

Cover Schemes, Frame-Valued Sets and Their Potential Uses in Spacetime Physics

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Abstract

In the present paper the concept of a covering is presented and developed. The relationship between cover schemes, frames (complete Heyting algebras), Kripke models, and frame-valued set theory is discussed. Finally cover schemes and frame-valued set theory are applied in the context of Markopoulou's account of discrete spacetime as sets "evolving" over a causal set. We observe that Markopoulou's proposal may be effectively realized by working within an appropriate frame-valued model of set theory. We go on to show that, within this framework, cover schemes may be used to force certain conditions to prevail in the associated models: for example, rendering the universe timeless, obliterating a given event or forcing it to become the universe's "beginning".

Preamble

The concept of *Grothendieck (pre)topology* or *covering* issued from the efforts of algebraic geometers to study "sheaf-like" objects defined on categories more general than the lattice of open sets of a topological space (see, e.g. [4]). A *Grothendieck pretopology* on a category \mathcal{C} with pullbacks is defined by specifying, for each object U of \mathcal{C} , a set $P(U)$ of arrows to U called *covering families* satisfying appropriate category

theoretic versions of the corresponding conditions for a family \mathcal{A} of sets to cover a set U , namely: (i) $\{U\}$ covers U , (ii) if \mathcal{A} covers U and $V \subseteq U$, then $\mathcal{A}|V = \{A \cap V : A \in \mathcal{A}\}$ covers V , and (iii) if \mathcal{A} covers U and, for each $A \in \mathcal{A}$, \mathcal{B}_A covers A , then $\bigcup_{A \in \mathcal{A}} \mathcal{B}_A$ covers U . In the present paper the covering concept—here called a *cover scheme*—is presented and developed in the simple case when the underlying category is a preordered set. The relationship between cover schemes, frames (complete Heyting algebras), Kripke models, and frame-valued set theory is discussed. Finally cover schemes and frame-valued set theory are applied in the context of Markopoulou’s [5] account of discrete spacetime as sets “evolving” over a causal set.

I. COVER SCHEMES ON PREORDERED SETS

A *preordered set* is a set equipped with a reflexive transitive relation \leq . Let (P, \leq) be a fixed but arbitrary preordered set: we shall use letters p, q, r, s, t to denote elements of P . We write $p \cong q$ for $(p \leq q \ \& \ q \leq p)$. A *meet* for a subset S of P is an element p of P such that $\forall q[\forall s \in S(q \leq s) \leftrightarrow q \leq p]$: if p and p' are both meets for S , then $p \cong p'$. If the empty subset \emptyset has a meet, any such meet m is necessarily a *largest* or *top* element of P , that is, satisfies $p \leq m$ for all p . We use the symbol 1 to denote a top element of P . A meet of a finite subset $\{p_1, \dots, p_n\}$ of P will be denoted by $p_1 \wedge \dots \wedge p_n$. P is a *lower semilattice* if each nonempty finite subset of P has a meet. A subset S of P is said to be a *sharpening of*, or to *sharpen*, a subset T of P , written $S \prec T$, if $\forall s \in S \exists t \in T(s \leq t)$. A *sieve* in P is a subset S such that $p \in S$ and $q \leq p$ implies $q \in S$. Each subset S of P generates a sieve \bar{S} given by $\bar{S} = \{p : \exists s \in S(p \leq s)\}$.

A *cover scheme* on P is a map \mathbf{C} assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p\downarrow = \{q: q \leq p\}$, called *(C-)covers of p* , such that, if $q \leq p$, any cover of p can be sharpened to a cover of q , i.e.,

$$\mathbf{(Cov)} \quad S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow \exists T \in \mathbf{C}(q) [\forall t \in T \exists s \in S (t \leq s)].$$

If P is a lower semilattice, a *coverage* (see [3]) on P is a map \mathbf{C} as above, satisfying, in place of **(Cov)**, the condition

$$S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow S \wedge q = \{s \wedge q : s \in S\} \in \mathbf{C}(p).$$

A cover scheme \mathbf{C} is said to be *normal* if every member of every $\mathbf{C}(p)$ is a sieve and whenever $S \in \mathbf{C}(p)$ and T is a sieve such that $S \subseteq T \subseteq p\downarrow$, we have $T \in \mathbf{C}(p)$. Any cover scheme \mathbf{C} on P induces a normal cover scheme $\bar{\mathbf{C}}$ (called its *normalization*) defined by

$$\bar{\mathbf{C}}(p) = \{X \subseteq p\downarrow : X \text{ is a sieve} \ \& \ \exists S \in \mathbf{C}(p). S \subseteq X\}.$$

Notice that a normal cover scheme on a lower semilattice is always a coverage. For if \mathbf{C} is such, then for $S \in \mathbf{C}(p)$ and $q \leq p$, any sharpening of S to a member of $\mathbf{C}(q)$ is easily seen to be included in $S \wedge q$, so that the latter is also in $\mathbf{C}(q)$.

Write $\mathcal{Cov}(P)$ for the set of all cover schemes on P . There is a natural partial ordering \triangleleft on $\mathcal{Cov}(P)$ defined by

$$\mathbf{C} \triangleleft \mathbf{D} \leftrightarrow \forall p \ \mathbf{C}(p) \subseteq \mathbf{D}(p).$$

With this ordering $\mathcal{Cov}(P)$ is a complete lattice in which the join $\bigvee_{i \in I} \mathbf{C}_i$ of any family $\{\mathbf{C}_i: i \in I\}$ is given by

$$\left(\bigvee_{i \in I} \mathbf{C}_i\right)(p) = \bigcup_{i \in I} \mathbf{C}_i(p).$$

There is also a natural *composition* \star defined on $\mathcal{Cov}(P)$. For $\mathbf{C}, \mathbf{D} \in \mathcal{Cov}(P)$, $\mathbf{D} \star \mathbf{C}$ is defined by decreeing that $(\mathbf{D} \star \mathbf{C})(p)$ is to consist of all subsets of $p\downarrow$ of the form $\bigcup_{s \in S} T_s$, where $S \in \mathbf{C}(p)$ and, for each $s \in S$, $T_s \in \mathbf{D}(s)$. That $\mathbf{D} \star \mathbf{C}$ is a cover scheme on P may be verified (using the axiom

of choice) as follows. Given $S \in \mathbf{C}(p)$, $\bigcup_{s \in S} T_s \in (\mathbf{D} \star \mathbf{C})(p)$ and $q \leq p$, there is $U \in \mathbf{C}(q)$ with $U \prec S$, so for each $u \in U$ there is $s(u) \in S$ for which $u \leq s(u)$. Then $T_{s(u)} \in \mathbf{D}(s(u))$ and we can choose $V_u \in \mathbf{D}(u)$ so that $V_u \prec T_{s(u)}$. Clearly $\bigcup_{u \in U} V_u \in (\mathbf{D} \star \mathbf{C})(q)$ and, since $V_u \prec T_{s(u)}$ for all $u \in U$, it follows immediately that $\bigcup_{u \in U} V_u \prec \bigcup_{s \in S} T_s$.

It is not hard to verify that \star is associative and that with this operation $\mathcal{Cov}(P)$ is actually a *quantale* (see, e.g., [6]) that is, for any \mathbf{D} , $\{\mathbf{C}_i; i \in I\}$ in $\mathcal{Cov}(P)$,

$$\mathbf{D} \star \bigvee_{i \in I} \mathbf{C}_i = \bigvee_{i \in I} (\mathbf{D} \star \mathbf{C}_i) \quad (\bigvee_{i \in I} \mathbf{C}_i) \star \mathbf{D} = \bigvee_{i \in I} (\mathbf{C}_i \star \mathbf{D}) \quad .$$

Also the element $\mathbf{1} \in \mathcal{Cov}(P)$ with $\mathbf{1}(p) = \{p\}$ acts as a quantal unit, since it is readily verified that $\mathbf{1} \star \mathbf{C} = \mathbf{C} \star \mathbf{1} = \mathbf{C}$ for all $\mathbf{C} \in \mathcal{Cov}(P)$.

In this connection a *Grothendieck pretopology*—which we shall abbreviate simply to *pretopology*—on P may be identified as a cover scheme \mathbf{C} on P satisfying $\mathbf{1} \triangleleft \mathbf{C}$ and $\mathbf{C} \star \mathbf{C} \triangleleft \mathbf{C}$, that is, $\{p\} \in \mathbf{C}(p)$ for all $p \in P$ and, if $S \in \mathbf{C}(p)$ and, for each $s \in S$, $T_s \in \mathbf{C}(s)$, then $\bigcup_{s \in S} T_s \in \mathbf{C}(p)$.

We observe that a normal pretopology \mathbf{C} has the additional properties: (i) each $\mathbf{C}(p)$ is a filter of sieves in $p \downarrow$, that is, satisfies $S, T \in \mathbf{C}(p) \leftrightarrow S \in \mathbf{C}(p) \ \& \ T \in \mathbf{C}(p)$; (ii) $S \in \mathbf{C}(p) \ \& \ q \leq p \rightarrow S \cap q \downarrow \in \mathbf{C}(q)$. For (ii), we observe that $S \cap q \downarrow$, including as it does any sharpening of S to a member of $\mathbf{C}(q)$, is itself a member of $\mathbf{C}(q)$. As for (i), the “ \rightarrow ” direction is obvious; conversely, if $S, T \in \mathbf{C}(p)$, then $S \cap T = S \cap \bigcup_{t \in T} (t \downarrow) = \bigcup_{t \in T} (S \cap t \downarrow)$.

But from (ii) we have $S \cap t \downarrow \in \mathbf{C}(t)$ for every $t \in T$, whence $\bigcup_{t \in T} (S \cap t \downarrow) \in \mathbf{C}(p)$, and so $S \cap T \in \mathbf{C}(p)$.

A normal pretopology is also called a *Grothendieck topology*. A normal cover scheme satisfying (i) and (ii) is called a *regular cover scheme*.

Each cover scheme \mathbf{C} generates a pretopology, and a Grothendieck topology in the following way. First, define \mathbf{C}^n for $n \in \omega$ recursively by $\mathbf{C}^0 = \mathbf{1}$ and $\mathbf{C}^{n+1} = \mathbf{C} \star \mathbf{C}^n$. Now put $\mathbf{G} = \bigvee_{n \in \omega} \mathbf{C}^n$. Then \mathbf{G} is a pretopology, for obviously $\mathbf{1} \triangleleft \mathbf{G}$, and

$$\mathbf{G} \star \mathbf{G} = \bigvee_{n \in \omega} \mathbf{C}^n \star \bigvee_{m \in \omega} \mathbf{C}^m = \bigvee_{n \in \omega} \bigvee_{m \in \omega} \mathbf{C}^{m+n} = \bigvee_{n \in \omega} \mathbf{C}^n = \mathbf{G}.$$

Also $\mathbf{C} \triangleleft \mathbf{G}$, and \mathbf{G} is evidently the \triangleleft -least such pretopology. \mathbf{G} is called the *pretopology generated by \mathbf{C}* . The normalization $\overline{\mathbf{G}}$ of \mathbf{G} is then a Grothendieck topology called the *Grothendieck topology generated by \mathbf{C}* .

Now let \mathbf{M} be a map assigning to each $p \in P$ a subset $\mathbf{M}(p)$ of subsets of $p \downarrow$. Since $\mathcal{Cov}(P)$ is a complete lattice, there is a \triangleleft -least cover scheme \mathbf{C} such that $\mathbf{M}(p) \subseteq \mathbf{C}(p)$ for all p . \mathbf{C} is called the cover scheme *generated by \mathbf{M}* ; the pretopology and Grothendieck topology generated in turn by \mathbf{C} are said to be *generated by \mathbf{M}* .

There are several naturally defined cover schemes on P which also happen to be pretopologies. First, each sieve A in P determines two cover schemes \mathbf{C}_A and \mathbf{C}^A defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \quad S \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A \subseteq S:$$

these are easily shown to be pretopologies. Notice that $\emptyset \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\emptyset \in \mathbf{C}^A(p) \leftrightarrow p \downarrow \cap A = \emptyset$.

Next, we have the *dense cover scheme \mathbf{Den}* given by:

$$(*) \quad S \in \mathbf{Den}(p) \leftrightarrow \forall q \leq p \exists s \in S \exists r \leq s (r \leq q):$$

it is a straightforward exercise to show that this is a pretopology. When S is a sieve, the above condition (*) is easily seen to be equivalent to the familiar condition of *density below p* : that is, $\forall q \leq p \exists s \in S (s \leq q)$.

Note that the following are equivalent for any cover scheme **C**:
(a) **C** \triangleleft **Den**, (b) $\emptyset \notin \mathbf{C}(p)$ for all p . For since $\emptyset \notin \mathbf{Den}(p)$, (a) clearly implies (b). Conversely, assume (b), and let $S \in \mathbf{C}(p)$. Then for each $q \leq p$ there is $T \in \mathbf{C}(q)$ for which $\forall t \in T \exists s \in S (t \leq s)$. Since (by (b)) $T \neq \emptyset$, we may choose $t_0 \in T$ and $s_0 \in S$ for which $t_0 \leq s_0$. Since $t_0 \leq q$, and $q \leq p$ was arbitrary, it follows that S satisfies the condition (*) above for membership in **Den**(p). This gives (a).

Finally, we have the *Beth cover scheme* **Bet**. This is defined as follows. First we define a *road* from p to be a maximal linearly preordered subset of $p \downarrow$: clearly any road from p contains p . Let us call a *rome* over p any subset of $p \downarrow$ intersecting every road from p . Now the Beth coverage has **Bet**(p) = collection of all romes over p . Let us check first that **Bet** is a cover scheme. Suppose that S is a rome over p and $q \leq p$. We claim that

$$T = \{t \leq q : \exists s \in S(t \leq s)\}$$

is a rome over q . For let Y be any road from q ; then, by Zorn's lemma, Y may be extended to a road X from p . We note that since $X \cap q \downarrow$ is linearly preordered and includes Y , it must coincide with Y . Since S is a rome over p , there must be an element $s \in S \cap X$. Since also $q \in Y \subseteq X$, we have $s \leq q$ or $q \leq s$. If $s \leq q$, then $s \in X \cap q \downarrow = Y$ and $s \in T$, so that $s \in Y \cap T$. If $q \leq s$, then $q \in T$; since $q \in Y$, it follows that $q \in Y \cap T$. So in either case $Y \cap T \neq \emptyset$; therefore T is a rome over q . Since clearly also $T \prec S$, we have shown that **Bet** is a cover scheme.

To show that **Bet** is a pretopology, we observe first that, for any p , $\{p\}$ is a rome over p . Now suppose that we are given: a rome S over p , for each $s \in S$, a rome T_s over s , and a road X from p . Then $s \in X \cap S$ for some s : we claim that $X \cap s \downarrow$ is a road from s . For suppose $t \leq s$ is comparable with every member of $X \cap s \downarrow$; now since $s \in X$, for each $x \in X$ either $s \leq x$ or $x \leq s$. In the first case $t \leq x$; in the second t is comparable with x by assumption. Hence t is comparable with every member of X ,

and so $t \in X$. Accordingly $X \cap s \downarrow$ is, as claimed, a road from s ; as such, it must meet the rome T_s , so X meets $\bigcup_{s \in S} T_s$, and the latter is therefore a rome over p . So **Bet** is indeed a pretopology.

Since clearly $\emptyset \notin \mathbf{Bet}(p)$ for any p , it follows from what we have noted above that **Bet** \triangleleft **Den**, a fact that can also be easily verified directly.

Any preordered set (P, \leq) generates a *free lower semilattice* \tilde{P} which may be described as follows. The elements of \tilde{P} are the finite subsets of P ; the preordering on \tilde{P} is the *refinement* relation \sqsubseteq , that is, for $F, G \in \tilde{P}$,

$$F \sqsubseteq G \leftrightarrow \forall q \in G \exists p \in F (p \leq q).$$

The meet operation \wedge in \tilde{P} is set-theoretic union; the canonical embedding of P into \tilde{P} is the map $p \mapsto \{p\}$. Notice also that \emptyset is the unique top element of \tilde{P} .

Now, suppose we are given a cover scheme **C** on P . This induces a cover scheme $\tilde{\mathbf{C}}$ on \tilde{P} defined in the following way. We start by setting $\tilde{\mathbf{C}}(\emptyset) = \{\{\emptyset\}\}$. Now fix a nonempty finite subset F of P , take any nonempty subset $\{p_1, \dots, p_n\}$ of F and any $S_1 \in \mathbf{C}(p_1), \dots, S_n \in \mathbf{C}(p_n)$. Define

$$S_1 \bullet \dots \bullet S_n = \{\{s_1, \dots, s_n\} \cup F : s_1 \in S_1, \dots, s_n \in S_n\}.$$

We decree that $\tilde{\mathbf{C}}(F)$ is to consist of all sets of the form $S_1 \bullet \dots \bullet S_n$, for $S_1 \in \mathbf{C}(p_1), \dots, S_n \in \mathbf{C}(p_n)$, and all nonempty finite subsets $\{p_1, \dots, p_n\}$ of F .

Let us check that $\tilde{\mathbf{C}}$ is a cover scheme on \tilde{P} . To begin with, the unique cover $\{\emptyset\}$ of \emptyset is clearly sharpenable to any cover $S_1 \bullet \dots \bullet S_n$ of any nonempty member of \tilde{P} . Now suppose that $S_1 \bullet \dots \bullet S_n$ is a $\tilde{\mathbf{C}}$ -cover of a nonempty member F of \tilde{P} and that $G = \{q_1, \dots, q_m\} \sqsubseteq F$. Then for each

$1 \leq i \leq n$ there is $q_i \in G$ for which $q_i \leq p_i$, hence $T_i \in \mathbf{C}(q_i)$ with $T_i \prec S_i$. Clearly $T_1 \bullet \dots \bullet T_n \in \tilde{\mathbf{C}}(G)$. Also $T_1 \bullet \dots \bullet T_n \prec S_1 \bullet \dots \bullet S_n$. For, given $t_1 \in T_1, \dots, t_n \in T_n$, then since $T_i \prec S_i$ for each i , there are $s_1 \in S_1, \dots, s_n \in S_n$ for which $t_1 \leq s_1, \dots, t_n \leq s_n$, whence $\{t_1, \dots, t_n\} \cup G \sqsubseteq \{s_1, \dots, s_n\} \cup F$. So $\tilde{\mathbf{C}}$ satisfies the conditions of a cover scheme.

The normalization $\bar{\mathbf{C}}$ of $\tilde{\mathbf{C}}$ is then a coverage on \tilde{P} called the coverage on \tilde{P} induced by \mathbf{C} .

We next show how cover schemes give rise to complete Heyting algebras, or frames (see, e.g. [3]).

A *Heyting algebra* is a lattice L with top and bottom elements $1, 0$ such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$ —of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$: it is called the *pseudocomplement* of x . A *Boolean algebra* is a Heyting algebra in which $\neg\neg x = x$ for all x , or equivalently, in which $x \vee \neg x = 1$ for all x .

If we think of the elements of a (complete) Heyting algebra as “truth values”, then $0, 1, \wedge, \vee, \neg, \Rightarrow, \bigvee, \bigwedge$ represent “true”, “false”, “and”, “or”, “not” and “implies”, “there exists” and “for all”, respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic*. In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

$$(*) \quad x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

And conversely, in any complete lattice satisfying (*), defining the operation \Rightarrow by $x \Rightarrow y = \bigvee\{z: z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame*.

Now we associate a frame with each cover scheme on P . First, we define \widehat{P} to be the set of sieves in P partially ordered by inclusion: \widehat{P} is then a frame—the *completion*¹ of P —in which joins and meets are just set-theoretic unions and intersections, and in which the operations \Rightarrow and \neg are given by

$$I \Rightarrow J = \{p : I \cap p \downarrow \subseteq J\} \quad \neg I = \{p : I \cap p \downarrow = \emptyset\}.$$

Given a cover scheme \mathbf{C} on P , a sieve I in P is said to be **C-closed** if

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \rightarrow p \in I.$$

We write $\widehat{\mathbf{C}}$ for the set of all **C-closed** sieves in P , partially ordered by inclusion.

Lemma. If $I \in \widehat{P}$, $J \in \widehat{\mathbf{C}}$, then $I \Rightarrow J \in \widehat{\mathbf{C}}$.

Proof. Suppose that $I \in \widehat{P}$, $J \in \widehat{\mathbf{C}}$, and $S \subseteq I \Rightarrow J$ with $S \in \mathbf{C}(p)$. Define $U = \{q \in I: \exists s \in S. q \leq s\}$. Then $U \subseteq J$. If $q \in I \cap p \downarrow$, then there is $T \in \mathbf{C}(q)$ for which $T \prec S$. Then for any $t \in T$, there is $s \in S$ for which $t \leq s$, whence $t \in U$. Accordingly $T \subseteq U \subseteq J$. Since J is a **C-closed**, it follows that $q \in J$. We conclude that $I \cap p \downarrow \subseteq J$, whence $p \in p \downarrow \subseteq I \Rightarrow J$. Therefore $I \Rightarrow J$ is **C-closed**. \square

¹ Writing **Lat** for the category of complete lattices and join preserving homomorphisms, \widehat{P} is in fact the object in **Lat** freely generated by P .

It follows from the lemma that $\widehat{\mathbf{C}}$ is a frame. For clearly an arbitrary intersection of \mathbf{C} -closed sieves is \mathbf{C} -closed. So $\widehat{\mathbf{C}}$ is a complete lattice. In view of the lemma the implication operation in \widehat{P} restricts to one in $\widehat{\mathbf{C}}$, making $\widehat{\mathbf{C}}$ a Heyting algebra, and so a frame.

Proposition 1. Suppose that \mathbf{C} is a pretopology. Then **(i)** the bottom element of $\widehat{\mathbf{C}}$ is $\mathbf{0} = \{p: \emptyset \in \mathbf{C}(p)\}$, **(ii)** the \mathbf{C} -closed sieve generated by a sieve A (that is, the smallest \mathbf{C} -closed sieve containing A) is $\{p: \exists S \in \mathbf{C}(p). S \subseteq A\}$, **(iii)** the join operation in $\widehat{\mathbf{C}}$ is given by $\bigvee_{i \in I} J_i = (\bigcup_{i \in I} J_i)^*$. If \mathbf{C} is a Grothendieck topology, then **(iv)** for any sieve $S \subseteq p \downarrow$, $p \in S^* \leftrightarrow S \in \mathbf{C}(p)$.

Proof. Suppose that \mathbf{C} is a pretopology. Then $\mathbf{0}$ is a \mathbf{C} -closed sieve. For it is easily seen to be a sieve; and it is \mathbf{C} -closed because if $S \in \mathbf{C}(p)$ and $S \subseteq \mathbf{0}$, then $\emptyset \in \mathbf{C}(s)$ for each $s \in S$, whence $\emptyset = \bigcup_{s \in S} \emptyset \in \mathbf{C}(p)$, and so $p \in \mathbf{0}$. Finally, $\mathbf{0} \subseteq I$ for any \mathbf{C} -closed sieve I , for if $\emptyset \in \mathbf{C}(p)$, then from $\emptyset \subseteq I$ we infer $p \in I$. This gives **(i)**. As for **(ii)**, suppose given a sieve A . Then $A \subseteq A^*$ follows from $\{p\} \in \mathbf{C}(p)$. A^* is a sieve, since if $p \in A^*$ and $q \leq p$, then there is $S \in \mathbf{C}(p)$ for which $S \subseteq A$, and $T \in \mathbf{C}(q)$ sharpening S ; clearly $T \subseteq A$ also, whence $q \in A^*$. And A^* is \mathbf{C} -closed, since if $S \subseteq A^*$ with $S \in \mathbf{C}(p)$, then for each $s \in S$ there is $T_s \in \mathbf{C}(s)$ with $T_s \subseteq A$; it follows that $\bigcup_{s \in S} T_s \subseteq A$ and $\bigcup_{s \in S} T_s \in \mathbf{C}(p)$, whence $p \in A^*$. Part **(iii)** is an immediate consequence of **(ii)**. Finally, if \mathbf{C} is a Grothendieck topology and $S \subseteq p \downarrow$ is a sieve, then $p \in S^* \leftrightarrow \exists T \in \mathbf{C}(p). T \subseteq S \leftrightarrow S \in \mathbf{C}(p)$, i.e. **(iv)**. \square

We observe parenthetically that $\widehat{\mathbf{Den}}$ is a Boolean algebra. To establish this it suffices to show that, for any $I \in \widehat{\mathbf{Den}}$, $\neg\neg I \subseteq I$. Now since $\emptyset \notin \mathbf{Den}(p)$, it follows from **(i)** of the proposition above that the bottom element of $\widehat{\mathbf{Den}}$ is \emptyset , so that, for any $I \in \widehat{\mathbf{Den}}$, $\neg I = \{p: I \cap p \downarrow = \emptyset\}$,

whence $\neg\neg I = \{p : \forall q \leq p \exists r \leq q. r \in I\}$. But it easily checked that the defining condition for I to be a member of $\widehat{\mathbf{Den}}$ is precisely that, if $\forall q \leq p \exists r \leq q. r \in I$, then $p \in I$. That is, $\neg\neg I \subseteq I$.

Cover schemes on P correspond to certain self-maps on \widehat{P} called (*weak*) *nuclei*. A *weak nucleus* on a frame H is a finite-meet-preserving map $j: H \rightarrow H$ such that $j(1) = 1$ and $a \leq j(a)$ for any $a \in H$. If in addition $j(j(a)) \leq j(a)$ (so that $j(j(a)) = j(a)$) for all $a \in H$, j is called a *nucleus* on H .

Proposition 2. Let \mathbf{C} be a cover scheme on P . For each $I \in \widehat{P}$ let I^* be the least \mathbf{C} -closed sieve containing I . Then the map $k_{\mathbf{C}}: I \mapsto I^*$ is a nucleus on \widehat{P} .

Proof. Clearly $I \subseteq I^*$ and $I^{**} = I^*$. It remains to be shown that, for $I, J \in \widehat{P}$, $(I \cap J)^* = I^* \cap J^*$. Since $*$ is obviously inclusion-preserving, $(I \cap J)^* \subseteq I^* \cap J^*$. For the reverse inclusion, note first that $I \in \widehat{\mathbf{C}} \leftrightarrow I^* = I$. Given $I, J \in \widehat{P}$, define $K = I \Rightarrow (I \cap J)^*$. By the Lemma above, $K \in \widehat{\mathbf{C}}$, so that $K^* = K$. Now $J^* \subseteq K$ since

$$J \cap I \subseteq (I \cap J)^* \rightarrow J \subseteq [I \Rightarrow (I \cap J)^*] = K,$$

whence $J^* \subseteq K^* \subseteq K$. Similarly, if we define $L = K \Rightarrow (I \cap J)^*$, then $I^* \subseteq L$. It follows that

$$I^* \cap J^* \subseteq K \cap L = K \cap [K \Rightarrow (I \cap J)^*] \subseteq (I \cap J)^*. \quad \square$$

Inversely, any *weak nucleus* j on \widehat{P} determines a *regular cover scheme* \mathbf{D}_j on P , given by

$$S \in \mathbf{D}_j(p) \leftrightarrow p \in j(S).$$

Let us check that \mathbf{D}_j is indeed a regular cover scheme. To do this it suffices to show that each $\mathbf{D}_j(p)$ is a filter of sieves and that, if $S \in \mathbf{D}_j(p)$, and $q \leq p$, then $S \cap q \downarrow \in \mathbf{D}_j(q)$. The first of these properties follows immediately from the fact that j preserves finite intersections, and the

second from the observation that, if $S \in \mathbf{D}_j(p)$, and $q \leq p$, then $p \in j(S)$, so that $q \in j(S)$, and $q \in q \downarrow \subseteq j(q \downarrow)$, whence $q \in j(S) \cap j(q \downarrow) = j(S \cap q \downarrow)$, i.e. $S \cap q \downarrow \in \mathbf{D}_j(q)$.

When j is a *nucleus*, \mathbf{D}_j is a *Grothendieck topology*. For under this assumption, if $S \in \mathbf{D}_j(p)$ and $T_s \in \mathbf{D}_j(s)$ for each $s \in S$, then $s \in j(T_s)$ for each $s \in S$, and it follows that

$$S \subseteq \bigcup_{s \in S} j(T_s) \subseteq j\left(\bigcup_{s \in S} T_s\right)$$

so that

$$p \in j(S) \subseteq j\left(j\left(\bigcup_{s \in S} T_s\right)\right) = j\left(\bigcup_{s \in S} T_s\right)$$

i.e., $\bigcup_{s \in S} T_s \in \mathbf{D}_j(p)$.

The correspondences $\mathbf{C} \mapsto k_{\mathbf{C}}$ and $j \mapsto \mathbf{D}_j$ between Grothendieck topologies on P and nuclei on \widehat{P} are mutually inverse. For if \mathbf{C} is a Grothendieck topology on P , then, by Proposition I **(iv)** we have

$$S \in \mathbf{D}_{k_{\mathbf{C}}}(p) \leftrightarrow p \in k_{\mathbf{C}}(S) = S^* \leftrightarrow S \in \mathbf{C}(p),$$

whence $\mathbf{D}_{k_{\mathbf{C}}} = \mathbf{C}$. And, for a nucleus j on \widehat{P} , we have, using Proposition I **(ii)**,

$$\begin{aligned} k_{\mathbf{D}_j}(I) &= \text{least } \mathbf{D}_j\text{-closed sieve } \supseteq I \\ &= \{p : \exists S \in \mathbf{D}_j(p). S \subseteq I\} \\ &= \{p : \exists S \subseteq I. p \in j(S)\} \\ &= j(I), \end{aligned}$$

whence $k_{\mathbf{D}_j} = j$.

II. COVER SCHEMES AND FRAMES

The relationship between cover schemes on a preordered set and (weak) nuclei on its completion can be extended to cover schemes on partially ordered sets and general frames. Accordingly let H be a frame: we write \vee , \wedge , \Rightarrow for the join, meet and implication operations,

respectively, in H . The partially ordered set (P, \leq) is said to be *dense* in H if P is a subset of H , the partial ordering on P is the restriction to P of that of H , and either of the two following equivalent conditions is satisfied: (i) for any $a \in H$, $a = \bigvee\{p: a \leq p\}$ (ii) for any $a, b \in H$, $a \leq b \leftrightarrow \forall p[p \leq a \rightarrow p \leq b]$. The canonical example of a frame in which P is dense is the frame \widehat{P} described in section I: here each $p \in P$ is identified with the $p\downarrow \in \widehat{P}$. \widehat{P} is easily seen to have the property that in it, for any $S \subseteq P$, $p \leq \bigvee S$ iff $p \in S$.

Now fix a frame H in which P is dense and a cover scheme \mathbf{C} on P . An element $a \in H$ is said to *cover* an element $p \in P$ if there exists a cover S of p for which $\bigvee S \leq a$. A \mathbf{C} -*element* of H is one which dominates every element of P that it covers—that is, an element $a \in H$ satisfying

$$\forall p \in P[(\exists S \in \mathbf{C}(p))\bigvee S \leq a \rightarrow p \leq a].$$

We write $H_{\mathbf{C}}$ for the set of all \mathbf{C} -elements of H . It is evident that $H_{\mathbf{C}}$ is closed under the meet operation of H . Notice that \mathbf{C} -elements and $\overline{\mathbf{C}}$ -elements coincide (recalling that $\overline{\mathbf{C}}$ is the normalization of \mathbf{C} .)

The *canonical H -cover scheme* \mathbf{C}_H on P is given by

$$S \in \mathbf{C}_H(p) \leftrightarrow \bigvee S = p.$$

Clearly \mathbf{C}_H is a pretopology, and every element of H is a \mathbf{C}_H -element.

Corresponding to the Lemma of §I, we have:

Lemma. If $a \in H, b \in H_{\mathbf{C}}$, then $a \Rightarrow b \in H_{\mathbf{C}}$.

Proof. Suppose $a \in H, b \in H_{\mathbf{C}}$, $S \in \mathbf{C}(p)$ and $\bigvee S \leq (a \Rightarrow b)$. Writing U for $\{q: q \leq a \ \& \ \exists s \in S(q \leq s)\}$, we have

$$\bigvee U \leq \bigvee\{s \wedge q: s \in S, q \leq a\} = \bigvee S \wedge \bigvee\{q: q \leq a\} = \bigvee S \wedge a \leq b.$$

Now if $q \leq p \wedge a$, there is $T \in \mathbf{C}(q)$ sharpening S . Then

$$t \in T \rightarrow t \leq a \ \& \ \exists s \in S(t \leq s),$$

so that $T \subseteq U$, and therefore $\bigvee T \leq \bigvee U \leq b$. Since $b \in H_{\mathbf{C}}$, it follows that $q \leq b$. Hence $q \leq p \wedge a \rightarrow q \leq b$, so that $p \wedge a \leq b$ and $p \leq (a \Rightarrow b)$. We conclude that $(a \Rightarrow b) \in H_{\mathbf{C}}$. \square

It follows from the lemma that $H_{\mathbf{C}}$ is itself a frame.

The nucleus on H associated with the cover scheme \mathbf{C} on P is the map $j = k_{\mathbf{C}}: H \rightarrow H$ defined by

$$j(a) = \bigwedge \{x \in H_{\mathbf{C}} : a \leq x\}.$$

That j is a nucleus results from the following observations. Evidently j is order preserving, maps H onto $H_{\mathbf{C}}$, is the identity on $H_{\mathbf{C}}$, and satisfies $j(1) = 1$ and $a \leq j(a)$ for all $a \in A$. Also it is easily shown that $j(j(a)) = j(a)$. Finally, j preserves finite meets. For clearly $j(a \wedge b) \leq j(a) \wedge j(b)$ since j is order preserving. For the reverse inequality, consider first the element $u = (a \Rightarrow j(a \wedge b))$: this is, by the Lemma above, an element of $H_{\mathbf{C}}$, so that $j(u) = u$. Also $j(b) \leq u$. For from $b \wedge a \leq j(a \wedge b)$ we deduce $b \leq (a \Rightarrow j(a \wedge b)) = u$, whence $j(b) \leq j(u) = u$. Similarly, $v = ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b))$ is an element of $H_{\mathbf{C}}$ and $j(a) \leq v$. Therefore

$$j(a) \wedge j(b) \leq v \wedge u \leq j(a \wedge b),$$

as required.

Notice that the nucleus associated with a cover scheme coincides with that associated with its normalization.

Accordingly we have shown that each cover scheme on P determines a nucleus on H . Conversely, we can show that any *weak nucleus* on H determines a cover scheme on P . For, starting with a weak nucleus j on H , define the map \mathbf{D}_j on P by

$$\mathbf{D}_j(p) = \{S \subseteq p \downarrow : p \leq j(\bigvee S)\}.$$

Then \mathbf{D}_j is a cover scheme on P . For suppose $q \leq p$ and $S \in \mathbf{D}_j(p)$. Then $q \leq p \leq j(\bigvee S)$; since $q \leq j(q)$ and j preserves finite meets, it follows that

$$(*) \quad q \leq j(q) \wedge j(\bigvee S) = j(q \wedge \bigvee S) = j(\bigvee \{s \wedge q : s \in S\}).$$

Now define $T \subseteq q \downarrow$ by

$$T = \{t : t \leq q \ \& \ \exists s \in S(t \leq s)\}.$$

We claim that T is a (\mathbf{D}_j) cover of q sharpening S . That T sharpens S is evident from its definition. To see that it is a cover of q we observe that, if $s \in S$, then

$$s \wedge q = \bigvee \{t : t \leq s \wedge q\} = \bigvee \{t : t \leq s \ \& \ t \leq q\} \leq \bigvee T.$$

Therefore $\bigvee \{s \wedge q : s \in S\} \leq \bigvee T$, so that, by (*),

$$q \leq j(\bigvee \{s \wedge q : s \in S\}) \leq j(\bigvee T),$$

that is, T covers q .

When j is a *nucleus*, the associated cover scheme \mathbf{D}_j is actually a *pretopology*. For in any case $\{p\} \in \mathbf{D}_j(p)$. Moreover, if j is a nucleus, $S \in \mathbf{D}_j(p)$ and $T_s \in \mathbf{D}_j(s)$ for each $s \in S$, then

$$p \leq j(\bigvee S) \leq j(\bigvee_{s \in S} j(\bigvee T_s)) \leq j(j(\bigvee_{s \in S} \bigvee T_s)) = j(\bigvee_{s \in S} T_s).$$

Therefore $\bigcup_{s \in S} T_s \in \mathbf{D}_j(p)$, and \mathbf{D}_j is a pretopology.

Starting with a weak nucleus j , we obtain the corresponding cover scheme \mathbf{D}_j . The latter in turn determines a nucleus j^* given by

$$j^*(a) = \bigwedge \{x \in H_{\mathbf{D}_j} : a \leq x\}.$$

Now by definition, we have

$$\begin{aligned} x \in H_{\mathbf{D}_j} &\leftrightarrow \forall p[(\exists S \in \mathbf{D}_j(p)) \bigvee S \leq x \rightarrow p \leq x] \\ &\leftrightarrow \forall p[(\exists S \subseteq p \downarrow)(p \leq j(\bigvee S) \ \& \ \bigvee S \leq x) \rightarrow p \leq x] \quad (\text{a}) \\ &\leftrightarrow \forall p[p \leq j(x) \rightarrow p \leq x] \quad (\text{b}) \\ &\leftrightarrow j(x) = x. \end{aligned}$$

(To see the equivalence between (a) and (b), we need to establish the equivalence between $(\exists S \subseteq p \downarrow)(p \leq j(\bigvee S) \ \& \ \bigvee S \leq x)$ and $p \leq j(x)$. Clearly the first of these implies the second. As for the converse, if $p \leq j(x)$, then since $p \leq j(p)$, we have

$$p \leq j(x) \wedge j(p) = j(x \wedge p) = j(\bigvee S),$$

where $S = \{q : q \leq x \wedge p\}$. Then $S \subseteq p\downarrow$, $p \leq j(\bigvee S)$ and $\bigvee S \leq x$, and the first statement follows.) Accordingly

$$(*) \quad j^*(a) = \bigwedge \{x \in H : a \leq x \ \& \ j(x) = x\}.$$

j^* is called the nucleus² *generated* by the weak nucleus j ; it is easily deduced from (*) that when j is a nucleus, j^* and j coincide.

The generation of nuclei by weak nuclei can itself be seen as an instance of a nuclear operation. For consider the set $W(H)$ of all weak nuclei on H . When $W(H)$ is partially ordered pointwise in the obvious way, it becomes a frame with implications, joins, and meets given by the following specifications: $(j \Rightarrow k)(a) = \bigwedge_{b \geq a} (j(b) \Rightarrow k(b))$ and for $S \subseteq W(H)$,

$$(\bigvee S)(a) = \bigvee_{s \in S} s(a), \quad (\bigwedge S)(a) = \bigwedge_{s \in S} s(a).$$

The subset $N(H)$ of $W(H)$ consisting of all nuclei can be shown to be a sublocale (see [3]) of $W(H)$, that is, it is closed under arbitrary meets in $W(H)$ and is such that $(j \Rightarrow k) \in N(H)$ whenever $j \in W(H)$, $k \in N(H)$. That being the case, the map $\varphi: W(H) \rightarrow N(H)$ defined by

$$\varphi(j) = \bigwedge \{k \in N(H) : j \leq k\}$$

is a nucleus on $W(H)$, and it is easily shown that $\varphi(j) = j^*$. So the generation of nuclei by weak nuclei is precisely the action of the nucleus φ .

Now start with a cover scheme \mathbf{C} on P , obtain the associated nucleus $k_{\mathbf{C}}$ on H , and consider its associated cover scheme $\mathbf{D}_{k_{\mathbf{C}}} = \mathbf{C}^*$ on P . By definition we have, for $S \subseteq p\downarrow$,

$$\begin{aligned} S \in \mathbf{C}^*(p) &\leftrightarrow p \leq j_{\mathbf{C}}(\bigvee S) \\ &\leftrightarrow p \leq \bigwedge \{x \in H_{\mathbf{C}} : \bigvee S \leq x\} \\ &\leftrightarrow \forall x \in H_{\mathbf{C}} (\bigvee S \leq x \rightarrow p \leq x) \end{aligned}$$

Recalling the definition of $H_{\mathbf{C}}$, we see immediately that this last assertion is implied by $S \in \mathbf{C}(p)$, so that always $\mathbf{C}(p) \subseteq \mathbf{C}^*(p)$. The reverse inclusion

² It can be verified directly that j^* is a nucleus.

will hold, and so \mathbf{C} will coincide with \mathbf{C}^* , precisely when the cover scheme \mathbf{C} is *saturated*, that is, coincides with its *saturate*, which we next proceed to define.

The (*H*-) *saturate* $\tilde{\mathbf{C}}$ of a cover scheme \mathbf{C} on P is defined by

$$\tilde{\mathbf{C}}(p) = \{S \subseteq p \downarrow : \forall x \in H_{\mathbf{C}} (\bigvee S \leq x \rightarrow p \leq x)\}.$$

Then $\tilde{\mathbf{C}}$ is a cover scheme. For if $S \in \tilde{\mathbf{C}}(p)$ and $q \leq p$, consider the subset T of $p \downarrow$ defined by

$$T = \{t \leq q : \exists s \in S (t \leq s)\}.$$

It is easily shown that $\bigvee T = (\bigvee S) \wedge q$. Now if $x \in H_{\mathbf{C}}$ and $\bigvee T \leq x$, then $\bigvee S \wedge q \leq x$, whence $\bigvee S \leq (q \Rightarrow x)$. But since x is an element of $H_{\mathbf{C}}$, so, by the lemma, is $q \Rightarrow x$, and since $S \in \tilde{\mathbf{C}}(p)$, it follows that $p \leq (q \Rightarrow x)$. Thus $q = p \wedge q \leq x$. Accordingly $T \in \tilde{\mathbf{C}}(q)$, and T obviously sharpens S . This shows that $\tilde{\mathbf{C}}$ is indeed a cover scheme.

It is readily shown that any cover scheme associated with a nucleus (as opposed to a weak nucleus) is saturated. Observe that, when H is \widehat{P} , every coverage on P is saturated, since in that case $H_{\mathbf{C}}$ is $\widehat{\mathbf{C}}$ and so we have, using Proposition I.1 (iv),

$$S \in \tilde{\mathbf{C}}(p) \leftrightarrow \forall I \in \widehat{\mathbf{C}}[S \subseteq I \rightarrow p \in I] \leftrightarrow p \in S^* \leftrightarrow S \in \mathbf{C}(p).$$

To sum up, each weak nucleus on H gives rise to a cover scheme on P and the cover scheme associated with a nucleus is saturated. Conversely, each cover scheme gives rise to a nucleus. This establishes mutually inverse correspondences between nuclei and saturated cover schemes.

Given $a \in H$, we define the nuclei j_a j^a on H by

$$j_a(x) = a \vee x \quad j^a(x) = a \Rightarrow x.$$

The associated cover schemes (easily seen to be pretopologies) on P are given by:

$$S \in \mathbf{C}_a(p) \leftrightarrow p \leq a \vee \forall S$$

$$S \in \mathbf{C}^a(p) \leftrightarrow p \wedge a \leq \forall S.$$

Notice that

$$p \leq a \leftrightarrow \emptyset \in \mathbf{C}_a(p)$$

$$p \leq \neg a \leftrightarrow \emptyset \in \mathbf{C}^a(p).$$

The *double negation operation* $\neg\neg$ is a nucleus on H , whose associated cover scheme is precisely the dense cover scheme **Den** (which accordingly is also known as the *double negation cover scheme*). An argument similar to the one above showing that $\widehat{\mathbf{Den}}$ is a Boolean algebra establishes that $H_{\mathbf{Den}}$ is a Boolean algebra: it is in fact the complete Boolean algebra of $\neg\neg$ -closed elements of H .

Proposition. Let j be a weak nucleus on H . Then the following are equivalent: (a) $j(0) = 0$ (b) $j \leq \neg\neg$ (in the pointwise ordering of $W(H)$) (c) $\emptyset \notin \mathbf{D}_j(p)$ for all p .

Proof. If $j \leq \neg\neg$ then $j0 \leq \neg\neg 0 = 0$. Conversely if $j0 = 0$ then, for any $a \in H$, $j(a) \wedge \neg a \leq j(a) \wedge j(\neg a) = j(a \wedge \neg a) = j0 = 0$. So $j(a) \leq \neg\neg a$. Finally, we have

$$j0 = 0 \leftrightarrow 0 \in H_{\mathbf{D}_j} \leftrightarrow \forall p[(\exists S \in \mathbf{D}_j(p)) \forall S = 0 \rightarrow p = 0]$$

$$\leftrightarrow \forall p[\neg(\exists S \in \mathbf{D}_j(p)) \forall S = 0]$$

$$\leftrightarrow \forall p[\emptyset \notin \mathbf{D}_j(p)]. \quad \square$$

III. COVER SCHEMES AND KRIPKE MODELS

We revert to the assumption that (P, \leq) is a preordered set. Recall that a *presheaf* on P is an assignment, to each $p \in P$, of a set $\mathcal{A}(p)$ and to each pair (p, q) with $q \leq p$ of a map $\mathcal{F}_{pq}: \mathcal{A}(p) \rightarrow \mathcal{A}(q)$ in such a way that \mathcal{F}_{pp} is the identity on $\mathcal{A}(p)$ and, for $r \leq q \leq p$, $\mathcal{F}_{pr} = \mathcal{F}_{qr} \circ \mathcal{F}_{pq}$. The set $V(\mathcal{A}) =$

$\bigcup_{p \in P} \mathcal{F}(p)$ is called the *universe* of \mathcal{F} . A *Kripke model* based on P is a presheaf \mathcal{K} for which $\mathcal{K}(p) \subseteq \mathcal{K}(q)$ whenever $q \leq p$ and each \mathcal{K}_{pq} is the corresponding insertion map. Put more simply, a Kripke model based on P is a map \mathcal{K} from P to a family of sets satisfying $\mathcal{K}(p) \subseteq \mathcal{K}(q)$ whenever $q \leq p$. A Kripke model \mathcal{K} based on P may be regarded as a set “evolving” or “growing” over P : each $\mathcal{K}(p)$ may be thought of as the “state” of the evolving set \mathcal{K} at “stage” p .

Now suppose that we are given a cover scheme \mathbf{C} on P . A Kripke model \mathcal{K} based on P satisfying

$$\mathcal{K}(p) = \bigcap_{s \in S} \mathcal{K}(s)$$

for any $p \in P$, $S \in \mathbf{C}(p)$ is said to be *compatible with \mathbf{C}* . (When P is directed downward, that is, whenever each pair of elements of P has a lower bound, and \mathbf{C} is a pretopology on P , a Kripke model compatible with \mathbf{C} is nothing other than a \mathbf{C} -sheaf.)

Each Kripke model \mathcal{K} based on P induces a Kripke model $\widetilde{\mathcal{K}}$ based on the free lower semilattice \widetilde{P} generated by P by setting

$$\widetilde{\mathcal{K}}(\emptyset) = \emptyset \quad \widetilde{\mathcal{K}}(\{p_1, \dots, p_n\}) = \mathcal{K}(p_1) \cup \dots \cup \mathcal{K}(p_n).$$

If, further, \mathcal{K} is compatible with the cover scheme \mathbf{C} on P , then $\widetilde{\mathcal{K}}$ is compatible with the cover scheme on \widetilde{P} induced by P (and hence also with the associated coverage on \widetilde{P} .) For suppose that \mathcal{K} is in fact compatible with the cover scheme \mathbf{C} on P . Given $F \in \widetilde{P}$, a nonempty subset $\{p_1, \dots, p_n\}$ of F , and $S_1 \in \mathbf{C}(p_1), \dots, S_n \in \mathbf{C}(p_n)$, we have

$$\begin{aligned}
\bigcap_{X \in \mathcal{S}_1 \bullet \dots \bullet \mathcal{S}_n} \widetilde{\mathcal{K}}(X) &= \bigcap_{s_1 \in \mathcal{S}_1, \dots, s_n \in \mathcal{S}_n} \widetilde{\mathcal{K}}(\{s_1, \dots, s_n\} \cup F) \\
&= \bigcap_{s_1 \in \mathcal{S}_1, \dots, s_n \in \mathcal{S}_n} (\mathcal{K}(s_1) \cup \dots \cup \mathcal{K}(s_n) \cup \bigcup_{p \in F} \mathcal{K}(p)) \\
&= \bigcap_{s_1 \in \mathcal{S}_1} \mathcal{K}(s_1) \cup \dots \cup \bigcap_{s_n \in \mathcal{S}_n} \mathcal{K}(s_n) \cup \bigcup_{p \in F} \mathcal{K}(p) \\
&= \mathcal{K}(p_1) \cup \dots \cup \mathcal{K}(p_n) \cup \bigcup_{p \in F} \mathcal{K}(p) \\
&= \widetilde{\mathcal{K}}(F).
\end{aligned}$$

Now suppose that the cover scheme \mathbf{C} is in fact a *pretopology*. Then any Kripke model \mathcal{K} based on P induces a Kripke model $\mathcal{K}_{\mathbf{C}}$ also based on P but in addition compatible with \mathbf{C} given by

$$\mathcal{K}_{\mathbf{C}}(p) = \bigcup_{S \in \mathbf{C}(p)} \bigcap_{s \in S} \mathcal{K}(s),$$

that is ,

$$a \in \mathcal{K}_{\mathbf{C}}(p) \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. a \in \mathcal{K}(s).$$

We note that $\mathcal{K}(p) \subseteq \mathcal{K}_{\mathbf{C}}(p)$ for every p . It is easily checked that this defines a Kripke model over P ; let us confirm its compatibility with \mathbf{C} . It suffices to show that, given $S \in \mathbf{C}(p)$, we have $\bigcap_{s \in S} \mathcal{K}_{\mathbf{C}}(s) \subseteq \mathcal{K}_{\mathbf{C}}(p)$. Indeed, if

$$a \in \bigcap_{s \in S} \mathcal{K}_{\mathbf{C}}(s), \text{ then for each } s \in S \text{ there is } T_s \in \mathbf{C}(s) \text{ with } a \in \bigcap_{t \in T_s} \mathcal{K}(t).$$

Writing $T = \bigcup_{s \in S} T_s$, we then have $T \in \mathbf{C}(p)$ and $a \in \bigcap_{t \in T} \mathcal{K}(t)$. It follows that $a \in \mathcal{K}_{\mathbf{C}}(p)$, as required.

Let us now examine some special cases. Let U be a subset of the universe V of \mathcal{K} , and let U^* be the sieve $\{p: U \subseteq \mathcal{K}(p)\}$. Now consider the Kripke model \mathcal{K}^U compatible with \mathbf{C}^{U^*} induced by \mathcal{K} . For arbitrary $p \in P$, we have $S = p \downarrow \cap U^* \in \mathbf{C}^{U^*}(p)$. and $U \subseteq \mathcal{K}(s) \subseteq \mathcal{K}^U(s)$ for every $s \in S$. Hence $U \subseteq \bigcap_{s \in S} \mathcal{K}^U(s) = \mathcal{K}^U(p)$. Thus, under these conditions, U is a subset of

every $\mathcal{K}^U(p)$. In other words, the passage from \mathcal{K} to \mathcal{K}^U forces U to be included in the state of \mathcal{K}^U at each stage. (Note that if $U^* = \emptyset$ then \mathcal{K}^U assumes the constant value V .)

Again let U be a subset of V ; this time define U^+ to be the sieve $\{p: U \cap \mathcal{K}(p) \neq \emptyset\}$. Now consider the Kripke model \mathcal{K}_U compatible with \mathbf{C}_{U^+} induced by \mathcal{K} . Then for any p we have

$$U \cap \mathcal{K}(p) \neq \emptyset \rightarrow p \in U^+ \rightarrow \emptyset \in \mathbf{C}_{U^+}(p) \rightarrow \mathcal{K}_U(p) = \bigcap_{s \in \emptyset} \mathcal{K}_U(s) = V.$$

That is, the passage from \mathcal{K} to \mathcal{K}_U forces each state of \mathcal{K}_U , apart from those already maximal, to be disjoint from U . (Notice that if $U^+ = P$, then \mathcal{K}_U assumes the constant value V .)

We next turn to *logic* in Kripke models. Each Kripke model \mathcal{K} based on P , with universe V , determines a map $\widehat{\mathcal{K}}: V \rightarrow \widehat{P}$ given by

$$\widehat{\mathcal{K}}(v) = \{p: v \in \mathcal{K}(p)\}.$$

This extends naturally to a homomorphism—also denoted by $\widehat{\mathcal{K}}$ —of the free Heyting algebra $F(V)$ generated by V into \widehat{P} . Think of the members of $F(V)$ as the formulas of intuitionistic propositional logic generated by the members of V regarded as propositional atoms. Introduce the familiar *forcing* relation $\Vdash_{\mathcal{K}}$ between P and $F(V)$ by defining

$$(*) \quad p \Vdash_{\mathcal{K}} \varphi \leftrightarrow p \in \widehat{\mathcal{K}}(\varphi).$$

Then the fact that $\widehat{\mathcal{K}}: F(V) \rightarrow \widehat{P}$ is a homomorphism of Heyting algebras translates into the usual rules for “Kripke semantics”, namely

- $p \Vdash_{\mathcal{K}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathcal{K}} \varphi \ \& \ p \Vdash_{\mathcal{K}} \psi$
- $p \Vdash_{\mathcal{K}} \varphi \vee \psi \leftrightarrow p \Vdash_{\mathcal{K}} \varphi \ \text{or} \ p \Vdash_{\mathcal{K}} \psi$
- $p \Vdash_{\mathcal{K}} \varphi \Rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_{\mathcal{K}} \varphi \rightarrow q \Vdash_{\mathcal{K}} \psi]$
- $p \Vdash_{\mathcal{K}} \neg \varphi \leftrightarrow \forall q \leq p \ q \not\Vdash_{\mathcal{K}} \varphi$

Equally, the map $\widehat{\mathcal{H}} : V \rightarrow \widehat{P}$ extends to a frame homomorphism (i.e., a map preserving top elements, \wedge , and \bigvee)—again denoted by $\widehat{\mathcal{H}}$ —of the *free frame* $\Phi(V)$ generated by V . Think of the members of $\Phi(V)$ as the formulas of *infinitary* intuitionistic propositional logic generated by the members of V regarded as propositional atoms. Such a formula φ is said to be *geometric* if it is generated from propositional atoms by applying just \wedge and \bigvee . Introducing the forcing relation $\Vdash_{\mathcal{H}}$ between P and $\Phi(V)$ as in (*) above, the fact that $\widehat{\mathcal{H}} : \Phi(V) \rightarrow \widehat{P}$ is a frame homomorphism translates into the semantical rules for *geometric* formulas:

- $p \Vdash_{\mathcal{H}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathcal{H}} \varphi \ \& \ p \Vdash_{\mathcal{H}} \psi$
- $p \Vdash_{\mathcal{H}} \bigvee_{i \in I} \varphi_i \leftrightarrow \exists i \in I \ p \Vdash_{\mathcal{H}} \varphi_i$.

Now suppose that \mathbf{C} is a pretopology on P . It is then easily seen that \mathcal{H} is compatible with \mathbf{C} iff each $\widehat{\mathcal{H}}(v)$ is a \mathbf{C} -closed sieve. So if \mathcal{H} is compatible with \mathbf{C} , the resulting map $\widehat{\mathcal{H}} : V \rightarrow \widehat{\mathbf{C}}$ can be extended to a homomorphism, which we shall denote by $\widehat{\mathcal{H}}_{\mathbf{C}}$, of $F(V)$ into $\widehat{\mathbf{C}}$. Introducing the forcing relation $\Vdash_{\mathcal{H}, \mathbf{C}}$ between P and $F(V)$ by

$$(**) \quad p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \leftrightarrow p \in \widehat{\mathcal{H}}_{\mathbf{C}}(\varphi),$$

we find that the fact that $\widehat{\mathcal{H}}_{\mathbf{C}} : F(V) \rightarrow \widehat{\mathbf{C}}$ is a homomorphism translates into the rules of “Beth-Kripke-Joyal” semantics for $\Vdash_{\mathcal{H}, \mathbf{C}}$ (see, e.g., [4]), viz.,

- $p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \ \& \ p \Vdash_{\mathcal{H}, \mathbf{C}} \psi$
- $p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \vee \psi \leftrightarrow \exists S \in \mathbf{C}(p) \ \forall s \in S \ [s \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \ \text{or} \ s \Vdash_{\mathcal{H}, \mathbf{C}} \psi]$
- $p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \Rightarrow \psi \leftrightarrow \forall q \leq p \ [q \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \rightarrow q \Vdash_{\mathcal{H}, \mathbf{C}} \psi]$
- $p \Vdash_{\mathcal{H}, \mathbf{C}} \neg \varphi \leftrightarrow \forall q \leq p \ [q \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \rightarrow \emptyset \in \mathbf{C}(p)].$

We verify the second and fourth of these. We have, using Proposition I 1. (iii),

$$\begin{aligned}
p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \vee \psi &\leftrightarrow p \in \widehat{\mathcal{H}_{\mathbf{C}}}(\varphi \vee \psi) = \widehat{\mathcal{H}_{\mathbf{C}}}(\varphi) \vee \widehat{\mathcal{H}_{\mathbf{C}}}(\psi) \text{ (in } \widehat{\mathbf{C}}) \\
&\leftrightarrow \exists S \in \mathbf{C}(p). S \subseteq \widehat{\mathcal{H}_{\mathbf{C}}}(\varphi) \cup \widehat{\mathcal{H}_{\mathbf{C}}}(\psi) \\
&\leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. s \in \widehat{\mathcal{H}_{\mathbf{C}}}(\varphi) \vee s \in \widehat{\mathcal{H}_{\mathbf{C}}}(\psi) \\
&\leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S [s \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \text{ or } s \Vdash_{\mathcal{H}, \mathbf{C}} \psi].
\end{aligned}$$

Also, using Proposition I 1. (ii) we have

$$\begin{aligned}
p \Vdash_{\mathcal{H}, \mathbf{C}} \neg \varphi &\leftrightarrow p \in \widehat{\mathcal{H}_{\mathbf{C}}}(\neg \varphi) = \neg \widehat{\mathcal{H}_{\mathbf{C}}}(\varphi) = (\widehat{\mathcal{H}_{\mathbf{C}}}(\varphi) \Rightarrow \mathbf{0}) \\
&\leftrightarrow \forall q \leq p [q \in \widehat{\mathcal{H}_{\mathbf{C}}}(\varphi) \rightarrow \emptyset \in \mathbf{C}(q)] \\
&\leftrightarrow \forall q \leq p [q \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \rightarrow \emptyset \in \mathbf{C}(q)].
\end{aligned}$$

Since **Den** is a Boolean algebra it follows that, when \mathcal{H} is compatible with **Den**, $p \Vdash_{\mathcal{H}, \mathbf{Den}} \varphi \vee \neg \varphi$ for every p , i.e., classical logic prevails in the Kripke model associated with $\widehat{\mathcal{H}_{\mathbf{Den}}}$.

When \mathcal{H} is compatible with \mathbf{C} , the map $\widehat{\mathcal{H}} : V \rightarrow \widehat{\mathbf{C}}$ can be extended to a frame homomorphism, which we shall again denote by $\widehat{\mathcal{H}_{\mathbf{C}}}$, of $\Phi(V)$ into $\widehat{\mathbf{C}}$. Introduce the forcing relation $\Vdash_{\mathcal{H}, \mathbf{C}}$, now between P and $\Phi(V)$, by the same equivalence (**) as above. When \mathbf{C} is a Grothendieck topology, a straightforward inductive argument shows that, for any geometric formula φ ,

$$(\dagger) \quad p \Vdash_{\mathcal{H}, \mathbf{C}} \varphi \leftrightarrow \exists S \in \mathbf{C}(p) \forall s \in S. s \Vdash_{\mathcal{H}} \varphi.$$

This may be applied to “force” any given set Σ of geometric formulas to become true in a Kripke model. For, starting with a Kripke model \mathcal{H} , let A be the sieve $\{p: \forall \sigma \in \Sigma. p \Vdash_{\mathcal{H}} \sigma\}$. Let \mathbf{G} be the Grothendieck topology generated by the coverage \mathbf{C}^A : it is easily verified that a sieve $S \subseteq p \downarrow$

satisfies the same condition for membership in $\mathbf{G}(p)$ as in $\mathbf{C}^A(p)$, viz., $p \downarrow \cap A \subseteq S$. Now by (\dagger) we have, for each $\sigma \in \Sigma$,

$$(\ddagger) \quad p \Vdash_{\mathcal{K}, \mathbf{G}} \sigma \leftrightarrow \exists S \in \mathbf{G}(p) \forall s \in S. s \Vdash_{\mathcal{K}} \sigma.$$

If we take S to be $p \downarrow \cap A$, then evidently $S \in \mathbf{G}(p)$ and $\forall s \in S. s \Vdash_{\mathcal{K}} \sigma$. It now follows from (\ddagger) that $p \Vdash_{\mathcal{K}, \mathbf{G}} \sigma$ for every $\sigma \in \Sigma$ and every $p \in P$. In this sense \mathbf{G} “forces” all the members of Σ to be true in the Kripke model associated with $\widehat{\mathcal{K}}_{\mathbf{G}}$.

IV. COVER SCHEMES AND FRAME-VALUED SET THEORY

We now set about relating what has been done so far to frame-valued set theory. Associated with each frame H is an H -valued model V^H of (intuitionistic) set theory (see, e.g. [1] or [2]): we recall some of its principal features.

- Each of the members of V^H —the *H-valued sets*—is a map from a subset of V^H to H .
- Corresponding to each sentence σ of the language of set theory (with names for all elements of V^H) is an element $\llbracket \sigma \rrbracket = \llbracket \sigma \rrbracket^H \in H$ called its *truth value in V^H* . These “truth values” satisfy the following conditions. For $a, b \in V^H$,

$$\begin{aligned} \llbracket b \in a \rrbracket &= \bigvee_{c \in \text{dom}(a)} \llbracket b = c \rrbracket \wedge a(c) \\ \llbracket b = a \rrbracket &= \bigvee_{c \in \text{dom}(a) \cup \text{dom}(b)} (\llbracket c \in b \rrbracket \leftrightarrow \llbracket c \in a \rrbracket) \\ \llbracket \sigma \wedge \tau \rrbracket &= \llbracket \sigma \rrbracket \wedge \llbracket \tau \rrbracket, \text{ etc.} \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{a \in V^H} \llbracket \varphi(a) \rrbracket \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{a \in V^H} \llbracket \varphi(a) \rrbracket \end{aligned}$$

A sentence σ is *valid*, or *holds*, in V^H , written $V^H \models \sigma$, if $\llbracket \sigma \rrbracket = 1$, the top element of H . The truth value $\llbracket \sigma \rrbracket$ “measures” the degree or

extent to which σ holds: the larger $\llbracket \sigma \rrbracket$ is, the “truer” σ is. In particular, when $\llbracket \sigma \rrbracket = 1$, σ is ‘universally’ or ‘absolutely’ true, and when $\llbracket \sigma \rrbracket = 0$, σ is “universally” or “absolutely” false.

- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in V^H . Accordingly the category $\mathcal{S}et^H$ of sets constructed within V^H is a topos: in fact $\mathcal{S}et^H$ can be shown to be equivalent to the topos of canonical sheaves on H .
- There is a canonical embedding $x \mapsto \hat{x}$ of the universe V of sets into V^H satisfying

$$\llbracket u \in \hat{x} \rrbracket = \bigvee_{y \in x} \llbracket u = \hat{y} \rrbracket \text{ for } x \in V, u \in V^{(H)}$$

$$x \in y \leftrightarrow V^{(H)} \vDash \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \vDash \hat{x} = \hat{y} \text{ for } x, y \in V$$

$$\varphi(x_1, \dots, x_n) \leftrightarrow V^{(H)} \vDash \varphi(x_1, \dots, x_n) \text{ for } x_1, \dots, x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

It follows from the last of these assertions that the canonical representative \hat{H} of H is a Heyting algebra in V^H . The *canonical prime filter* in \hat{H} is the H -set Φ_H defined by

$$\text{dom}(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \text{ for } a \in H.$$

Clearly $V^H \vDash \Phi_H \subseteq \hat{H}$, and it is easily verified that

$$V^H \vDash \Phi_H \text{ is a (proper) prime filter}^3 \text{ in } \hat{H}.$$

It can also be shown that Φ_H is *V-generic* in the sense that, for any subset $A \subseteq H$,

³ We recall that a filter F in a lattice is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$.

$$V^{(H)} \models \widehat{\bigvee} A \in \Phi_H \leftrightarrow \Phi_H \cap \widehat{A} \neq \emptyset.$$

Moreover, for any $a \in H$ we have $\llbracket \widehat{a} \in \Phi_H \rrbracket = a$, and in particular, for any sentence σ , $\llbracket \sigma \rrbracket = \llbracket \llbracket \widehat{\sigma} \rrbracket \in \Phi_H \rrbracket$. Thus $V^{(H)} \models \sigma \leftrightarrow V^{(H)} \models \llbracket \widehat{\sigma} \rrbracket \in \Phi_H$ —in this sense Φ_H is the filter of “true” sentences in $V^{(H)}$.

This suggests that we define a *truth set* in $V^{(H)}$ to be an H -set F for which

$$V^{(H)} \models F \text{ is a filter in } \widehat{H} \text{ such that } F \supseteq \Phi_H.$$

There is a natural bijective correspondence between truth sets in $V^{(H)}$ and weak nuclei on H . With each weak nucleus j on H we associate the H -set T_j defined by $\text{dom}(T_j) = \text{dom}(\Phi_H)$ and $T_j(\widehat{a}) = j(a)$ for $a \in H$. It is easily verified that T_j is a truth set—the requirement that T_j be a filter corresponds exactly to the condition that j preserve finite meets and that it contain Φ_H to the condition that j satisfy $a \leq j(a)$. Inversely, given a truth set F in $V^{(H)}$, we define the map $j_F : H \rightarrow H$ by $j_F(a) = \llbracket \widehat{a} \in F \rrbracket$. Again, it is readily verified that j_F is a weak nucleus on H . These correspondences are evidently mutually inverse and in fact establish an isomorphism between the frame $W(H)$ of weak nuclei on H and the internal frame of filters in \widehat{H} containing Φ_H . Under this isomorphism *nuclei* correspond precisely to *reflexive truth sets*, that is, truth sets satisfying the additional condition (evidently met by Φ_H)

$$V^{(H)} \models \llbracket \widehat{\widehat{a} \in F} \rrbracket \in F \rightarrow \widehat{a} \in F.$$

It is of interest to examine the familiar case in which H is a complete *Boolean algebra* B . In this case the canonical prime filter Φ_B is an *ultrafilter* in \widehat{B} , so that, in $V^{(B)}$, the only filters in \widehat{B} containing Φ_B —the only truth sets—are Φ_B itself and \widehat{B} . It follows that, for truth sets F and G in $V^{(B)}$

$$V^{(B)} \models F = G \leftrightarrow [\widehat{0} \in F \leftrightarrow \widehat{0} \in G].$$

In other words, the truth value $\llbracket \hat{0} \in F \rrbracket$, which can be an arbitrary member of B , determines the identity of F . This means that truth sets in V^B , and so equally weak nuclei on B , are in bijective correspondence with the members of B . In fact it is readily shown directly that any weak nucleus on a Boolean algebra B is of the form j_a for some $a \in B$. For given a weak nucleus j on B , observe: $\neg x \leq j(\neg x)$, whence $\neg j(\neg x) \leq \neg\neg x = x$. Also $j(x) \wedge j(\neg x) = j(x \wedge \neg x) = j(0)$, whence $j(x) \leq j(\neg x) \Rightarrow j(0) = \neg j(\neg x) \vee j(0) \leq x \vee j(0)$. But clearly $x \vee j(0) \leq j(x)$, so that $j(x) = x \vee j(0)$.

Consider now the special case in which H is the completion \widehat{P} of a preordered set P . We have already established a bijective correspondence between Grothendieck topologies on P and nuclei on \widehat{P} . This leads in turn to a bijective correspondence between Grothendieck topologies on P and reflexive truth sets in $V^{\widehat{P}}$. Explicitly, this correspondence assigns to each Grothendieck topology \mathbf{C} on P the reflexive truth set $T_{\mathbf{C}}$ in $V^{\widehat{P}}$ given by $T_{\mathbf{C}}(S) = S^*$ for $S \in \widehat{P}$, and to each reflexive truth set F in $V^{\widehat{P}}$ the Grothendieck topology \mathbf{C}_F on P defined by $S \in \mathbf{C}_F(p) \leftrightarrow p \in \llbracket \hat{S} \in T \rrbracket$.

The topos $\mathcal{S}et^{\widehat{P}}$ of sets in $V^{\widehat{P}}$ is, as we have observed, equivalent to the topos of canonical sheaves on \widehat{P} , which is itself, as is well known, equivalent to the topos $\mathcal{S}et^{P^{op}}$ of presheaves on P . Moreover, Grothendieck topologies on P are known (see [4]) to correspond bijectively to internal Lawvere-Tierney topologies—that is, internal nuclei—on the truth-value object Ω in $\mathcal{S}et^{P^{op}}$. How this fact related to the representation of Grothendieck topologies as reflexive truth sets in $V^{\widehat{P}}$? It turns out that in a general V^H there is a natural bijection between truth sets/reflexive truth sets and weak nuclei/nuclei on $\Omega = \{u: u \subseteq \hat{1}\}$. The representation of Grothendieck topologies as truth sets in V^H , while equivalent to that through Lawvere-Tierney topologies, seems especially perspicuous.

The *forcing* relation \Vdash_P in $V^{(\bar{P})}$ between sentences and elements of P is defined by

$$p \Vdash_P \sigma \leftrightarrow p \in \llbracket \sigma \rrbracket^{\bar{P}}.$$

Note that we then have

$$\llbracket \sigma \rrbracket^{\bar{P}} = \{p : p \Vdash_P \sigma\}.$$

\Vdash_P satisfies the usual rules governing Kripke semantics for predicate sentences, viz.,

- $p \Vdash_P \varphi \wedge \psi \leftrightarrow p \Vdash_P \varphi \ \& \ p \Vdash_P \psi$
- $p \Vdash_P \varphi \vee \psi \leftrightarrow p \Vdash_P \varphi \ \text{or} \ p \Vdash_P \psi$
- $p \Vdash_P \varphi \rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_P \varphi \rightarrow q \Vdash_P \psi]$
- $p \Vdash_P \neg \varphi \leftrightarrow \forall q \leq p \ q \not\Vdash_K \varphi$
- $p \Vdash_P \forall x \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for every $a \in V^{(\bar{P})}$
- $p \Vdash_P \exists x \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for some $a \in V^{(\bar{P})}$.

We note also that \Vdash_P is *persistent* in the sense that, if $p \Vdash_P \varphi$ and $q \leq p$, then $q \Vdash_P \varphi$.

If \mathbf{C} be a pretopology on P , the forcing relation $\Vdash_{\mathbf{C}}$ in the model $V^{(\bar{\mathbf{C}})}$ is similarly defined by

$$p \Vdash_{\mathbf{C}} \sigma \leftrightarrow p \in \llbracket \sigma \rrbracket_{\mathbf{C}}.$$

As for Kripke models, this relation can be shown to satisfy the rules of Beth-Kripke-Joyal semantics, viz.,

- $p \Vdash_{\mathbf{C}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathbf{C}} \varphi \ \& \ p \Vdash_{\mathbf{C}} \psi$
- $p \Vdash_{\mathbf{C}} \varphi \vee \psi \leftrightarrow \exists S \in \mathbf{C}(p) \ \forall s \in S [s \Vdash_{\mathbf{C}} \varphi \ \text{or} \ s \Vdash_{\mathbf{C}} \psi]$
- $p \Vdash_{\mathbf{C}} \varphi \Rightarrow \psi \leftrightarrow \forall q \leq p [q \Vdash_{\mathbf{C}} \varphi \rightarrow q \Vdash_{\mathbf{C}} \psi]$
- $p \Vdash_{\mathbf{C}} \neg \varphi \leftrightarrow \forall q \leq p [q \Vdash_{\mathbf{C}} \varphi \rightarrow \emptyset \in \mathbf{C}(p)]$
- $p \Vdash_{\mathbf{C}} \forall x \varphi \leftrightarrow p \Vdash_{\mathbf{C}} \varphi(a)$ for every $a \in V^{(\bar{\mathbf{C}})}$
- $p \Vdash_{\mathbf{C}} \exists x \varphi \leftrightarrow \exists S \in \mathbf{C}(p) \ \forall s \in S \ s \Vdash_{\mathbf{C}} \varphi(a)$ for some $a \in V^{(\bar{\mathbf{C}})}$.

**V. POTENTIAL APPLICATIONS OF COVER SCHEMES, KRIPKE MODELS, AND FRAME-
VALUED SET THEORY IN SPACETIME PHYSICS**

In spacetime physics any set \mathcal{C} of events—a *causal set*—is taken to be partially ordered by the relation \leq of *possible causation*: for $p, q \in \mathcal{C}$, $p \leq q$ means that q is in p 's future light cone. In her groundbreaking paper [5] Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by “sets evolving over \mathcal{C} ”—that is, in essence, by the topos $\mathit{Set}^{\mathcal{C}}$ of presheaves on \mathcal{C}^{op} . To enable what she has done to be the more easily expressed within the framework presented here, we will reverse the causal ordering, that is, \mathcal{C} will be replaced by \mathcal{C}^{op} , and the latter written as P —which will, moreover, be required to be no more than a *preordered* set. Thus P is a set of events preordered by the relation \leq , where $p \leq q$ is intended to mean that p is in q 's future light cone—that q *could* be the cause of p . In requiring that \leq be no more than a preordering—in dropping, that is, the antisymmetry of \leq —we are, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Specifically, then, we fix a preordered set (P, \leq) , which we shall call the *universal causal set*; its members will be called *events* and $p \leq q$ understood to mean that p is in q 's causal *future*, or q 's future light cone, in short, that p is a possible *effect* of q . (Thus, for each event p , the set $p \downarrow$ is p 's future light cone.) Markopoulou, in essence, suggests that viewing the universe “from the inside” amounts to placing oneself within the topos of presheaves or “evolving universe” $\mathit{Set}^{P^{\text{op}}}$. Since, as we have already observed, $\mathit{Set}^{P^{\text{op}}}$ is equivalent to the topos of sets in $V^{(\bar{P})}$, Markopoulou's proposal may be effectively realized by working within

$V^{(\hat{P})}$. Let us do so, writing for simplicity H for \hat{P} : we think of $V^{(H)}$ as an *evolving universe*, and describing what the universe looks like “from the inside” will then amount to reporting the view from $V^{(H)}$. Each sentence σ of the language of set theory will be construed as an *assertion* concerning the evolving universe $V^{(H)}$.

The fact that each truth value $\llbracket \sigma \rrbracket^H$ (which we shall normally abbreviate to $\llbracket \sigma \rrbracket$) is a sieve in P — that is, satisfies $p \in \llbracket \sigma \rrbracket$ and $q \leq p \rightarrow q \in \llbracket \sigma \rrbracket$ may be understood as asserting that truth values in the evolving universe are “closed under potential effects”, or “causally closed”.

The forcing relation \Vdash_P (which we will usually write simply as \Vdash) defined in the previous section now links events p and assertions σ : $p \Vdash \sigma$ will be taken to mean that σ *holds* as a result of (the occurrence of) event p , or that p *induces* the assertion σ to hold. The persistence of \Vdash —i.e. the fact that, if $p \Vdash \sigma$ and $q \leq p$, then $q \Vdash \sigma$ —amounts to the observation that, once an event p induces an assertion to hold, that assertion continues to hold throughout p ’s causal future⁴.

Define the set $K \in V^{(H)}$ by $\text{dom}(K) = \{\hat{p} : p \in P\}$ and $K(\hat{p}) = p \downarrow$. Then, in $V^{(H)}$, K is a subset of \hat{P} and for $p \in P$, $\llbracket \hat{p} \in K \rrbracket = p \downarrow$. K is the counterpart in $V^{(H)}$ of the “evolving” set *Past* Markopoulou defines by $\text{Past}(p) = p \downarrow$, with insertions as transition maps. (\hat{P} , incidentally, is the $V^{(H)}$ - counterpart of the constant presheaf on P with value P which Markopoulou calls *World*.) Accordingly the “causal past” of any “event” p is represented by the truth value in $V^{(H)}$ of the statement $\hat{p} \in K$. The fact that, for any $p, q \in P$ we have

⁴ It follows that assertions must be taken as being implicitly in the past tense: “such and such *was* the case”.

$$q \Vdash_P \widehat{p} \in K \leftrightarrow q \leq p$$

may be construed as asserting that *the events in the causal future of an event p are precisely those forcing (the canonical representative of) p to be a member of K* . For this reason we shall call K the *causal set in $V^{(H)}$* .

If we identify each $p \in P$ with $p \downarrow \in H$, P may then be regarded as a subset of H so that, in $V^{(H)}$, \widehat{P} is a subset of \widehat{H} . It is not hard to show that, in $V^{(H)}$, K generates the canonical prime filter Φ_H in \widehat{H} . Using the V -genericity of Φ_H , and the density of P in H , one can show that $\llbracket \sigma \rrbracket = \llbracket \exists p \in K. p \leq \widehat{\sigma} \rrbracket$, so that, with moderate abuse of notation,

$$V^{(H)} \models [\sigma \leftrightarrow \exists p \in K. p \Vdash \sigma].$$

That is, in $V^{(H)}$, *a sentence holds precisely when it is forced to do so at some “causal past stage” in K* . This establishes the centrality of the causal set K —and, correspondingly, that of the “evolving” set *Past*—in determining the truth of sentences “from the inside”, that is, inside the universe $V^{(H)}$.

Markopoulou also considers the complement of *Past*. In the present setting, this is the $V^{(H)}$ -set $\neg K$ —the *complement* of the causal set K —for which $\llbracket \widehat{p} \in \neg K \rrbracket = \llbracket p \notin K \rrbracket = \neg p \downarrow$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p \downarrow$ those *beyond p 's causal horizon*, in that no observer at p can ever receive “information” from any event in $\neg p \downarrow$. Since clearly we have

$$(\star) \quad q \Vdash \widehat{p} \in \neg K \leftrightarrow q \in \neg p \downarrow,$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of $\neg K$* . In this sense $\neg K$ reflects, or “measures” the causal structure of P .

In this connection it is natural to call $\neg \neg p \downarrow = \{q : \forall r \leq q \exists s \leq r. s \leq p\}$ the *causal horizon* of p : it consists of those events q

for which an observer placed at p could, in its future, receive information from any event in the future of an observer placed at q . Since

$$q \Vdash \widehat{p} \in \neg\neg K \leftrightarrow q \in \neg\neg p \downarrow,$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of $\neg\neg K$.*

It is easily shown that $\neg K$ is *empty* (i.e. $V^{(H)} \models \neg K = \emptyset$) if and only if P is *directed downwards* in the sense that for any $p, q \in P$ there is $r \in P$ for which $r \leq p$ and $r \leq q$; that is, if *the future light cones of any pair of events have nonempty intersection or “overlap”*. This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which P is the opposite \mathbb{N}^{op} of the totally ordered set \mathbb{N} of natural numbers. Here the corresponding complete Heyting algebra H is the family of all downward-closed sets of natural numbers. Interestingly, in this case, the causal set K is *neither finite nor actually infinite*.

To see this, first note that, for any natural number n , we have, $\llbracket \neg(\widehat{n} \in \neg K) \rrbracket = \mathbb{N}$. It follows that $V^{(H)} \models \neg\neg \forall n \in \widehat{\mathbb{N}}. n \in K$. But, working in $V^{(H)}$, if $\forall n \in \widehat{\mathbb{N}}. n \in K$, then K is not finite, so if K is finite, then $\neg \forall n \in \widehat{\mathbb{N}}. n \in K$, and so $\neg\neg \forall n \in \widehat{\mathbb{N}}. n \in K$ implies the non-finiteness of K .

But, in $V^{(H)}$, K is not actually infinite. For (again working in $V^{(H)}$), if K were actually infinite (i.e., if there existed an injection of $\widehat{\mathbb{N}}$ into K), then the statement

$$\forall x \in K \exists y \in K. x > y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$\begin{aligned}
\llbracket \forall x \in K \exists y \in K. x > y \rrbracket &= \bigcap_{m \in \mathbb{N}^{op}} [m \downarrow \Rightarrow \bigcup_{n \in \mathbb{N}^{op}} n \downarrow \cap \llbracket \hat{m} > \hat{n} \rrbracket] \\
&= \bigcap_m [m \downarrow \Rightarrow \bigcup_{n < m} n \downarrow] \\
&= \bigcap_m [m \downarrow \Rightarrow (m+1) \downarrow] \\
&= \bigcap_m (m+1) \downarrow \\
&= \emptyset
\end{aligned}$$

So $\forall x \in K \exists y \in K. x > y$ is false in $V^{(H)}$ and therefore K is not actually infinite.

In other words, in evolving Newtonian spacetime, the set K representing past time is potentially, but not actually infinite: this is, in essence, what Kant asserted of time.

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables*. This amounts to specifying a presheaf of C^* -algebras on P , which, in the present framework, corresponds to specifying a set \mathcal{A} in $V^{(H)}$ satisfying

$$V^{(H)} \models \mathcal{A} \text{ is a } C^*\text{-algebra.}$$

The “internal” C^* -algebra \mathcal{A} is then subject to the intuitionistic internal logic of $V^{(H)}$: *any* theorem concerning C^* -algebras—provided only that it be constructively proved—automatically applies to \mathcal{A} . Reasoning with \mathcal{A} is more direct and simpler than reasoning with \mathcal{A} .

This same procedure of “internalization” can be performed with any causally evolving object: each such object of type \mathcal{I} corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \models S \text{ is of type } \mathcal{I}.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event p , $\text{Antichains}(p)$ consists of all sets of causally unrelated events in $\text{Past}(p)$,

while $Graphs(p)$ is the set of all graphs supported by elements of $Antichains(p)$. In the present framework $Antichains$ is represented by the $V^{(H)}$ -set $Anti = \{X \subseteq \hat{P} : X \text{ is an antichain}\}$ and $Graphs$ by the $V^{(H)}$ -set $Grph = \{G : \exists X \in A . G \text{ is a graph supported by } X\}$. Again, both $Anti$ and $Grph$ can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Finally let us examine the role of cover schemes on causal sets. Suppose we are given a cover scheme \mathbf{C} on the universal causal set P . Each \mathbf{C} -cover of an event p may be thought of as a “sampling” of the events in p ’s causal future, a “survey” of p ’s potential effects—in a word, a *survey of p* . Using this language the defining condition (**Cov**) for cover schemes laid down in section I becomes: *for any survey S of a given event p , and any event q which is a possible effect of p , there exists a survey of q each event in which is the possible effect of some event in S .*

As we have seen, cover schemes may be used to force certain conditions to prevail in the associated models. Let us consider, for example, the cover scheme **Den** in P . We know that the associated frame $\widehat{\mathbf{Den}}$ is a Boolean algebra—let us denote it by B . The corresponding causal set K_B in $V^{(B)}$ then has the property

$$\llbracket \hat{p} \in K_B \rrbracket = \neg\neg p \downarrow;$$

so that,

$$\begin{aligned} q \Vdash_B \hat{p} \in K_B &\leftrightarrow q \in \neg\neg p \downarrow \\ &\leftrightarrow q \text{ is in } p\text{'s causal horizon.} \end{aligned}$$

Comparing this with (**★**) above, we see that moving to the universe $V^{(B)}$ —“Booleanizing” it, so to speak—amounts to replacing causal futures by causal horizons. When P is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of P , B is the two-element Boolean algebra $\mathbf{2}$, so that $V^{(B)}$ is just the

universe V of “static” sets. In this case, then, the effect of “Booleanization” is to *render the universe timeless*.

The universes associated with the cover schemes \mathbf{C}^A and \mathbf{C}_A seem also to have a rather natural physical meaning. Consider, for instance the case in which A is the sieve $p\downarrow$ —the causal future of p . In the associated universe $V(\widehat{\mathbf{C}^A})$ the corresponding causal set K^A satisfies

$$\llbracket \widehat{q} \in K^A \rrbracket = \text{least } \mathbf{C}^A\text{-closed sieve containing } q$$

so that, in particular

$$\begin{aligned} \llbracket \widehat{p} \in K^A \rrbracket &= \text{least } \mathbf{C}^A\text{-closed sieve containing } p \\ &= P. \end{aligned}$$

This means that, for every event q ,

$$q \Vdash_{\widehat{\mathbf{C}^A}} \widehat{p} \in K^A.$$

Comparing this with (\star) , we see that in $V(\widehat{\mathbf{C}^A})$ that every event has been “forced” into p ’s causal future: in short, that p now marks the “beginning” of the universe as viewed from inside $V(\widehat{\mathbf{C}^A})$.

Similarly, we find that the causal set K_A in the universe $V(\widehat{\mathbf{C}_A})$ has the property

$$q \leq p \rightarrow \forall r [r \Vdash_{\widehat{\mathbf{C}_A}} \widehat{q} \in \neg K_A].$$

a comparison with (\star) above reveals that, in $V(\widehat{\mathbf{C}^A})$, every event—including p itself—has been placed beyond p ’s causal horizon. In effect, the event p has been obliterated, effaced from the universe—like the extraordinary events in H.G. Wells’s *The Man Who Could Work Miracles*, the event p never occurred!

As a final possibility consider the universe $V(\widehat{\widetilde{P}})$ associated with the free lower semilattice \widetilde{P} generated by P . In this case the elements of \widetilde{P} are finite sets of events, preordered by the relation \sqsubseteq : for $F, G \in \widetilde{P}$, $F \sqsubseteq G$ iff

every event in G is in the causal past of an event in F . The empty set of events is the top element of \tilde{P} . The causal set \tilde{K} in $V^{(\tilde{P})}$ has the property that its complement $\neg\tilde{K}$ is empty (so that, in this universe, the light cones of any pair of “events” overlap) and $\hat{\mathcal{O}}$ is an initial event in the sense that $F \Vdash_{\tilde{P}} \hat{\mathcal{O}} \in \tilde{K}$ for every “event” F . In this case passage to the new universe $V^{(\hat{P})}$ preserves the original causal relations in the sense that

$$\{q\} \Vdash_{\tilde{P}} \{\hat{p}\} \in \tilde{K} \leftrightarrow q \Vdash_P \hat{p} \in K.$$

In other words, in passing to the new universe the initial event $\hat{\mathcal{O}}$ and the new light cone overlaps have been “freely adjoined” to the original universe.

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