than finite and infinite decimals. These “others”, according to du Bois-Reymond, are infinitesimal segments: there are infinitely many of these in any line segment, however short.

Du Bois-Reymond provides just a few rules of calculation for infinitesimal segments, reminiscent of those used by 17th and 18th mathematicians. To wit:

¹ Quoted *ibid.*, p. 109.

² Quoted *ibid.*, p. 109.

³ Quoted *ibid.*, p. 109.

⁴ *Ibid.*, p. 110.

⁵ *Ibid.*, p. 114.

⁶ Quoted *ibid.*, p. 114.

A finite number of infinitely small segments joined to one another do not form a finite segment, but again an infinitely small segment ... no upper bound can be specified either for the finite or for the infinitely small.

I say two finite segments are equal when there is no finite difference between them ... Two finite quantities whose difference is infinitely small are equal to one another ... A finite quantity does not change if an infinitely small quantity is added to it or taken away from it. ¹

While we may be incapable of forming a mental image of the *relation* of the infinitesimal to the finite, according to du Bois-Reymond we can visualize the infinitesimal *in itself*, and when we do we find that it behaves just like the finite:

The infinitely small is a mathematical quantity, and has all its properties in common with the finite. ²

The admission of the infinitesimal in relation to the *finite* opens the way to the infinitesimal in relation to the *infinitesimal* (so entailing, reciprocally, the presence of the infinitely large):

In this way, there arises a series of types of quantities, whose successive relation always is that a finite number of quantities of one kind never yields a quantity of the preceding kinds. ³

Such quantities accordingly form a nonarchimedean domain. Moreover,

If within one and the same of these types of quantities, the properties of ordinary mathematical quantities hold, hence the same types of calculation as in the finite, then the comparison of the different types of quantities with each other is the object of the so-called infinitary calculus. This calculus reckons with the relations of the infinitely large or infinitely small from type to type, and these types show connections with each other that do not fall under the ordinary concept of equality. The passages of one type into another do not show, for example, the continuity of change of mathematical quantities, although no jump changes result. ⁴

Du Bois-Reymond concludes his musings on the infinitesimal with the observation that there is an imbalance between belief in the infinitely large and belief in the infinitely small. A majority of educated people, he says, will admit an “infinite” (i.e., actual infinite) in space and time, and not just an “unboundedly large” (i.e., potential infinite). But only with difficulty will they accept the infinitely small, despite the fact that it has the same “right to existence” as the infinitely large. ⁵ In sum,

A belief in the infinitely small does not triumph easily. Yet when one thinks boldly and freely, the initial mistrust will soon mellow into a pleasant certainty.⁶ ... Were the the sight of the starry sky lacking to mankind; had the race arisen and developed

¹ Quoted *ibid.*, p. 115.

² Quoted *ibid.*, p. 115. Du Bois-Reymond took a dim view of the conception of infinitesimals as being ordinary magnitudes continually in a state of flux towards zero, remarking sarcastically

As long as the book is closed there is perfect repose, but as soon as I open it there commences a race of all the magnitudes which are provided with the letter d towards the zero limit. (Quoted in Ehrlich (1994), pp. 9-10.)

³ Quoted *ibid.*, p. 115.

⁴ Quoted *ibid.*, p. 115.

⁵ *Ibid.*, p. 116.

⁶ *Ibid.*, p. 116.

*troglodytically in enclosed spaces; had its scholars, instead of wandering through the distant places of the universe telescopically, only looked for the smallest constituents of form and so were used in their thoughts to advancing into the boundless in the direction of the unmeasurably small: who would doubt that then the infinitely small would take the same place in our system of concepts that the infinitely large does now? Moreover, hasn't the attempt in mechanics to go back down to the smallest active elements long ago introduced into science the atom, the embodiment of the infinitely small? And don't as always skilful attempts to make it superfluous for physics face with certainty the same fate as Lagrange's battle against the differential?*¹

VERONESE

While du Bois-Reymond's conception of the infinite and infinitesimal derived from his work as an analyst, that of the second of Cantor's critical targets, **Giuseppe Veronese** (1854-1917) originated in geometry. An outstanding member of the Italian school of geometry in the last quarter of the 19th century, Veronese in 1891 published his exhaustive work on the foundations of geometry, whose title in approximate English translation reads: *Foundations of geometry of several dimensions and several kinds of linear unit, presented in elementary form*. In this work Veronese develops n -dimensional projective geometry, including non-Euclidean geometries, in a synthetic and unified way from first principles. Controversially, he also introduces "non-Archimedean" geometries containing both infinitesimal and infinitely large segments. On publication this work attracted the scathing criticism not only of Cantor, but also of Peano and Killing. Yet Hilbert later called it "profound", and incorporated some of Veronese's ideas into his own later *Grundlagen der Geometrie*.

As a geometer Veronese naturally took an essentially geometric view of the continuum. He begins his *Foundations* with a complaint about the use of real numbers as the basis of geometry. Spatial intuition, he says, is what furnishes us with the basal geometric objects and their inherent properties, so that the proper procedure in geometry is a synthetic one,

*which always treats figures as figures, works directly with the elements of the figures and separates and unites them so that each truth and each step of a proof is accompanied as far as possible by intuition.*²

In answer to the question "What is the *continuum*?" Veronese writes, in striking contrast with Cantor:

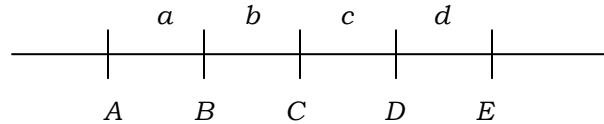
This is a word whose meaning we understand without any mathematical definition, since we intuit the continuum in its simplest form as the common characteristic of many concrete things, such as, for example, to give some of the simplest, the time and the place occupied in the external neighbourhood of the object sketched here, or by a plumb line, if one takes no account of its physical properties and its thickness (in the empirical sense). Noting the particulars of this intuitive continuum, we should approach an abstract definition of the continuum in which intuition or perceived representation of it doesn't enter any more as a necessary part, in such a way that, conversely, this definition can serve abstractly, with complete logical rigour, for the deduction of other properties of this intuitive continuum. That one can give this mathematically abstract definition, we shall see later. On the other hand, if the definition of the continuum is not merely nominal and we want it instead to conform

¹ Quoted *ibid.*, p. 116.

² Quoted in Fisher (1994), p. 135.

to the intuitive one, it must clearly arise from investigating the intuitive one, even if later the abstract definition, conforming to mathematically possible principles, contains this continuum as a special case.¹

Veronese continues by considering a rectilinear continuum L , which, “within certain limits of observation”, is seen to be divided into a sequence of consecutive identical parts $a, b, c, d, \text{etc.}$, placed from left to right:



He continues:

We see further that we can experimentally (that is, with a bounded natural sequence [i.e., a finite set] of decompositions) as well as abstractly (that is, according to any mathematically possible hypothesis or operation which doesn't contradict the results of experience) arrive at a part which is not further decomposable into parts (indivisible) of which the continuum is composed (as an instant is for time).

It is then experience itself which moves us to look for the indivisible in such a way that we cannot obtain it experimentally, because it shows that a part considered indivisible with respect to one observation is not indivisible with respect to other observations with more exact instruments or under other conditions. If we assume that an indivisible part exists, we see that we can also experimentally consider it indeterminate, and therefore smaller than any given part of the rectilinear object.²

Veronese thus considers that any given linear continuum L contains what may be termed “relative” indivisibles, i.e. parts which are, *with respect to a given means of observation*, smaller than any other given part. Any such indivisible I will be infinitesimal by comparison with the whole continuum L . On the other hand I , as a part of L , is itself a continuum and so subject to the same analysis as was the latter. So a more acute observational technique will yield parts J of I which are indivisible with respect to that technique and infinitesimal relative to J . Clearly no relative indivisible can be a mathematical point, since points are *absolutely* indivisible.

In fact for Veronese points are nothing more than signs indicating “positions of the uniting of two parts” of a (rectilinear) continuum. They are, as they were for Aristotle,

...a product of the function of abstracting in our mind... not parts of the rectilinear object.³

To elucidate the nature of points Veronese offers two thought experiments. In the first of these it is supposed that

...the part a of the rectilinear object is painted red, the remaining part α white, and suppose further that there is no other colour between the white and the red. That

¹ Quoted *ibid.*, pp. 136-137. In a footnote Veronese observes:

In order to establish the mathematical concepts, we can very well fall back on empirically obtained knowledge without therefore having to make any use of it later in the definitions themselves and in the proof.

² Quoted *ibid.*, p. 138.

³ Quoted *ibid.*, p. 138.

which separates the white from the red can be coloured neither white nor red, and therefore cannot be a part of the object, since by assumption all its parts are white or red. And this sign of separation of uniting can be considered as belonging either to the white or to the red, if one considers them independently of one another. If we now abstract from the colours, we can assume that the sign of separation between the parts a and a' belongs to the object itself.¹

Accordingly a point can belong to a continuum thorough “assignment”, but cannot be a part of it.²

In the second thought experiment, Veronese invites the reader to

cut a very fine thread at the place indicated by X with the blade of an extremely sharp knife, [so that] the two parts a and a' separate [fig.1] and we assume that one can put the thread back together without seeing where the cut was [fig. 2], in other

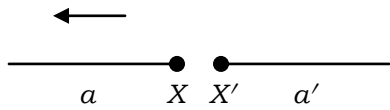


Fig. 1

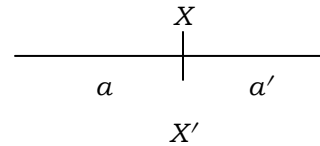


Fig. 2

words, without a particle of the thread being lost. One produces this, apparently, if one looks at the thread from a certain distance. If one now considers the part a from right to left as the arrow above a indicates, then what one sees of the cut is surely not part of the thread, just as what one sees from a body is not part of the body itself. It happens analogously if one looks at the part a' from left to right. If the sign of separation X of the parts a and a' , which by assumption belongs to the thread itself, were part of the thread, then looking at a from right to left, one would not see all of this part, since that which separates the part a from a' is only that which one sees in the way indicated above when one supposes the thread put back together.³

The premise of this argument is the assumption that the thread, or linear continuum, can be separated into two parts which, upon being rejoined, reconstitute the continuum in its entirety—in a word, that the continuum is *decomposable*. From this assumption Veronese infers that the point, or sign, of separation cannot be part of the continuum. The argument may be clearer in its contrapositive form, namely, if points of separation are parts of (linear) continua, then such continua are *indecomposable* in the sense of being inseparable into two parts which, upon being rejoined, reconstitute the continuum in its entirety.

Suppose then that any point of separation of a linear continuum L is part of L , and suppose that L is separated into two parts a , a' with point of separation X . By assumption X is part of L ; but on the other hand X cannot be part either of a or a' , since if it were part of one it would, by symmetry, have to be part of the other so that

¹ Quoted *ibid.*, p. 139

² *Ibid.*

³ *Ibid.*, pp. 139-40.

the parts a and a' would overlap contrary to assumption. So rejoining a and a' would not yield L , since the result would lack its part X . The indecomposability of L follows¹.

From the “hypothesis” that the point is not part of the linear continuum, and is itself partless, Veronese then draws the inference:

*that all the points we can imagine in it, however many that may be, do not constitute the continuum when they are joined together, and choosing a part (XX') as small as one wants of the object (for time, an instant), however indeterminate, which is to say without X and X' being fixed in our thoughts, intuition tells us that this part is always continuous.*²

As presented in the *Foundations*, the linear continuum is subject to what Veronese calls “the hypothesis on the existence of bounded infinitely large segments”, namely that, if any segment is selected as unit, and one generates the *scale* based on it, consisting of the multiples of the unit segment by natural numbers, there is always an element of the continuum lying beyond the region covered by this scale.³ In connection with this hypothesis Veronese writes:

*In order to distinguish the segments bounded by ends which generate the region of a scale with arbitrary unit (AA_1) from those which don't generate the scale and are larger than them, we call the first finite and the second actually infinitely large or infinitely large with respect to the unit [of the scale]. However, if the second is smaller than the first, we call it actually infinitely small or infinitely small with respect to the given unit. For example the unit (AA_1) or an arbitrary bounded segment of a given scale is infinitely small with respect to an infinitely large segment (AA_∞).*⁴

Veronese's segments observe the expected order relations: for example, a segment is either finite or infinitely small or large with respect to a given segment, and the sum of two segments sharing just an endpoint which are finite or infinitely small or large with respect to a given segment bear the same relation to that segment.⁵

Veronese contrasts his own account of the continuum with that of Cantor and Dedekind in the following words:

Cantor and Dedekind...assert in their valuable works that ... the one-one relation between the points of [a] line and the points forming the real continuum is arbitrary. They certainly obtain this continuum by means of a sequence of abstract definitions of symbols which, although possible, are arbitrary... According to Dedekind, the numerical continuum is necessary in order to clarify the idea of the continuum of space. According to us, however, it is the intuitive rectilinear continuum which, by means of a point without parts, that serves to give us abstract definitions with respect to the continuum itself, of which the numerical continuum is only a special case. In this way, the definitions appear not as a force which keeps our mind in check, but finds its complete justification in the perceptual representation of the continuum. One must take some account of this representation in the discussion of basic concepts, but without leaving the field of pure mathematics. Moreover, it would be truly marvellous if an abstract form as complicated as the numerical continuum obtained not only

¹ In turning Veronese's argument around in this way I do mean to imply that he would have entertained the idea that continua are indecomposable. But in both intuitionistic and smooth infinitesimal analysis (see below) continua are indecomposable and in a certain sense contain points as parts. See Chapters 9 and 10 below.

² Quoted *ibid.*, p. 140.

³ *Ibid.*, pp. 121-2.

⁴ Quoted *ibid.*, p. 123. Here A_∞ is an element outside the scale generated by (AA_1).

⁵ *Ibid.*, p. 123.

without being guided by the intuitive, but, as is done nowadays by some authors, from mere definitions of symbols, should then find itself in agreement with a representation as simple and primitive as that of the rectilinear continuum. ¹

Veronese's claim here is that Cantor and Dedekind's numerical continuum, which they regard as the "real" continuum, is itself no less arbitrary than the "arbitrary" correspondence they identify between the points of their continuum and those of a line. For Veronese the geometer it is the intuitive geometric continuum which must furnish the basis for any precise determination of the mathematical continuum, even though, as he says, such intuition "ought not [to] enter as a necessary component either in the statements of properties or of definitions, or in proofs."²

As for the points from which the Cantor-Dedekind discretized continuum is constructed, Veronese, again echoing Aristotle, has this to say:

The rectilinear continuum is independent of a system of points which we can imagine here. A system of points, if we think of a point as a sign of separation of two consecutive parts of a line or as the end of one of these parts, can never give the whole intuitive continuum, because a point has no parts. We find only that a system of points can represent sufficiently in geometrical investigations. The rectilinear continuum is never composed of its points but of segments, which the points join two by two, and which themselves are still continuous. In this way the mystery of continuity is pushed back from a given and constant part of the line to an indeterminate part as small as one likes, which is still always continuous, into which we are not permitted to enter with our representation. ... But it is well to mention that mathematically this mystery has no influence, because for us a determination of the continuum by means of a well-defined ordered system of points is sufficient. On the other hand, one should observe that a determination by points is incidental, because we have the intuition of the continuum just as well without it. If in fact one considers a point to be without parts, then.. even if we make the points of a line correspond to starting from an origin, we don't get the whole continuum. ³

Finally Veronese remarks that, as far as he knows, "it has not yet been demonstrated that there are discontinuous systems of points which satisfy all the properties of space given by experience."⁴ Given the likelihood of his knowing of Cantor's 1882 demonstration that continuous motion was possible in discontinuous spaces (see above), it would seem that Veronese did not regard the possibility of continuous motion alone as constituting a sufficient condition for a domain to possess all the properties of the space of experience. In any case, Veronese says, even if the properties of empirical space could be fully reproduced by some discontinuous system of points, "this would say nothing against the continuity of space."

BRENTANO

In his later years the Austrian philosopher **Franz Brentano** (1838-1917) became preoccupied with the nature of the continuous. Much of Brentano's philosophy has its starting-point in Aristotelian doctrine, and his conception of the continuum constitutes no exception. Aristotle's theory of the continuum, it will be recalled, rests upon the assumption that all change is continuous and that continuous variation of quality, of quantity and of position are inherent features of perception and intuition. Aristotle

¹ Quoted *ibid.*, p. 142.

² Quoted *ibid.*, p. 144.

³ Quoted *ibid.*, pp. 142-3.

⁴ Quoted *ibid.*, pp. 144.

considered it self-evident that a continuum cannot consist of points. Any pair of unextended points, he observes, are such that they either touch or are totally separated: in the first case, they yield just a single unextended point, in the second, there is a definite gap between the points. Aristotle held that any continuum—a continuous path, say, or a temporal duration, or a motion—may be divided *ad infinitum* into other continua but not into what might be called “discreta”—parts that cannot themselves be further subdivided. Accordingly, paths may be divided into shorter paths, but not into unextended points; durations into briefer durations but not into unextended instants; motions into smaller motions but not into unextended “stations”. Nevertheless, this does not prevent a continuous line from being divided at a point constituting the common border of the line segments it divides. But such points are, according to Aristotle, just *boundaries*, and not to be regarded as actual *parts* of the continuum from which they spring. If two continua have a common boundary, that common border unites them into a single continuum. Such boundaries exist only *potentially*, since they come into being when they are, so to speak, marked out as connecting parts of a continuum; and the parts in their turn are similarly dependent as parts upon the existence of the continuum.

In its fundamentals Brentano’s account of the continuous is akin to Aristotle’s. Brentano regards continuity as something given in perception, primordial in nature, rather than a mathematical construction. Brentano held that the idea of the continuous is a primordial notion abstracted from sensible intuition:

Thus I affirm that... the concept of the continuous is acquired not through combinations of marks taken from different intuitions and experiences, but through abstraction from unitary intuitions...Every single one of our intuitions—both those of outer perception as also their accompaniments in inner perception, and therefore also those of memory—bring to appearance what is continuous.¹

Brentano suggests that the continuous is brought to appearance by sensible intuition in three phases. First, sensation presents us with objects having parts that coincide. From such objects the concept of *boundary* is abstracted in turn, and then one grasps that these objects actually *contain* coincident boundaries. Finally one sees that this is all that is required in order to have grasped the concept of a continuum.

Continuity is manifested in sensation in a variety of ways. In *visual* sensation, we are presented with *extension*, something possessing length and breadth, and hence with something such that between any two of its parts, provided these be separated, there is a third part. Every sensation possesses a certain *qualitative* continuity in that the object presented in the sensation could have a given manifested quality (colour, for example) in a greater or less degree, and between any two degrees of that quality lies still another degree of that quality. Finally, each sensation manifests *temporal* continuity: this is most evident when we perceive something as moving or at rest.

Brentano recognizes that continua have *qualities* which cause them to possess *multiplicity*—a continuum may manifest continuity in several ways simultaneously. This led him to classify continua into *primary* and *secondary*: a secondary continuum being one whose manifestation is dependent upon another continuum. Here is Brentano himself on the matter:

Imagine, for example, a coloured surface. Its colour is something from which the geometer abstracts. For him there comes into consideration only the constantly changing manifold of spatial differences. But the colour, too, appears extended with the spatial surface, whether it manifests no specific colour-differences of its own—as in the case of a red colour which fills out a surface uniformly—or whether it varies in its colouring—perhaps in the manner of a rectangle which

¹ Brentano (1988), p. 6.

*begins on one side with red and ends on the other side with blue, progressing uniformly through all colour-differences from violet to pure blue in between. In both cases we have to do with a multiple continuum, and it is the spatial continuum which appears thereby as primary, the colour-continuum as secondary. A similar double continuum can also be established in the case of a motion from place to place or of a rest, in which case it is a temporal continuum as such that is primary, the temporally constant or varying place that is the secondary continuum. Even when one considers a boundary of a mathematical body as such, for example a curved or straight line, a double continuity can be distinguished. The one presents itself in the totality of the differences of place that are given in the line, which always grows uniformly, whether in the case of straight, bent, or curved lines, and is that which determines the length of the line. The other resides in the direction of the line, and is either constant or alternating, and may vary continuously, or now more strongly, now less. It is constant in the case of the straight line, changing in the case of the broken line, and continuously varying in every line that is more or less curved. The direction-continuum here is to be compared with the colour-continuum discussed earlier and with the continuum of place in the case of rest or motion of a corporeal point in time. In the double continuum that presents itself to us in the line it is this continuum of directions that is to be referred to as the secondary, the manifold of differences of place as such as the primary continuum.*¹

For Brentano the essential feature of a continuum is its inherent capacity to engender boundaries, and the fact that such boundaries can be grasped as coincident. Boundaries themselves possess a quality which Brentano calls *plerosis* (“fullness”). *Plerosis* is the measure of the number of directions in which the given boundary actually bounds. Thus, for example, within a temporal continuum the endpoint of a past episode or the starting point of a future one bounds in a single direction, while the point marking the end of one episode and the beginning of another may be said to bound doubly. In the case of a spatial continuum there are numerous additional possibilities: here a boundary may bound in all the directions of which it is capable of bounding, or it may bound in only some of these directions. In the former case, the boundary is said to exist in *full plerosis*; in the latter, in *partial plerosis*. Brentano writes:

...the spatial nature of a point differs according to whether it serves as a limit in all or only in some directions. Thus a point located inside a physical thing serves as a limit in all directions, but a point on a surface or an edge or a vertex serves as a limit in only some direction. And the point in a vertex will differ in accordance with the directions of the edges that meet at the vertex... I call these specific distinctions differences of plerosis. Like any manifold variation, plerosis admits of a more and a less. The plerosis of the centre of a cone is more complete than that

¹ *Ibid.*, p. 21f. It is worth noting that Brentano’s distinction of primary and secondary continua can be neatly represented within *category theory*: to put it succinctly, a *primary continuum* is a *domain*, a *secondary continuum* a *codomain*. We form a category \mathcal{C} —the *category of continua*—by taking continua as objects and *correlations* between continua as arrows. Then, given any arrow $A \xrightarrow{f} B$ in \mathcal{C} , the domain A of f may be taken as a “primary” continuum and its codomain B as a “secondary” continuum. In Brentano’s example of a coloured surface, for instance, the primary continuum A is the given spatial surface, the secondary continuum B is the colour spectrum, and the correlation f assigns to each place in A its colour as a position in B . In the case of a corporeal point moving in space, the primary continuum A is an interval of time, the secondary continuum B a region of space, and the correlation f assigns to each instant in A the position in B occupied by the corporeal point. Finally, in the case of the varying direction of a curve the primary continuum A is the curve itself, the secondary continuum is the continuum of measures of angles, and the correlation f assigns to each point on the curve the slope of the tangent there: thus f is nothing other than the first derivative of the function associated with the curve.

*of a point on its surface; the plerosis of a point on its surface is more complete than that of a point on its edge, or that of its vertex. Even the plerosis of the vertex is the more complete the less the cone is pointed.*¹

Brentano believed that the concept of plerosis enabled sense to be made of the idea that a boundary possesses “parts”, even when the boundary lacks dimensions altogether, as in the case of a point. Thus, while the present or “now” is, according to Brentano, temporally unextended and exists only as a boundary between past and future, it still possesses two “parts” or aspects: it is both the end of the past and the beginning of the future. It is worth mentioning that for Brentano it was not just the “now” that existed only as a boundary; since, like Aristotle he held that “existence” in the strict sense means “existence *nou*”, it necessarily followed that existing things exist only as boundaries of what has existed or of what will exist, or both.

Brentano ascribes particular importance to the fact that points in a continuum can *coincide*. On this matter he writes:

*Various other thorough studies could be made [on the continuum concept] such as a study of the impossibility of adjacent points and the possibility of coincident points, which have, despite their coincidence, distinctness and full relative independence. [This] has been and is misunderstood in many ways. It is commonly believed that if four different-coloured quadrants of a circular area touch each other at its centre, the centre belongs to only one of the coloured surfaces and must be that colour only. Galileo’s judgment on the matter was more correct; he expressed his interpretation by saying paradoxically that the centre of the circle has as many parts as its periphery. Here we will only give some indication of these studies by commenting that everything which arises in this connection follows from the point’s relativity as involves a continuum and the fact that it is essential for it to belong to a continuum. Just as the possibility of the coincidence of different points is connected with that fact, so is the existence of a point in diverse or more or less perfect plerosis. All of this is overlooked even today by those who understand the continuum to be an actual infinite multiplicity and who believe that we get the concept not by abstraction from spatial and temporal intuitions but from the combination of fractions between numbers, such as between 0 and 1.*²

Brentano’s doctrines of plerosis and coincidence of points are well illustrated by applying them to the traditional philosophical problem of the initiation of motion: if a thing begins to move, is there a last moment of its being at rest or a first moment of its being in motion? The usual objection to the claim that both moments exist is that, if they did, there would be a time between the two moments, and at that time the thing could be said neither to be at rest nor to be in motion—in violation of the law of excluded middle. Brentano’s response would be to say that both moments do exist, but that they *coincide*, so that there are no times between them; the violation of the law of excluded middle is thereby avoided. More exactly, Brentano would assert that the temporal boundary of the thing’s being at rest—the end of its being at rest—is the same as the temporal boundary of the thing’s being in motion—the beginning of its being in motion—, but the boundary is twofold in respect of its plerosis. The boundary is, in fact, in half plerosis at rest and in half plerosis in motion.

Brentano took a somewhat dim view of the efforts of mathematicians to construct the continuum from numbers. His attitude varied from rejecting such attempts as

¹ Quoted *ibid.*, p. xvii.

² Brentano (1974), p. 357.

inadequate to according them the status of “fictions”¹. This is not surprising given his Aristotelian inclination to take mathematical and physical theories to be genuine descriptions of empirical phenomena rather than idealizations: in his view, if such theories were to be taken as literal descriptions of experience, they would amount to nothing better than “misrepresentations”. Indeed, Brentano writes:

*We must ask those who say that the continuum ultimately consists of points what they mean by a point. Many reply that a point is a cut which divides the continuum into two parts. The answer to this is that a cut cannot be called a thing and therefore cannot be a presentation in the strict and proper sense at all. We have, rather, only presentations of contiguous parts. ... The spatial point cannot exist or be conceived of in isolation. It is just as necessary for it to belong to a spatial continuum as for the moment of time to belong to a temporal continuum.*²

Concerning Poincaré’s approach to the continuum³ Brentano has this to say:

*Poincaré ... follows extreme empiricists in the in the area of sensory psychology and therefore does not believe that there is granted to us an intuition of a continuous space. Poincaré’s entire mode of procedure reveals that he also denies that we are in possession of an intuition of a continuous time. We saw how first of all he inserted between 0 and 1 fractions having a whole number as numerator and a whole power of 2 as denominator. In similar fashion, he then inserted all proper fractions whose denominator is a whole power of 3, and then also all those whose denominators are powers of every other whole number. He obtained thereby a series containing all rational fractions which, as he said, already has a certain continuity about. He then inserted ... a series of irrational fractions. To these one now adds the series of fractions involving transcendental ratios... . Poincaré was prepared to admit that this process will never come to an end... . But he believed that he could be satisfied with the insertions already made. And nothing is more self-evident than that we have here a confession that the attempt to obtain a true continuum in this way has broken down.*⁴

And this on Dedekind’s:

Dedekind differs from Poincaré already in the fact that he does not wish to deny that we have an intuition of a continuum—he simply does not want to make any use thereof. ... Dedekind’s and Poincaré’s constructions share in common that they fail to recognise the essential character of the continuum, namely that it allows the distinguishing of boundaries, which are nothing in themselves, but yet in conjunction make a contribution to the continuum. Dedekind believes that either the number $\frac{1}{2}$ forms the beginning of the series $\frac{1}{2}$ to 1, so that the series 0 to $\frac{1}{2}$ would thereby be spared a final member, i.e. an end point which would belong to it, or conversely. But this is not how things are in the case of a true continuum. Much rather it is the case that, when one divides a line, every part has a starting point, but in half plerosis.⁵ ... If a red and a blue surface are in contact with each other then a red and a blue line coincide, each with different plerosis. And if a circular area is made up of three sectors, a red, a blue and a yellow, then the mid-point is a whole which consists to an

¹ In a letter to Husserl drafted in 1905, Brentano asserts that “I regard it as absurd to interpret a continuum as a set of points.”

² Brentano (1974), p. 354.

³ See below.

⁴ Brentano (1988), p. 39.

⁵ Here Brentano appears to be saying that when one divides a closed interval $[a, b]$ at an intermediate point c , one necessarily obtains the closed intervals $[a, c]$, $[c, b]$, with the common point c (in half plerosis). In that case, Brentano have probably have regarded a continuous line as indecomposable, into disjoint intervals at least.

*equal extent of a red, a blue and a yellow part. According to Dedekind this point would belong to just one of the three colour-segments, and we should have to say that it could be separated from this while the segment in question remained otherwise unchanged. Indeed the whole circular surface would then be conceivable as having been deprived of its mid-point, like Dedekind's number-series from which only the number $\frac{1}{2}$ has fallen away. One sees immediately that this is absurd if one keeps in mind that the true concept of the continuum is obtained through abstraction from an intuition, and thus also that the entire conception has missed its target.*¹

In conclusion,

*One sees that in this entire putative construction of the concept of what is continuous the goal has been entirely missed; for that which is above all else characteristic of a continuum, namely the idea of a boundary in the strict sense (to which belongs the possibility of a coincidence of boundaries), will be sought after entirely in vain. Thus also the attempt to have the concept of what is continuous spring forth out of the combination of individual marks distilled from intuition is to be rejected as entirely mistaken, and this implies further that what is continuous must be given to us in individual intuition and must therefore have been extracted therefrom.*²

Brentano's analysis of the continuum centred on its phenomenological and qualitative aspects, which are by their very nature incapable of reduction to the discrete. Brentano's rejection of the mathematicians' attempts to construct it in discrete terms is thus hardly surprising.

PEIRCE

The American philosopher-mathematician **Charles Sanders Peirce's** (1839–1914) view of the continuum was, in a sense, intermediate between that of Brentano and the arithmetizers. Like Brentano, he held that the cohesiveness of a continuum rules out the possibility of it being a mere collection of discrete individuals, or points, in the usual sense:

The very word continuity implies that the instants of time or the points of a line are everywhere welded together.

*[The] continuum does not consist of indivisibles, or points, or instants, and does not contain any except insofar as its continuity is ruptured.*³

And even before Brouwer⁴ Peirce seems to have been aware that a faithful account of the continuum will involve questioning the law of excluded middle:

Now if we are to accept the common idea of continuity ... we must either say that a continuous line contains no points or ... that the principle of excluded middle does not

¹ Brentano (1988), pp. 40 – 41. That Brentano considered “absurd” the idea of removing a single point from a continuum seems to indicate that his continuum has the same “syrupy” property as those of intuitionistic and smooth infinitesimal analysis. See Chapters (and 10 below.

² Brentano (1988), pp. 4-5.

³ Peirce (1976), p. 925.

⁴ See below.

*hold of these points. The principle of excluded middle applies only to an individual ... but places being mere possibilities without actual existence are not individuals.*¹

But Peirce also held that any continuum harbours an *unboundedly large* collection of points—in his colourful terminology, a *supermultitudinous* collection—what we would today call a *proper class*. Peirce maintained that if “enough” points were to be crowded together by carrying insertion of new points between old to its ultimate limit they would—through a *logical* “transformation of quantity into quality”—lose their individual identity and become fused into a true continuum². Here are his observations on the matter:

It is substantially proved by Euclid that there is but one assignable quantity which is the limit of a convergent series. That is, if there is an increasing convergent series, A say, and a decreasing convergent series, B say, of which every approximation exceeds every approximation of A, and if there is no rational quantity which is at once greater than every approximation of A and less than every approximation of B, then there is but one surd quantity so intermediate... There is one surd quantity and only one for each convergent series, calling two series the same if their approximations all agree after a sufficient number of terms, or if their difference approximates toward zero. But this is only to say that the multitude of surds equals the multitude of denumerable sets of rational numbers which is... the primipostnumeral³ multitude.

... We remark that there is plenty of room to insert a secundipostnumeral multitude of quantities between [a] convergent series and its limit. Any one of those quantities may likewise be separated from its neighbours, and we thus see that between it and its nearest neighbours there is ample room for a tertipostnumeral multitude of other quantities, and so on through the whole denumerable series of postnumeral quantities.

But if we suppose that all such orders of systems of quantities have been inserted, there is no longer any room for inserting any more. For to do so we must select some quantity to be thus isolated in our representation. Now whatever one we take, there will always be quantities of higher order filling up the spaces on the two sides.

¹ Peirce (1976), p. xvi: the quotation is from a note written in 1903.

² In their Introduction to Peirce [1992], Ketner and Putnam characterize Peirce’s conception of the continuum as “a possibility of repeated division which can never be exhausted in any possible world, not even in a possible world in which one can complete [nondenumerably] infinite processes.” There is some resemblance between this conception and John Conway’s system of *surreal numbers* (see Ehrlich 1994a). Conway’s system may be characterized as being an η_α -field for every ordinal α , that is, a real-closed ordered field S which satisfies the condition that, for any pair of subsets X, Y for which every member of X is less than every member of Y , there is an element of S strictly between X and Y . (In their Introduction to Peirce [1992], Ketner and Putnam characterize Peirce’s conception of the continuum as “a possibility of repeated division which can never be exhausted in any possible world, not even in a possible world in which one can complete [nondenumerably] infinite processes. This description would seem to apply equally well to Conway’s conception.) It is not hard to show that, between any pair of members of S there is a *proper class* of members of S —in Peirce’s terminology, a *supermultitudinous* collection. Nevertheless, S is still discrete: its elements, while *supermultitudinous*, remain distinct and unfused (were it not for this fact, Conway would scarcely be justified in calling the members of S “numbers”). On the face of it the discreteness of S would seem to imply that the presence of superabundant quantity in Peirce’s sense is not enough to ensure continuity. Of course, Brentano would have dismissed this idea altogether, in view of his critical attitude towards any construction of the continuum by repeated insertion of points.

³ Peirce assumed what amounts to the generalized continuum hypothesis in supposing that each possible infinite set has one of the cardinalities $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots$. These he termed *denumerable*, *primipostnumeral*, *secundipostnumeral*, etc.

We therefore see that such a supermultitudinous collection sticks together by logical necessity. Its constituent individuals are no longer distinct and independent subjects. They have no existence—no hypothetical existence—except in their relations to one another. They are not subjects, but phrases expressive of the properties of the continuum.

...Supposing a line to be a supermultitudinous collection of points, ... to sever a line in the middle is to disrupt the logical identity of the point there, and make it two points. It is impossible to sever a continuum by separating the connections of the points, for the points only exist by virtue of those connections. The only way to sever a continuum is to burst it, that is, to convert what was one into two.¹

Peirce's conception of the number continuum is also notable for the presence in it of an abundance of *infinitesimals*, a feature it shares with du Bois-Reymond's and Veronese's nonarchimedean number systems². In defending infinitesimals, Peirce remarks that

It is singular that nobody objects to $\sqrt{-1}$ as involving any contradiction, nor, since Cantor, are infinitely great quantities much objected to, but still the antique prejudice against infinitely small quantities remains.³

Peirce actually held the view that the conception of infinitesimal is suggested by introspection—that the *specious present* is in fact an infinitesimal:

It is difficult to explain the fact of memory and our apparently perceiving the flow of time, unless we suppose immediate consciousness to extend beyond a single instant. Yet if we make such a supposition we fall into grave difficulties, unless we suppose the time of which we are immediately conscious to be strictly infinitesimal.⁴

We are conscious of the present time, which is an instant, if there be any such thing as an instant. But in the present we are conscious of the flow of time. There is no flow in an instant. Hence, the present is not an instant.⁵

Peirce championed the retention of the infinitesimal concept in the foundations of the calculus, both because of what he saw as the efficiency of infinitesimal methods, and because he regarded infinitesimals as constituting the “glue” causing points on a continuous line to lose their individual identity.

POINCARÉ

The idea of continuity played a central role in the thought of the great French mathematician **Henri Poincaré** (1854–1912). But sorting out his views on the

¹ Peirce (1976), p. 95.

² I do not know whether Peirce was acquainted with their work.

³ Peirce (1976), p. 123. In this connection it is worth quoting from a letter addressed by Peirce in 1900 to the editor of *Science* in which he defends his views on infinitesimals against the strictures of Josiah Royce:

Professor Royce remarks that my opinion that differentials may quite logically be considered as true infinitesimals, if we like, is shared by no mathematician “outside of Italy”. As a logician, I am more comforted by corroboration in the clear mental atmosphere of Italy than I could be by any seconding from a tobacco-clouded and bemused land (if any such there be) where no philosophical eccentricity misses its champion, but where sane logic has not found favor.

⁴ *Ibid.*, p. 124.

⁵ *Ibid.*, p. 925.

continuum, concerning which he made numerous scattered remarks, is by no means an easy task. Indeed there seems to be an inconsistency in his attitude towards the set-theoretical, or arithmetized, continuum. On the one hand, he rejected actual infinity and impredicative¹ definition—both cornerstones of the Cantorian theory of sets which underpins the construction of the arithmetized continuum. And yet in his mathematical work he employs variables ranging over all the points of an interval of the set-theoretical continuum, and he “accepts the standard account of the least upper bound, which is impredicative.”² But beneath this apparent inconsistency lies his belief that what ultimately underpins mathematics, creating its linkage with objective reality, is *intuition*—that “intuition is what bridges the gap between symbol and reality.”³ His view of the continuum, in particular, is informed by this credo. For Poincaré the continuum and the range of points on it is grasped in intuition in something like the Kantian sense, and yet the continuum cannot be treated as a completed mathematical object, as a “mere set.”⁴ ,

Of the arithmetical continuum Poincaré remarks:

The continuum so conceived is only a collection of individuals ranged in a certain order, infinite to one another, it is true, but exterior to one another. This is not the ordinary conception, wherein is supposed between the elements of the continuum a sort of intimate bond which makes of them a whole, where the point does not exist before the line, but the line before the point. Of the celebrated formula “the continuum is unity in multiplicity”, only the multiplicity remains, the unity has disappeared. The analysts are none the less right in defining the continuum as they do, for they always reason on just this as soon as they pique themselves on their rigor. But this is enough to apprise us that the veritable mathematical continuum is a very different thing from that of the physicists and the metaphysicians. ⁵

But despite Poincaré’s apparent acceptance of the arithmetic definition of the continuum, he questions the fact that (as with Dedekind and Cantor’s formulations) the (irrational) numbers so produced are mere symbols, detached from their origins in intuition:

*But to be content with this [fact] would be to forget too far the origin of these symbols; it remains to explain how we have been led to attribute to them a sort of concrete existence, and, besides, does not the difficulty begin even for the fractional numbers themselves? Should we have the notion of these numbers if we had not known a matter that we conceive as infinitely divisible, that is to say, a continuum?*⁶

That being the case, Poincaré asks whether the notion of the mathematical continuum is “simply drawn from experience.” To this he responds in the negative, for the reason that our sensations, the “raw data of experience”, cannot be brought under an acceptable scheme of measurement:

¹ Impredicativity is a form of circularity: a definition of a term is impredicative if it contains a reference to a totality to which the term under definition belongs. See, e.g., Fraenkel, Bar-Hillel and Levy (1973), pp. 193-200..

² Folina (1992), p. xv.

³ *Ibid.*, p. 113. And yet Poincaré also remarks, in connection with the continuous nowhere differentiable functions of analysis:

Instead of seeking to reconcile intuition with analysis, we have been content to sacrifice one of the two, and as analysis must remain impeccable, we have decided against intuition (1946, p. 52).

⁴ Folina (1992), p. xvi.

⁵ Poincaré (1946), pp. 43-44.

⁶ *Ibid.*, pp. 45-6.

It has been observed, for example, that a weight A of 10 grams and a weight B of 11 grams produce identical sensations, that the weight B is just as indistinguishable from a weight C of 12 grams, but that the weight A is easily distinguished from the weight C. Thus the raw results of experience may be expressed by the following relations:

$$A = B, B = C, A < C,$$

which may be regarded as the formula of the physical continuum.

According to Poincaré it is the “intolerable discord with the principle of contradiction” of this formula¹ which has forced the invention of the mathematical continuum. This latter is obtained in two stages. First, formerly indistinguishable terms are distinguished and a new term, indistinguishable from both, inserted between them. Repeating this procedure indefinitely gives rise to what Poincaré calls a *first-order continuum*, in essence the rational number line. A second stage now becomes necessary because two first-order continua, for example the diagonal of a square and its inscribed circle, need not intersect. This second stage, in which are added all possible “boundary” points between first-order continua leads to the second-order or mathematical continuum. Here is how Poincaré describes the process:

But conceive of a straight line divided into two rays. Each of these rays will appear to our imagination as a band of a certain breadth; these bands moreover will encroach one on the other, since there must be no interval between them. The common part will appear to us as a point which will always remain when we try to imagine our bands narrower and narrower, so that we admit as an intuitive truth that if a straight line is cut into two rays their common boundary is a point; we recognize here the conception of Dedekind, in which an incommensurable number was regarded as the common boundary of two classes of rational numbers.

Such is the origin of the continuum of second order, which is the mathematical continuum so called.²

Poincaré goes on to discuss continua of higher dimensions. To obtain these he considers *aggregates* of sensations. As with single sensations, any given pair of these aggregates may or may not be distinguishable. He remarks that, while these aggregates, which he terms *elements*, are analogous to mathematical points, they are not in fact quite the same thing, for

we cannot say that our element is without extension, since we cannot distinguish it from neighbouring elements and it is thus surrounded by a sort of haze. If the astronomical comparison may be allowed, our ‘elements’ would be like nebulae, whereas the mathematical points would be like stars.³

This leads to a definition of a physical continuum:

a system of elements will form a continuum if we can pass from any one of them to any other, by a series of consecutive elements such that each is indistinguishable

¹ This formula ceases to be contradictory if the identity relation = is replaced by a symmetric, reflexive, but *nontransitive* relation \approx : here $x \approx y$ is taken to assert that the sensations or perceptions x and y are indistinguishable. See Appendix below.

² *Ibid.*, p. 49.

³ *Ibid.*, p. 52.

*from the preceding. This linear series is to the line of the mathematician what an isolated element was to the point.*¹

Poincaré defines a *cut* in a physical continuum C to be a set of elements removed from it “which for an instant we shall regard as no longer belonging to this continuum.” Such a cut may happen to subdivide C into several distinct continua, in which case C will contain two distinct elements A and B that must be regarded as belonging to two distinct continua. This becomes necessary

*because it will be impossible to find a linear series of consecutive elements of C , each of these elements indistinguishable from the preceding, the first being A and the last B , without one of the elements of this series being indistinguishable from one of the elements of the cut.*²

On the other hand, it may happen that the cut fails to subdivide the continuum C , in which case it becomes necessary to determine precisely which cuts will subdivide it. Poincaré calls a continuum *one-dimensional* if it can be subdivided by a cut reducing to a finite number of elements all distinguishable from one another (and so forming neither a continuum nor several continua). When C can be subdivided only by cuts which are themselves continua, C is said to possess several dimensions:

*If cuts which are continua of one dimension suffice, we shall say that C has two dimensions; if cuts of two dimensions suffice, we shall say that C has three dimensions, and so on.*³

Thus is defined the concept of a multidimensional physical continuum, based on “the very simple fact that two aggregates of sensations are distinguishable or indistinguishable.”

Unlike Cantor, Poincaré accepted the infinitesimal, even if he did not regard all of the concept’s manifestations as useful. This emerges from his answer to the question: “Is the creative power of the mind exhausted by the creation of the mathematical continuum?”. He responds:

No; the works of Du Bois-Reymond demonstrate it in a striking way. We know the mathematicians distinguish between infinitesimals and that those of second order are infinitesimal, not only in an absolute way, but also in relation to those of first order. It is not difficult to imagine infinitesimals of fractional and even irrational order, and thus we find again that scale of the mathematical continuum which has been dealt with in the preceding pages.

Further, there are infinitesimals which are infinitely small in relation to those of the first order, and, on the contrary, infinitely great in relation to those of order $1 + \varepsilon$, and that however small ε may be. Here, then, are new terms intercalated in our series ... I shall say that thus has been created a sort of continuum of the third order.

It would be easy to go further, but that would be idle; one would only be imagining symbols without possible application, and no one would think of doing that. The continuum of the third order, to which the consideration of the different orders of infinitesimals leads, is itself not useful enough to have won citizenship, and

¹ *Ibid.*

² *Ibid.*, p. 53.

³ *Ibid.*

*geometers regard it as a mere curiosity. The mind uses its creative faculty only when experience requires it.*¹

Poincaré's attitude towards the continuum resembles in certain respects that of the intuitionists (see below): while the continuum exists, and is knowable intuitively, it is not a "completed" set-theoretical object. It is geometric intuition, not set theory, upon which the totality of real numbers is ultimately grounded.

BROUWER

The Dutch mathematician **L. E. J. Brouwer** (1881–1966) is best known as the founder of the philosophy of (neo)*intuitionism*. Brouwer's highly idealist views on mathematics bore some resemblance to Kant's. For Brouwer, mathematical concepts are admissible only if they are adequately grounded in intuition, mathematical theories are significant only if they concern entities which are constructed out of something given immediately in intuition, and mathematical demonstration is a form of construction in intuition. Brouwer's insistence that mathematical proof be constructive in this sense required the jettisoning of certain received principles of classical logic, notably the *law of excluded middle*. Brouwer maintained, in fact, that the applicability of the law of excluded middle to mathematics

*was caused historically by the fact that, first, classical logic was abstracted from the mathematics of the subsets of a definite finite set, that, secondly, an a priori existence independent of mathematics was ascribed to the logic, and that, finally, on the basis of this supposed apriority it was unjustifiably applied to the mathematics of infinite sets.*²

Thus Brouwer held that much of modern mathematics is based on an illicit extension of procedures valid only in the restricted domain of the finite. He therefore embarked on the radical course of jettisoning virtually all of the mathematics of his day—in particular the set-theoretical construction of the continuum—and starting anew, using only concepts and modes of inference that could be given clear intuitive justification. In the process it would become clear precisely what are the logical laws that intuitive, or constructive, mathematical reasoning actually obeys, making possible a comparison of the resulting *intuitionistic*, or *constructive logic* with classical logic³.

While admitting that the emergence of noneuclidean geometry had discredited Kant's view of space, Brouwer maintained, in opposition to the logicians (whom he called "formalists") that arithmetic, and so all mathematics, must derive from *temporal intuition*. In his own words:

Neointuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-ness. This intuition of two-ness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-ness may be thought of as a new two-ness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal ω . Finally this basal intuition of mathematics,

¹ *Ibid.*, pp. 50–1.

² Quoted in Kneebone (p. 246.

³This is not to say that Brouwer was primarily interested in *logic*, far from it: indeed, his distaste for formalization caused him to be quite dismissive of subsequent codifications of intuitionistic logic.

in which the connected and the separate, the Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the “between”, which is not exhaustible by the interposition of new units and which can therefore never be thought of as a mere collection of units. In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry, and not only for elementary two- and three-dimensional geometry, but for non-euclidean and n-dimensional geometries as well. For since Descartes we have learned to reduce all these geometries to arithmetic by means of coordinates.¹

Brouwer maintained that it is the awakening of awareness of the temporal continuum in the subject, an event termed by him “The Primordial Happening” or “The Primordial Intuition of Time”, that engenders the fundamental concepts and methods of mathematics. In “Mathematics, Science and Language” (1929), he describes how the notion of number—the discrete—emerges from the awareness of the continuous:

Mathematical Attention as an act of the will serves the instinct for self-preservation of individual man; it comes into being in two phases; time awareness and causal attention. The first phase is nothing but the fundamental intellectual phenomenon of the falling apart of a moment of life into two qualitatively different things of which one is experienced as giving away to the other and yet is retained by an act of memory. At the same time this split moment of life is separated from the Ego and moved into a world of its own, the world of perception. Temporal twofoldness, born from this time awareness, or the two-membered sequence of time phenomena, can itself again be taken as one of the elements of a new twofoldness, so creating temporal threedom, and so on. In this way, by means of the self-unfolding of the fundamental phenomenon of the intellect, a time sequence of phenomena is created of arbitrary multiplicity.²

But in his doctoral dissertation of 1907 he regards continuity and discreteness as complementary notions, neither of which is reducible to the other:

...We shall go further into the basic intuition of mathematics (and of every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a “between”, which is never exhausted by the insertion of new entities. Since continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear, it is impossible to avoid [regarding each one of them as a primitive entity... Having recognized that the intuition of continuity, of “fluidity” is as primitive as that of several things conceived as forming together a unit, the latter being at the basis of every mathematical construction, we are able to state properties of the continuum as “a matrix of points to be thought of as a whole”³

In that work Brouwer states unequivocally that the continuum is not constructible from discrete points:

...The continuum as a whole [is] given to us by intuition; a construction for it, an action which would create from the mathematical intuition ‘all’ its points as individuals, is inconceivable and impossible.⁴

¹ Brouwer, *Intuitionism and Formalism*, in Benacerraf and Putnam (1977), p. 80.

² Brouwer, *Mathematics Science and Language*. In Mancosu (1998), p.45

³ Brouwer (1975), p. 17.

⁴ *Ibid.* p. 45.

Later Brouwer was to modify this doctrine. In his mature thought, he radically transformed the concept of point, endowing points with sufficient fluidity to enable them to serve as generators of a “true” continuum. This fluidity was achieved by admitting as “points”, not only fully defined discrete numbers such as $\sqrt{2}$, π , e , and the like—which have, so to speak, already achieved “being”—but also “numbers” which are in a perpetual state of becoming in that their the entries in their decimal (or dyadic) expansions are the result of free acts of choice by a subject operating throughout an indefinitely extended time. The resulting *choice sequences* cannot be conceived as finished, completed objects: at any moment only an initial segment is known¹. In this way Brouwer obtained the mathematical continuum in a way compatible with his belief in the primordial intuition of time—that is, as an unfinished, indeed unfinishable entity in a perpetual state of growth, a “medium of free development”. In this conception, the mathematical continuum is indeed “constructed”, not, however, by initially shattering, as did Cantor and Dedekind, an intuitive continuum into isolated points, but rather by assembling it from a complex of continually changing overlapping parts.

The mathematical continuum as conceived by Brouwer displays a number of features that seem bizarre to the classical eye. For example, in the Brouwerian continuum the usual law of comparability, namely that for any real numbers a , b either $a < b$ or $a = b$ or $a > b$, fails. Even more fundamental is the failure of the law of excluded middle in the form that for any real numbers a , b , either $a = b$ or $a \neq b$. The failure of these seemingly unquestionable principles in turn vitiates the proofs of a number of basic results of classical analysis, for example the Bolzano-Weierstrass theorem, as well as the theorems of monotone convergence, intermediate value, least upper bound, and maximum value for continuous functions².

While the Brouwerian continuum may possess a number of negative features from the standpoint of the classical mathematician, it has the merit of corresponding more closely to the continuum of intuition than does its classical counterpart. Hermann Weyl pointed out a number of respects in which this is so:

*In accordance with intuition, Brouwer sees the essential character of the continuum, not in the relation between element and set, but in that between part and whole. The continuum falls under the notion of the ‘extensive whole’, which Husserl characterizes as that “which permits a dismemberment of such a kind that the pieces are by their very nature of the same lowest species as is determined by the undivided whole.”*³

Far from being bizarre, the failure of the law of excluded middle for points in the intuitionistic continuum is seen by Weyl as “fitting in well with the character of the intuitive continuum”:

*For there the separateness of two places, upon moving them toward each other, slowly and in vague gradations passes over into indiscernibility. In a continuum, according to Brouwer, there can be only continuous functions. The continuum is not composed of parts.*⁴

¹ For an illuminating informal account of choice sequences, see Fraenkel, Bar-Hillel and Levy (1973), pp. 255-261.

² The failure of these important results of classical analysis in caused most mathematicians of the day to shun intuitionistic, and even constructive mathematics. It was not until the 1960s that adequate constructive versions were worked out. See Chapter 9 below.

³ Weyl (1949), p. 52.

⁴ *Ibid.*, p. 54.

For Brouwer had indeed shown, in 1924, that every function defined on a closed interval of his continuum is uniformly continuous¹. As a consequence the intuitionistic continuum is *indecomposable*, it is what we have termed an *Aristotelian* continuum². In contrast with a discrete entity, the indecomposable Brouwerian continuum cannot be composed of its parts. Brouwer's vision of the continuum has in recent years become the subject of intensive investigation by logicians and category-theorists.

WEYL

Hermann Weyl (1885–1955), one of most versatile mathematicians of the 20th century, was unusual among scientists in being attracted to idealist philosophy. In his youth he inclined towards the idealism of Kant and Fichte, and later came to be influenced by Husserl's phenomenology. His idealist leanings can be seen particularly in his work on the foundations of mathematics.

Towards the end of his *Address on the Unity of Knowledge*, delivered at the 1954 Columbia University bicentennial celebrations, Weyl enumerates what he considers to be the essential constituents of knowledge. At the top of his list³ comes

*...intuition, mind's ordinary act of seeing what is given to it.*⁴

Throughout his life Weyl held to the view that intuition, or *insight*, not *proof*, furnishes the ultimate foundation of *mathematical* knowledge. Thus in *Das Kontinuum* of 1918 he writes:

*In the Preface to Dedekind (1888) we read that "In science, whatever is provable must not be believed without proof." This remark is certainly characteristic of the way most mathematicians think. Nevertheless, it is a preposterous principle. As if such an indirect concatenation of grounds, call it a proof though we may, can awaken any "belief" apart from assuring ourselves through immediate insight that each individual step is correct. In all cases, this process of confirmation—and not the proof—remains the ultimate source from which knowledge derives its authority; it is the "experience of truth".*⁵

While Weyl held that the roots of mathematics lay in the intuitively given, he recognized at the same time that it would be unreasonable to require all mathematical knowledge to possess intuitive immediacy. In *Das Kontinuum*, for example, he says:

*The states of affairs with which mathematics deals are, apart from the very simplest ones, so complicated that it is practically impossible to bring them into full givenness in consciousness and in this way to grasp them completely.*⁶

But Weyl did not think that this fact furnished justification for extending the bounds of mathematics to embrace notions which cannot be given fully in intuition *even in principle* (e.g., the actual infinite). He held, rather, that this extension of

¹ One might be inclined to regard this claim as impossible: is not a counterexample provided by, for example, the function f given by $f(0) = 0$, $f(x) = |x|/x$ otherwise? No, because from the intuitionistic standpoint this function is not everywhere defined on the interval $[-1, 1]$, being undefined at those arguments x for which it is unknown whether $x = 0$ or $x \neq 0$.

² See Chapter 9 below.

³ The others, in order, are: *understanding and expression*; *thinking the possible*; and finally, in science, *the construction of symbols or measuring devices*.

⁴ Weyl [1954], p. 629.

⁵ Weyl [1987], p. 119.

⁶ *Ibid.*, p. 17.

mathematics into the transcendent—the realm of being not fully accessible to intuition—is necessitated by the fact that mathematics plays an indispensable role in the physical sciences, where intuitive evidence is *necessarily* transcended. As he says in *The Open World*:

*... if mathematics is taken by itself, one should restrict oneself with Brouwer to the intuitively cognizable truths ... nothing compels us to go farther. But in the natural sciences we are in contact with a sphere which is impervious to intuitive evidence; here cognition necessarily becomes symbolical construction. Hence we need no longer demand that when mathematics is taken into the process of theoretical construction in physics it should be possible to set apart the mathematical element as a special domain in which all judgments are intuitively certain; from this higher standpoint which makes the whole of science appear as one unit, I consider Hilbert to be right.*¹

In *Consistency in Mathematics* (1929), Weyl characterized the mathematical method as

*the a priori construction of the possible in opposition to the a posteriori description of what is actually given.*²

The problem of mapping the limits on constructing “the possible” in this sense occupied Weyl a great deal. He was greatly exercised by the concept of the mathematical *infinite*, which he believed to elude “construction” in the idealized sense of set theory. Again to quote a passage from *Das Kontinuum*:

*No one can describe an infinite set other than by indicating properties characteristic of the elements of the set. ... The notion that a set is a “gathering” brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; “inexhaustibility” is essential to the infinite.*³

It is the necessity of bridging the gap between mathematics and external reality that compels the former to embody a conception of the actual infinite, as Weyl attests towards the end of *The Open World*:

*The infinite is accessible to the mind intuitively in the form of a field of possibilities open to infinity, analogous to the sequence of numbers which can be continued indefinitely, but the completed, the actual infinite as a closed realm of actual existence is forever beyond its reach. Yet the demand for totality and the metaphysical belief in reality inevitably compel the mind to represent the infinite as closed being by symbolical construction.*⁴

As a fundamental, but at the same time perplexing “possible” in mathematics, the *continuum* became the subject of what was arguably Weyl’s most searching mathematico-philosophical analysis. In his *Philosophy of Mathematics and Natural Science* he reflects on what he calls the “inwardly infinite” nature of a continuum:

The essential character of the continuum is clearly described in this fragment of Anaxagoras: “Among the small there is no smallest, but always something smaller. For what is cannot cease to be no matter how small it is being subdivided.” The

¹ Weyl [1932], p. 82.

² Weyl [1929], p. 249.

³ Weyl [1987], p. 23.

⁴ Weyl [1932], p. 83.

*continuum is not composed of discrete elements which are “separated from one another as though chopped off by a hatchet.” Space is infinite not only in the sense that it never comes to an end; but at every place it is, so to speak, inwardly infinite, inasmuch as a point can only be fixed as step-by-step by a process of subdivision which progresses ad infinitum. This is in contrast with the resting and complete existence that intuition ascribes to space. The “open” character is communicated by the continuous space and the continuously graded qualities to the things of the external world. A real thing can never be given adequately, its “inner horizon” is unfolded by an infinitely continued process of ever new and more exact experiences; it is, as emphasized by Husserl, a limiting idea in the Kantian sense. For this reason it is impossible to posit the real thing as existing, closed and complete in itself. The continuum problem thus drives one to epistemological idealism. Leibniz, among others, testifies that it was the search for a way out of the “labyrinth of the continuum” which first suggested to him the conception of space and time as orders of phenomena.*¹

Weyl identifies three attempts in the history of thought “to conceive of the continuum as Being in itself.”² These are, respectively, atomism, the infinitely small, and set theory. In Weyl’s view, despite atomism’s brilliant success in unravelling the structure of matter, it had failed in that regard as to space, time, and mathematical extension because it “never achieved sufficient contact with reality.” As for the infinitely small, it was not so much supplanted as rendered superfluous by the limit concept. Weyl saw the limit concept as providing the necessary link between the microcosm of the infinitely small and the realm of macroscopic objects. Without that link, the fact that the microcosm is governed by “elementary laws” making for ease of calculation, would remain entirely useless in drawing conclusions about the macrocosm.

Weyl believed that the ground of mathematics lies in what he calls *constructive cognition*, which unfolds in the three stages:

1. *We ascribe to that which is given certain characters which are not manifest in the phenomena but are arrived at as the result of certain mental operations. It is essential that the performance of these operations be universally possible and that their result is held to be uniquely determined by the given. But it is not essential that the operations which define the character be actually carried out.*
2. *By the introduction of symbols the assertions are split so that one part of the operations is shifted to the symbols and thereby made independent of the given and its continued existence. Thereby the free manipulation of concepts is contrasted with their application, ideas become detached from reality and acquire a relative independence.*
3. *Characters are not individually exhibited as they actually occur, but their symbols are projected onto the background of an ordered manifold of possibilities which can be generated by a fixed process and is open into infinity.*³

This threefold process is above all manifested in the generation of the infinite sequence of natural numbers. But then, says Weyl, cognition makes “a leap into the beyond” by turning the number sequence “that is never complete but remains open into the infinite into “a closed aggregate of objects existing in themselves.”⁴ This potentially dangerous move is compounded in the third attempt at hypostatizing the continuum, set theory, for it ascribes an analogous closure to “the places in the continuum, i.e. to the possible sequences or sets of natural numbers.”⁵ In Weyl’s view this was a double error, for

¹ Weyl (1949), p. 41.

² *Ibid.*, p. 42

³ *Ibid.*, pp. 37- 8.

⁴ *Ibid.*, p. 38.

⁵ *Ibid.*, p. 46.

neither the aggregate of sets of natural numbers, nor (in general) individual such sets can be considered finished entities. Rather the continuum should be considered as an essentially incompletable “field of constructive possibilities”¹ To suppose otherwise is to risk running up against set-theoretic paradoxes such as Russell’s.

During the period 1918–1921 Weyl wrestled with the problem of providing the continuum with an exact mathematical formulation free of objectionable set-theoretic assumptions. As he saw it in 1918, there is an unbridgeable gap between intuitively given continua (e.g. those of space, time and motion) on the one hand, and the discrete exact concepts of mathematics (e.g. that of real number) on the other. For Weyl the presence of this split meant that the construction of the mathematical continuum could not simply be “read off” from intuition. Rather, he believed at this time that the mathematical continuum must be treated as if it were an element of the transcendent realm, and so, in the end, justified in the same way as a physical theory. In Weyl’s view, it was not enough that the mathematical theory be *consistent*; it must also be *reasonable*.

Das Kontinuum (1918) embodies Weyl’s attempt at formulating a theory of the continuum which satisfies the first, and, as far as possible, the second, of these requirements. In the following passages from this work he acknowledges the difficulty of the task:

*... the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd.*²

*... the continuity given to us immediately by intuition (in the flow of time and of motion) has yet to be grasped mathematically as a totality of discrete “stages” in accordance with that part of its content which can be conceptualized in an exact way.*³

*Exact time- or space-points are not the ultimate, underlying atomic elements of the duration or extension given to us in experience. On the contrary, only reason, which thoroughly penetrates what is experientially given, is able to grasp these exact ideas. And only in the arithmetico-analytic concept of the real number belonging to the purely formal sphere do these ideas crystallize into full definiteness.*⁴

*When our experience has turned into a real process in a real world and our phenomenal time has spread itself out over this world and assumed a cosmic dimension, we are not satisfied with replacing the continuum by the exact concept of the real number, in spite of the essential and undeniable inexactness arising from what is given.*⁵

However much he may have wished it, in *Das Kontinuum* Weyl did not aim to provide a mathematical formulation of the continuum as it is presented to intuition, which, as the quotations above show, he regarded as an impossibility (at that time at least). Rather, his goal was first to achieve *consistency* by putting the *arithmetical* notion of real number on a firm logical basis, and then to show that the resulting theory is

¹ *Ibid.*, p. 50.

² Weyl [1987], p. 108.

³ *Ibid.*, p. 24.

⁴ *Ibid.*, p. 94.

⁵ *Ibid.*, p. 93.

reasonable by employing it as the foundation for a plausible account of continuous process in the objective physical world.¹

Weyl had come to believe that mathematical analysis at the beginning of the 20th century would not bear logical scrutiny, for its essential concepts and procedures involved vicious circles to such an extent that, as he says, “every cell (so to speak) of this mighty organism is permeated by contradiction.” In *Das Kontinuum* he tries to overcome this by providing analysis with a *predicative* formulation—not, as Russell and Whitehead had attempted in their *Principia Mathematica*, by introducing a hierarchy of logically ramified types, which Weyl seems to have regarded as too complicated—but rather by confining the basic principle of set formation to formulas whose bound variables range over just the initial given entities (numbers). Thus he restricts analysis to what can be done in terms of natural numbers with the aid of three basic logical operations, together with the operation of substitution and the process of “iteration”, i.e., primitive recursion. Weyl recognized that the effect of this restriction would be to render unprovable many of the central results of classical analysis—e.g., Dirichlet’s principle that any bounded set of real numbers has a least upper bound²—but he was prepared to accept this as part of the price that must be paid for the security of mathematics.

In '6 of *Das Kontinuum* Weyl presents his conclusions as to the relationship between the intuitive and mathematical continua. He poses the question: Does the mathematical framework he has erected provide an adequate representation of physical or temporal continuity as it is *actually experienced*? He begins his investigation by noting that, according to his theory, if one asks whether a given function is continuous, the answer is not fixed once and for all, but is, rather, dependent on the extent of the domain of real numbers which have been defined up to the point at which the question is posed. Thus the continuity of a function must always remain *provisional*; the possibility always exists that a function deemed continuous *now* may, with the emergence of “new” real numbers, turn out to be discontinuous *in the future*.³

To reveal the discrepancy between this formal account of continuity based on real numbers and the properties of an intuitively given continuum, Weyl next considers the experience of seeing a pencil lying on a table before him throughout a certain time interval. The position of the pencil during this interval may be taken as a function of the time, and Weyl takes it as a fact of observation that during the time interval in question this function is continuous and that its values fall within a definite range. And so, he says,

This observation entitles me to assert that during a certain period this pencil was on the table; and even if my right to do so is not absolute, it is nevertheless reasonable and well-grounded. It is obviously absurd to suppose that this right can be undermined by “an expansion of our principles of definition”—as if new moments of time, overlooked by my intuition could be added to this interval, moments in which the pencil was, perhaps, in the vicinity of Sirius or who knows where. If the temporal

¹ The connection between mathematics and physics was of course of paramount importance for Weyl. His seminal work on relativity theory, *Space-Time-Matter*, was published in the same year (1918) as *Das Kontinuum*; the two works show subtle affinities.

² In this connection it is of interest to note that on 9 February 1918 Weyl and George Pólya made a bet in Zürich in the presence of twelve witnesses (all of whom were mathematicians) that “within 20 years, Pólya, or a majority of leading mathematicians, will come to recognize the falsity of the least upper bound property.” When the bet was eventually called, everyone—with the single exception of Gödel—agreed that Pólya had won.

³ This fact would seem to indicate that in Weyl’s theory the domain of definition of a function is not unambiguously determined by the function, so that the continuity of such a “function” may vary with its domain of definition. (This would be a natural consequence of Weyl’s definition of a function as a certain kind of relation.) A simple but striking example of this phenomenon is provided in classical analysis by the function f which takes value 1 at each rational number, and 0 at each irrational number. Considered as a function defined on the rational numbers, f is constant and so continuous; as a function defined on the real numbers, f fails to be continuous anywhere.

*continuum can be represented by a variable which “ranges over” the real numbers, then it appears to be determined thereby how narrowly or widely we must understand the concept “real number” and the decision about this must not be entrusted to logical deliberations over principles of definition and the like.*¹

To drive the point home, Weyl focuses attention on the fundamental continuum of *immediately given phenomenal time*, that is, as he characterizes it,

*... to that constant form of my experiences of consciousness by virtue of which they appear to me to flow by successively. (By “experiences” I mean what I experience, exactly as I experience it. I do not mean real psychical or even physical processes which occur in a definite psychic-somatic individual, belong to a real world, and, perhaps, correspond to the direct experiences.)*²

In order to correlate mathematical concepts with phenomenal time in this sense Weyl grants the possibility of introducing a rigidly punctate “now” and of identifying and exhibiting the resulting temporal points. On the collection of these temporal points is defined the relation of *earlier than* as well as a congruence relation of *equality of temporal intervals*, the basic constituents of a simple mathematical theory of time. Now Weyl observes that the discrepancy between phenomenal time and the concept of real number would vanish if the following pair of conditions could be shown to be satisfied:

1. *The immediate expression of the intuitive finding that during a certain period I saw the pencil lying there were construed in such a way that the phrase “during a certain period” was replaced by “in every temporal point which falls within a certain time span OE. [Weyl goes on to say parenthetically here that he admits “that this no longer reproduces what is intuitively present, but one will have to let it pass, if it is really legitimate to dissolve a period into temporal points.”]*

2. *If P is a temporal point, then the domain of rational numbers to which t belongs if and only if there is a time point L earlier than P such that $OL = t.OE$ can be constructed arithmetically in pure number theory on the basis of our principles of definition, and is therefore a real number in our sense.*³

Condition 2 means that, if we take the time span *OE* as a unit, then each temporal point *P* is correlated with a definite real number. In an addendum Weyl also stipulates the converse.

But can temporal intuition itself provide evidence for the truth or falsity of these two conditions? Weyl thinks not. In fact, he states unequivocally that

*... everything we are demanding here is obvious nonsense: to these questions, the intuition of time provides no answer—just as a man makes no reply to questions which clearly are addressed to him by mistake and, therefore, are unintelligible when addressed to him.*⁴

The grounds for this assertion are by no means immediately evident, but one gathers from the passages following it that Weyl regards the experienced *continuous flow* of phenomenal time as constituting an insuperable barrier to the whole enterprise of representing this continuum in terms of individual points, and even to the characterization of “individual temporal point” itself. As he says,

¹ Weyl [1987], p. 88.

² *Ibid.*

³ *Ibid.*, p. 89.

⁴ *Ibid.*, p. 90.

The view of a flow consisting of points and, therefore, also dissolving into points turns out to be mistaken: precisely what eludes us is the nature of the continuity, the flowing from point to point; in other words, the secret of how the continually enduring present can continually slip away into the receding past.

*Each one of us, at every moment, directly experiences the true character of this temporal continuity. But, because of the genuine primitiveness of phenomenal time, we cannot put our experiences into words. So we shall content ourselves with the following description. What I am conscious of is for me both a being-now and, in its essence, something which, with its temporal position, slips away. In this way there arises the persisting factual extent, something ever new which endures and changes in consciousness.*¹

Weyl sums up what he thinks can be affirmed about “objectively presented time”—by which I take it he means “phenomenal time described in an objective manner”—in the following two assertions, which he claims apply equally, *mutatis mutandis*, to every intuitively given continuum, in particular, to the continuum of spatial extension:

1. *An individual point in it is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a “point of transition” (which, of course, can in no way be understood mathematically);*
2. *it is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible.*²

The fact that single points in a true continuum “cannot be exhibited” arises, Weyl continues, from the fact that they are not genuine individuals and so cannot be characterized by their properties. In the physical world they are never defined absolutely, but only in terms of a coordinate system, which, in an arresting metaphor, Weyl describes as “the unavoidable residue of the eradication of the ego.” This metaphor, which Weyl was to employ more than once³, reflects the continuing influence of phenomenological doctrine: in this case, the thesis that the existent is given in the first instance as the contents of a consciousness⁴. By 1919 Weyl had come to embrace

¹ *Ibid.*, p. 91-92.

² *Ibid.*, p. 92.

³ E.g. in Weyl [1950], 8 and [1949], p. 123.

⁴ Many years later, in *Insight and Reflection*, Weyl expanded the metaphor into a full-fledged analogy: In Weyl [1969], objects, subjects (or egos), and the appearance of an object to a subject are correlated respectively with *points on a plane*, (*barycentric*) *coordinate systems in the plane*, and *coordinates of a point with respect to a such a coordinate system*. In Weyl’s analogy, a coordinate system *S* consists of the vertices of a fixed nondegenerate triangle *T*; each point *p* in the plane determined by *T* is assigned a triple of numbers summing to 1—its *barycentric coordinates* relative to *S*—representing the magnitudes of masses of total weight 1 which, placed at the vertices of *T*, have centre of gravity at *p*. Thus objects, i.e. points, and subjects i.e., coordinate systems or triples of points belong to the same “sphere of reality.” On the other hand, the *appearances* of an object to a subject, i.e., triples of numbers, lie, Weyl asserts, in a different sphere, that of *numbers*. These *number-appearances*, as Weyl calls them, correspond to the experiences of a subject, or of pure consciousness.

From the standpoint of naïve realism the points (objects) simply exist as such, but Weyl indicates the possibility of constructing geometry (which under the analogy corresponds to external reality) solely in terms of number-appearances, so representing the world in terms of the experiences of pure consciousness, that is, from the standpoint of idealism. Thus suppose that we are given a coordinate system *S*. Regarded as a subject or “consciousness”, from its point of view a point or object now corresponds to what was originally an appearance of an object, that is, a triple of numbers summing to 1; and, analogously, any coordinate system *S'* (that is, another subject or “consciousness”) corresponds to three such triples determined by the vertices of a nondegenerate triangle. Each point or object *p* may now be *identified* with its coordinates relative to *S*. The

Brouwer's views¹ on the intuitive continuum. The latter's influence looms large in Weyl's next paper on the subject, *On the New Foundational Crisis of Mathematics*, written in 1920. Here Weyl identifies two distinct views of the continuum: "atomistic" or "discrete"; and "continuous". In the first of these the continuum is composed of individual real numbers which are well-defined and can be sharply distinguished. Weyl describes his earlier attempt at reconstructing analysis in *Das Kontinuum* as atomistic in this sense:

*Existential questions concerning real numbers only become meaningful if we analyze the concept of real number in this extensionally determining and delimiting manner. Through this conceptual restriction, an ensemble of individual points is, so to speak, picked out from the fluid paste of the continuum. The continuum is broken up into isolated elements, and the flowing-into-each other of its parts is replaced by certain conceptual relations between these elements, based on the "larger-smaller" relationship. This is why I speak of the atomistic conception of the continuum.*²

coordinates of p relative to any other coordinate system S' can be determined by a straightforward algebraic transformation: these coordinates represent the *appearance* of the object corresponding to p to the subject represented by S' . Now these coordinates will, in general, *differ* from those assigned to p by our given coordinate system S , and will in fact coincide for all p if and only if S' is what is termed by Weyl the *absolute* coordinate system consisting of the three triples $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, that is, the coordinate system which corresponds to S itself. Thus, for this coordinate system, "object" and "appearance" coincide, which leads Weyl to term it the *Absolute I*. (This term Weyl borrows from Fichte, whom he quotes as follows: "The I demands that it comprise all reality and fill up infinity. This demand is based, as a matter of necessity, on the idea of the infinite I; this is the absolute I—which is not the I given in real awareness.")

Weyl points out that this argument takes place entirely within the realm of numbers, that is, for the purposes of the analogy, the *immanent consciousness*. In order to do justice to the claim of objectivity that all "I"s are equivalent, he suggests that only such numerical relations are to be declared of interest as remain unchanged under passage from an "absolute" to an arbitrary coordinate system, that is, those which are invariant under arbitrary linear coordinate transformations. When this scheme is given a purely axiomatic formulation, Weyl sees a third viewpoint emerging in addition to that of realism and idealism, namely, a *transcendentalism* which "postulates a transcendental reality but is satisfied with modelling it in symbols."

Interestingly, by the time this was written, Weyl seems to have moved away somewhat from the phenomenology that originally suggested the geometric analogy. For he asserts that a number of Husserl's theses become "demonstratively false" when translated into the context of the analogy, "something which," he opines, "gives serious cause for suspecting them." Unfortunately, he does not specify which of Husserl's theses he has in mind.

Weyl goes on to emphasize:

Beyond this, it is expected of me that I recognize the other I—the you—not only by observing in my thought the abstract norm of invariance or objectivity, but absolutely: you are for you, once again, what I am for myself: not just an existing but a conscious carrier of the world of appearances.

This recognition of the *Thou*, according to Weyl, can be presented within his geometric analogy only if it is furnished with a purely *axiomatic* formulation. In taking this step Weyl sees a third viewpoint emerging in addition to that of realism and idealism, namely, a *transcendentalism* which "postulates a transcendental reality but is satisfied with modelling it in symbols."

But Weyl, ever-sensitive to the claims of subjectivity, hastens to point out that this scheme by no means resolves the enigma of selfhood. In this connection he refers to Leibniz's attempt to resolve the conflict between human freedom and divine predestination by having God select for existence, on the grounds of sufficient reason, certain beings, such as Judas and St. Peter, whose nature thereafter determines their entire history. Concerning this solution Weyl remarks characteristically:

[it] may be objectively adequate, but it is shattered by the desperate cry of Judas: Why did I have to be Judas! The impossibility of an objective formulation to this question strikes home, and no answer in the form of an objective insight can be given. Knowledge cannot bring the light that is I into coincidence with the murky, erring human being that is cast out into an individual fate.

For further discussion of Weyl's philosophical views see Bell (2003).

¹ Weyl described Brouwer's system of mathematics as refraining from making the "leap into the beyond" required by classical set theory (Weyl 1949, p. 50).

² Weyl [1998], p. 91.

Weyl now repudiated atomistic theories of the continuum, including that of *Das Kontinuum*. He writes:

In traditional analysis, the continuum appeared as the set of its points; it was considered merely as a special case of the basic logical relationship of element and set. Who would not have already noticed that, up to now, there was no place in mathematics for the equally fundamental relationship of part and whole? The fact, however, that it has parts, is a fundamental property of the continuum; and so (in harmony with intuition, so drastically offended against by today's "atomism") this relationship is taken as the mathematical basis for the continuum by Brouwer's theory. This is the real reason why the method used in delimiting subcontinua and in forming continuous functions starts out from intervals and not points as the primary elements of construction. Admittedly a set also has parts. Yet what distinguishes the parts of sets in the realm of the "divisible" is the existence of "elements" in the set-theoretical sense, that is, the existence of parts that themselves do not contain any further parts. And indeed, every part contains at least one "element". In contrast, it is inherent in the nature of the continuum that every part of it can be further divided without limitation. The concept of a point must be seen as an idea of a limit, "point" is the idea of a limit of a division extending in infinitum. To represent the continuous connection of the points, traditional analysis, given its shattering of the continuum into isolated points, had to have recourse to the concept of a neighbourhood. Yet, because the concept of continuous function remained mathematically sterile in the resulting generality, it became necessary to introduce the possibility of "triangulation" as a restrictive condition.¹

Like Brentano, Weyl knew that to "shatter a continuum into isolated points" would be to eradicate the very feature which characterizes a continuum—the fact that its cohesiveness is inherited by every one of its parts.

While intuitive considerations, together with Brouwer's influence, must certainly have fuelled Weyl's rejection of atomistic conceptions of the continuum, it also had a logical basis. For Weyl had come to regard as meaningless the formal procedure—employed in *Das Kontinuum*—of negating universal and existential statements concerning real numbers conceived as developing sequences or as sets of rationals. This had the effect of undermining the whole basis on which his theory had been erected, and at the same time rendered impossible the very formulation of a "law of excluded middle" for such statements. Thus Weyl found himself espousing a position considerably more radical than that of Brouwer, for whom negations of quantified statements had a perfectly clear constructive meaning, under which the law of excluded middle is simply not generally affirmable.

Of existential statements Weyl says:

An existential statement—e.g., "there is an even number"—is not a judgement in the proper sense at all, which asserts a state of affairs; existential states of affairs are the empty invention of logicians.²

Weyl termed such pseudostatements "judgement abstracts", likening them to "a piece of paper which announces the presence of a treasure, without divulging its location." Universal statements, although possessing greater substance than existential ones, are still mere intimations of judgements, "judgement instructions", for which Weyl provides the following metaphorical description:

If knowledge be compared to a fruit and the realization of that knowledge to the

¹ Weyl [1998], p. 115.

² Weyl [1998], p. 97.

*consumption of the fruit, then a universal statement is to be compared to a hard shell filled with fruit. It is, obviously, of some value, however, not as a shell by itself, but only for its content of fruit. It is of no use to me as long as I do not open it and actually take out a fruit and eat it.*¹

Above and beyond the claims of logic, Weyl welcomed Brouwer's construction of the continuum by means of sequences generated by free acts of choice, thus identifying it as a "medium of free Becoming" which "does not dissolve into a set of real numbers as finished entities".² Weyl felt that Brouwer, through his doctrine of intuitionism³, had come closer than anyone else to bridging that "unbridgeable chasm" between the intuitive and mathematical continua. In particular, he found compelling the fact that the Brouwerian continuum is not the union of two disjoint nonempty parts—that it is *indecomposable*. "A genuine continuum," Weyl says, "cannot be divided into separate fragments." In later publications he expresses this more colourfully by quoting Anaxagoras to the effect that a continuum "defies the chopping off of its parts with a hatchet."⁴

Weyl also agrees with Brouwer that all functions everywhere defined on a continuum are continuous, but here certain subtle differences of viewpoint emerge. Weyl contends that what mathematicians had taken to be discontinuous functions actually consist of several continuous functions defined on separated continua. For example, the "discontinuous" function defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$ in fact consists of the two functions with constant values 0 and 1 respectively defined on the separated continua $\{x: x < 0\}$ and $\{x: x \geq 0\}$. The union of these two continua fails to be the whole of the real continuum because of the failure of the law of excluded middle: it is not the case that, for any real number x , either $x < 0$ or $x \geq 0$.

Brouwer, on the other hand, had not dismissed the possibility that discontinuous functions could be defined on proper parts of a continuum, and still seems to have been searching for an appropriate way of formulating this idea.⁵ In particular, at that time Brouwer would probably have been inclined to regard the above function f as a genuinely discontinuous function defined on a *proper part* of the real continuum. For Weyl, it seems to have been a self-evident fact that all functions defined on a continuum are continuous, but this is because Weyl confines attention to functions which turn out to be continuous by definition. Brouwer's concept of function is less restrictive than Weyl's and it is by no means immediately evident that such functions must always be continuous.

Weyl defined real functions as mappings correlating each interval in the choice sequence determining the argument with an interval in the choice sequence determining the value "interval by interval" as it were, the idea being that approximations to the input of the function should lead effectively to corresponding approximations to the output. Such functions are continuous by definition. Brouwer, on the other hand, considers real functions as correlating choice sequences with choice sequences, and the continuity of these is by no means obvious. The fact that Weyl refused to grant (free) choice sequences—whose identity is in no way predetermined—sufficient individuality to admit them as arguments of functions perhaps betokens a commitment to the conception of the continuum as a "medium of free Becoming" even deeper than that of Brouwer.

¹ *Ibid.*, p. 98.

² See Chapter 9 below.

³ For my remarks on Weyl's relationship with Intuitionism I have drawn on the illuminating paper van Dalen [1995].

⁴ See Chapter 1 above.

⁵ Brouwer established the continuity of functions fully defined on a continuum in 1904, but did not publish a definitive account until 1927. In that account he also considers the possibility of partially defined functions.

There thus being only minor differences between Weyl's and Brouwer's accounts of the continuum, Weyl accordingly abandoned his earlier attempt at the reconstruction of analysis and "joined Brouwer." At the same time, however, Weyl recognized that the resulting gain in intuitive clarity had been bought at a considerable price:

Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the greater part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes.¹

Although he later practiced intuitionistic mathematics very rarely, Weyl remained an admirer of intuitionism. And the "riddle of the continuum" retained its fascination for him: in one of his last papers, *Axiomatic and Constructive Procedures in Mathematics*, written in 1954, we find the observation that

... the constructive transition to the continuum of real numbers is a serious affair... and I am bold enough to say that not even to this day are the logical issues involved in that constructive concept completely clarified and settled.²

It seems a pity that Weyl did not live to see the emergence in the 1970s of *smooth infinitesimal analysis*³, a mathematical framework within which his vision of a true continuum, not "synthesized" from discrete elements, is realized, at least in a formal sense. Although the underlying logic of smooth infinitesimal analysis is intuitionistic—the law of excluded middle not being generally affirmable—mathematics developed within avoids the "unbearable awkwardness" to which Weyl refers above. And while it unquestionably falls within the compass of "symbolic construction", as opposed to immediate intuition, the simple and elegant way in which it gives expression to the indecomposability of the continuum and to the automatic continuity of functions defined thereon would have been recognized by Weyl as creating a natural bridge between the intuitive and the mathematical continua. And central to smooth infinitesimal analysis is a concept of *infinitesimal quantity* governed by axioms which serve to establish the link—indispensable in Weyl's view—between the microcosm and the macrocosm, so making possible the development of a truly infinitesimal *physics*.

¹ Weyl [1949], p. 54.

² Weyl [1985], p. 17.

³ See Chapter 10.