DISSENTING VOICES

Divergent Conceptions of the Continuum in 19th and Early 20th Century Mathematics and Philosophy

John L. Bell

IN BOOK VI OF THE *PHYSICS* ARISTOTLE characterizes the continuous as

> *capable of being divided into parts that can in their turn be divided again, and so on without limit.*

Continuous entities are accordingly distinguished by the feature that—in principle at least— they can be *divided indefinitely* without altering their essential nature. So, for instance, the water in a bucket may be indefinitely halved and yet remain water. Aristotle nowhere to my knowledge defines discreteness as such but we may take the notion as signifying the opposite of continuity—that is, incapable of being indefinitely divided into parts. Thus discrete entities, typically, cannot be divided without effecting a change in their nature: half a wheel is plainly no longer a wheel[1]. Thus we have two contrasting properties: on the one hand, the property of being indivisible, separate or discrete, and, on the other, the property of being indefinitely divisible and continuous although not actually divided into parts. Still, one and the same object can, in a sense, possess both of these properties. For example, if the wheel is regarded simply as a piece of matter, it remains so on being divided in half. In other words, the wheel *qua* wheel is discrete, but as a piece of matter, it is continuous. Examples like this show that continuity and discreteness are complementary attributes originating through the mind's ability to perform acts of abstraction, the one arising by abstracting an object's divisibility and the other its self-identity.

In mathematics it is the concept of whole number, later elaborated into the set concept, that provides an embodiment of the idea of pure discreteness, that is, of the idea of a collection of separate individual objects, all of whose properties—apart from their distinctness—have been refined away. The basic mathematical representation of the idea of continuity, on the other hand, is the geometric figure, and more particularly the straight line. By their very nature geometric figures are continuous; discreteness is injected into geometry, the realm of the

[1] This observation is suggestive of Frege's identification of the idea of "unit" with "concept".

continuous, through the concept of a point, that is, a discrete entity marking the boundary of a line.

The opposition between continuity and discreteness has animated the development of mathematics since antiquity. Indeed, tradition defined mathematics as "the science of discrete and continuous magnitude". A striking example of this opposition—amounting, one might say, to a collision—is the Pythagorean discovery of incommensurable magnitudes. Here the realm of continuous geometric magnitudes resisted the Pythagorean attempt to reduce it to the discrete form of pure number. The theory of proportions later invented by Eudoxus to resolve the problem of incommensurability was in essence an extension of the idea of number—i.e., of the discrete— adequate to the task of expressing the relations between continuous magnitudes.

The opposition between continuity and discreteness arose also in connection with the *method of exhaustion*. We are told by Archimedes that, using his principle of convergence, Eudoxus successfully proved that the volume of a cone is one third that of the circumscribed cylinder. Archimedes also claims that Democritus originally discovered the result, but was unable to prove it rigorously. The obstacle was that he could see no way of actually building the cone from circular segments, each one of which would differ slightly in area from the two flanking it (the method he had apparently used in discovering the result). The atomist Democritus, with his belief in ultimate finite units, would presumably have understood this "slightly" as entailing a *discrete* difference between the areas of these circular segments, which would produce, not a smooth cone, but instead a ziggurat-like figure with a surface consisting of a series of tiny steps. If, on the other hand, this "slightly" were to be taken to mean "continuously", or "infinitesimally", then the difference between the areas of the segments would seem as a result to be nonexistent, and one would end up, not with a cone, but a cylinder. Eudoxus later surmounted this difficulty by taking the *limit* of the volumes in a manner essentially similar to the method employed in the integral calculus.

The opposition resurfaced with renewed vigour in the seventeenth century with the emergence of the differential and integral calculus. Here the controversy centred on the concept of *infinitesimal*. According to one school of thought, the infinitesimal was to be regarded as a real, infinitely small, indivisible element of a continuum, similar to the atoms of Democritus, except that now their number was considered to be infinite. Calculation of areas and volumes, i.e., integration, was thought of as summation of an infinite number of these infinitesimal elements. An area, for example, was taken to be the "sum of the lines of which it is formed". Thus the continuous was, in a way, again reduced to the discrete, but now, with the intrusion of the concept of the infinite, in a subtler and more complex way than before.

Infinitesimals enjoyed a considerable vogue among seventeenth and eighteenth century mathematicians. In the guise of the charmingly named “linelets” and “timelets”, they played an essential role in Isaac Barrow’s “method for finding tangents by calculation”, which appears in his *Lectiones Geometricae* of 1670. As “evanescent quantities” they were instrumental (although later abandoned) in Newton's development of the calculus, and, as “inassignable quantities”, in Leibniz’s. The Marquis de l'Hospital, who in 1696 published the first treatise on the differential calculus (entitled *Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes*), invokes the concept in postulating that “a curved line may be regarded as being made up of infinitely small straight line segments,” and that “one can take as equal two quantities differing by an infinitely small quantity.”

However, the conception of infinitesimals as real entities suffered from imprecision and even, on occasion, logical inconsistency. Memorably derided by Berkeley as “ghosts of departed quantities” (and in the twentieth century roundly condemned by Bertrand Russell as “unnecessary, erroneous, and self-contradictory”), this conception of infinitesimal was gradually displaced by the idea—originally suggested by Newton—of the infinitesimal as a *continuous variable* which becomes arbitrarily small. By the start of the nineteenth century, when the rigorous theory of limits was in the process of being created, this conception of infinitesimal had been accepted by the majority of mathematicians. A line, for instance, was now understood as consisting not of “points” or “indivisibles”, but as the domain of values of a continuous variable, in which separate points are to be considered as locations. At this stage, then, the discrete had given way to the continuous.

But the development of mathematical analysis in the latter half of the nineteenth century led mathematicians to demand still greater precision in the theory of continuous variables, and above all in fixing the concept of *real number* as the value of an arbitrary such variable. As a result, the modern *arithmetico-set-theoretical* conception of real number emerged in the 1870s, largely at the hands of Dedekind and Cantor. The newly fashioned ordered field of real numbers became known as the *arithmetical continuum* because it was held that this number system is entirely adequate for the analytical representation of all types of continuous phenomena. In particular, a line, or the domain of values of a continuous variable, is represented as a set of distinct real numbers, identified as “points”. In this scheme of things there was no place for the concept of infinitesimal, which accordingly departed the scene for a time. Thus the continuous was, once again, reduced to an assemblage of separate discrete points. This last reduction, underpinned by the development of *set theory*, has, as we all know, become the prevailing orthodoxy among mathematicians.

Even so, the doctrine that the continuous is fully explicable in terms of the discrete has never remained unchallenged. Typically, the doctrine's opponents accept that a continuum is an inexhaustible source of points, but deny that it can be "reconstituted" from these latter. Witness, for example:

Aristotle

> *No continuum can be made up of indivisibles, as for instance a line out of points, granting that the line is continuous and the point indivisible*[2]

Leibniz

> *A point may not be a constitutive part of a line.*[3]

Kant

> *Space and time are* quanta continua...*points and instants mere positions.. and out of mere positions viewed as constituents capable of being given prior to space and time neither space nor time can be constructed.*[4]

Among philosophers of the late 19th and early 20th centuries, *Bergson* and *Whitehead* emphasized the primordial nature of the phenomenon of continuity. But six figures of this period—*du Bois-Reymond, Veronese, Brentano, Peirce, Weyl and Brouwer*—stand out as champions of the irreducibility of the continuum concept to discreteness. My remarks here will be chiefly devoted to Brentano, Peirce, and Weyl. I begin, however, with some brief observations on the first pair of members of our sextet.

Paul du Bois-Reymond was a prominent mathematician of the later 19th century who made significant contributions to real analysis, differential equations, mathematical physics and the foundations of mathematics. While accepting many of the methods of the Dedekind-Cantor school, and indeed embracing the idea of the actual infinite, he rejected its associated philosophy of the continuum on the grounds that it was committed to the reduction of the continuous to the discrete. So in 1882 he writes:

> *The conception of space as static and unchanging can never generate the notion of a sharply defined, uniform line from a series of points however dense, for, after all, points are devoid of size, and hence no matter how dense a series of points may be, it can never become an interval, which must always be regarded as the sum of intervals between points.*

Du Bois-Reymond took a somewhat mystical view of the continuum, asserting that its true nature, being beyond the limits of human cognition, would forever elude the understanding of

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[2] Aristotle (1980), Ch. 1

[3] Quoted in Rescher (1967), p. 109.

[4] Kant (1964), p. 204.

mathematicians. Nevertheless this did not prevent him from developing his own theory of the mathematical continuum, his so-called "calculus of infinities", during the 1870s and 80s. As well as offering an account of mathematical magnitude and a scheme for comparing and distinguishing actual infinite quantities, it also incorporates a theory of actual infinitesimals, a notion that du Bois-Reymond had long championed. He writes:

> *The infinitely small is a mathematical quantity and has all its properties in common with the finite… A belief in the infinitely small does not triumph easily. Yet when one thinks boldly and freely, the initial distrust will soon mellow into a pleasant certainty…A majority of educated pepole will admite an infinite in space and time, and not just an "unboundedly large". But they will only with difficulty believe in the infinitely small, despite the fact that the infinitely small has the same right to existence as the infinitely large…*
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> *Were the sight of the starry sky lacking to mankind; had the race arisen and developed troglodytically in enclosed spaces; had its scholars instead of wandering through the distant places of the universe telescopically, only looked for the smallest constituents of form and so were used in their thoughts to advancing into the boundless in the direction of the unmeasurably small: who would doubt then that the infinitely small would take the same place in our system of concepts that the infinitely large does now? Moreover, hasn't the attempt in mechanics to go back down to the smallest active elements long ago introduced into science the atom, the embodiment of the infinitely small? And don't as always skillful attempts to make it superfluous for physics face with certainty the same fate as Lagrange's battle against the differential?*

One of du Bois-Reymond's severest critics was Georg Cantor, who fought an unceasing battle against the concept of infinitesimal, and more generally against the idea that the continuous was in some essential way "irreducible" to the discrete. While du Bois-Reymond did not hesitate to employ geometric and visual intuition whenever he felt it necessary, Cantor, by origin a number-theorist, was naturally inclined to the discrete, as his work, from analysis of discontinuities to set theory, shows. Cantor's strictures against the work of the next member of our sextet, *Giuseppe Veronese*, were, if anything, even more virulent.

Veronese was an outstanding member of the Italian school of geometry in the last quarter of the 19th century. In 1891 he published his exhaustive work on the foundations of geometry, whose title in approximate English translation reads: *Foundations of geometry of several dimensions and several kinds of linear unit, presented in elementary form.* In this work Veronese develops *n*-dimensional projective geometry, including non-Euclidean geometries, from first principles in a synthetic and unified way. Controversially, he also introduces "non-Archimedean" geometries containing both infinitesimal and infinitely large segments. On publication this work attracted the scathing criticism not only of Cantor, but also of Peano and Killing. Yet Hilbert later called it "profound", and incorporated some of Veronese's ideas into his own later *Grundlagen der Geometrie.*

As a geometer Veronese naturally took an essentially geometric view of the continuum. He begins his *Foundations* with a complaint about the use of real numbers as the basis of geometry. Spatial intuition, he says, is what furnishes us with the basal geometric objects and their inherent properties, so that the proper procedure in geometry is a synthetic one

> *which always treats figures as figures, works directly with the elements of the figures and separates and unites them so that each truth and each step of a proof is accompanied as far as possible by intuition.*

In answer to the question "What is the *continuum*?" Veronese writes:

> *This is a word whose meaning we understand without any mathematical definition, since we* intuit *the continuum in its simplest form as the common characteristic of many concrete things, such as, for example, to give some of the simplest, the* time *and the* place occupied *in the external neighbourhood of the object sketched here, or by a plumb line, if one takes no account of its physical properties and its thickness (in the empirical sense).*

For Veronese points are nothing more than signs indicating "positions of the uniting of two parts" of a (rectilinear) continuum. They are, as for Aristotle,

> *…a product of mental abstraction… not parts of the rectilinear object.*

And so, Veronese writes,

> *The hypothesis that the point is not part of the rectilinear continuum (and also has no parts in itself) means that all the points we can imagine in it, however many that may be, do not* constitute *the continuum when they are joined together, and choosing a part (XX′) as small as one wants of the object (for time, an instant), however indeterminate, which is to say without X and X′ being fixed in our thoughts, intuition tells us that* this part is always continuous.

Veronese contrasts his own account of the continuum with that of Cantor and Dedekind in the following words:

> *Cantor and Dedekind…assert in their valuable works that… the one-one relation between the points of [a] line and the points forming the real continuum is arbitrary. They certainly obtain this continuum by means of a sequence of abstract definitions of symbols which, although possible, are arbitrary… According to Dedekind, the numerical continuum is necessary in order to clarify the idea of the continuum of space. According to us, however, it is the intuitive rectilinear continuum which, by means of a point without parts, that serves to give us abstract definitions with respect to the continuum itself, of which the numerical*

> *continuum is only a special case. In this way, the definitions appear not as a force which keeps our mind in check, but finds its complete justification in the perceptual representation of the continuum. One must take some account of this representation in the discussion of basic concepts, but without leaving the field of pure mathematics. Moreover, it would be truly marvellous if an abstract form as complicated as the numerical continuum obtained not only without being guided by the intuitive, but, as is done nowadays by some authors, from mere definitions of symbols, should then find itself in agreement with a representation as simple and primitive as that of the rectilinear continuum.*

Franz Brentano (1838-1917), the modern philosopher most concerned with the nature of the continuous, was also a critic of the idea of an "discretized" continuum. The greater part of Brentano's philosophy has its starting-point in Aristotelian doctrine, and his conception of the continuum constitutes no exception. Aristotle's theory of the continuum rests upon the assumption that all change is continuous and that continuous variation of quality, of quantity and of position are inherent features of perception and intuition. Aristotle considered it self-evident that a continuum cannot consist of points. Any pair of unextended points, he observes, are such that they either touch or are totally separated: in the first case, they yield just a single unextended point, in the second, there is a definite gap between the points. Aristotle held that any continuum—a continuous path, say, or a temporal duration, or a motion—may be divided *ad infinitum* into other continua but not into what might be called "discreta"—parts that cannot themselves be further subdivided. Accordingly, paths may be divided into shorter paths, but not into unextended points; durations into briefer durations but not into unextended instants; motions into smaller motions but not into unextended "stations". Nevertheless, this does not prevent a continuous line from being divided at a point constituting the common border of the line segments it divides. But such points are, according to Aristotle, just *boundaries,* and not to be regarded as actual *parts* of the continuum from which they spring. If two continua have a common boundary, that common border unites them into a single continuum. Such boundaries exist only *potentially*, since they come into being when they are, so to speak, marked out as connecting parts of a continuum; and the parts in their turn are similarly dependent as parts upon the existence of the continuum.

In its fundamentals Brentano's synechology—his theory of the continuum—is akin to Aristotle's. Brentano regards continuity as an essentially perceptual phenomenon, rather than as a mathematical construction. Indeed, Brentano took a somewhat dim view of the efforts of mathematicians to "arithmetize" the continuum—that is, to construct it from points. His attitude varied from rejecting them as inadequate to according them the status of "fictions". (In a letter to Husserl drafted in 1905, Brentano asserts that "I regard it as absurd to interpret a continuum as a set of points.") This is not surprising given his

Aristotelian inclination to take mathematical and physical theories to be genuine descriptions of empirical phenomena rather than idealizations: certainly, if such theories were to be taken as literal descriptions of experience, they would amount to nothing more than "misrepresentations". Indeed, Brentano writes:

> *We must ask those who say that the continuum ultimately consists of points what they mean by a point. Many reply that a point is a cut which divides the continuum into two parts. The answer to this is that a cut cannot be called a thing and therefore cannot be a presentation in the strict and proper sense at all. We have, rather, only presentations of contiguous parts. … The spatial point cannot exist or* be conceived of *in isolation. It is just as necessary for it to belong to a spatial continuum as for the moment of time to belong to a temporal continuum.*[5]

Brentano held that the idea of the continuous is derived from primitive sensible intuition:

> *Thus I affirm that... the concept of the continuous is acquired not through combinations of marks taken from different intuitions and experiences, but through abstraction from unitary intuitions...Every single one of our intuitions—both those of outer perception as also their accompaniments in inner perception, and therefore also those of memory—bring to appearance what is continuous.*[6]

Brentano suggests that the continuous is brought to appearance by sensible intuition in three phases. First, sensation presents us with objects having parts that coincide. From such objects the concept of *boundary* is then abstracted, and then one grasps that these objects actually *contain* coincident boundaries. Finally one sees that this is all that is required in order to have grasped the concept of a continuum.

Continuity is manifested in sensation in a variety of ways. In *visual* sensation, we are presented with *extension*, something possessing length and breadth, and hence with something such that between any two of its parts, provided these be separated, there is a third part. Every sensation possesses a certain *qualitative* continuity in that the object presented in the sensation could have a given manifested quality (colour, for example) in a greater or less degree, and between any two degrees of that quality lies still another degree of that quality. Finally, each sensation manifests *temporal* continuity: this is most evident when we perceive something as moving or at rest.

Brentano recognizes that continua have *qualities* which cause them to possess *multiplicity*—a continuum may manifest continuity in several ways simultaneously. This led him to classify continua into *primary* and *secondary*: a secondary continuum being one whose manifestation is dependent upon another continuum. Here is Brentano himself on the matter:

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[5] Brentano (1974), p. 354.

[6] Brentano (1988), p. 6.

> *Imagine, for example, a coloured surface. Its colour is something from which the geometer abstracts. For him there comes into consideration only the constantly changing manifold of spatial differences. But the colour, too, appears extended with the spatial surface, whether it manifests no specific colour-differences of its own—as in the case of a red colour which fills out a surface uniformly—or whether it varies in its colouring—perhaps in the manner of a rectangle which begins on one side with red and ends on the other side with blue, progressing uniformly through all colour-differences from violet to pure blue in between. In both cases we have to do with a multiple continuum, and it is the spatial continuum which appears thereby as primary, the colour-continuum as secondary. A similar double continuum can also be established in the case of a motion from place to place or of a rest, in which case it is a temporal continuum as such that is primary, the temporally constant or varying place that is the secondary continuum. Even when one considers a boundary of a mathematical body as such, for example a curved or straight line, a double continuity can be distinguished. The one presents itself in the totality of the differences of place that are given in the line, which always grows uniformly, whether in the case of straight, bent, or curved lines, and is that which determines the length of the line. The other resides in the direction of the line, and is either constant or alternating, and may vary continuously, or now more strongly, now less. It is constant in the case of the straight line, changing in the case of the broken line, and continuously varying in every line that is more or less curved. The direction-continuum here is to be compared with the colour-continuum discussed earlier and with the continuum of place in the case of rest or motion of a corporeal point in time. In the double continuum that presents itself to us in the line it is this continuum of directions that is to be referred to as the secondary, the manifold of differences of place as such as the primary continuum.* [7]

[It may be remarked parenthetically that Brentano's distinction of primary and secondary continua can be neatly represented within *category theory*: to put it succinctly, a *primary continuum* is a *domain,* a *secondary continuum* a *codomain.* We form a category $\mathscr{C}$—the *category of continua*—by taking continua as objects and *correlations* between continua as arrows. Then, given any arrow $A \xrightarrow{f} B$ in $\mathscr{C}$, the domain A of f may be taken as a "primary" continuum and its codomain B as a "secondary" continuum. In Brentano's example of a coloured surface, for instance, the primary continuum A is the given spatial surface, the secondary continuum B is the colour spectrum, and the correlation f assigns to each place in A its colour as a position in B. In the case of a corporeal point moving in space, the primary continuum A is an interval of time, the secondary continuum B a region of space, and the correlation f assigns to each instant in A the position in B occupied by the corporeal point. Finally, in the case of the varying direction of a curve the primary continuum A is the curve itself, the secondary continuum is the continuum of measures of angles, and the correlation f assigns to each point on the curve the slope of the tangent there: thus f is nothing other than the first derivative of the function associated with the curve.]

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[7] *Ibid.,* p. 21f.

For Brentano the essential feature of a continuum is its inherent capacity to engender boundaries, and the fact that such boundaries can be grasped as coincident. Boundaries themselves possess a quality which Brentano calls *plerosis* ("fullness"). Plerosis is the measure of the number of directions in which the given boundary actually bounds. Thus, for example, within a temporal continuum the endpoint of a past episode or the starting point of a future one bounds in a single direction, while the point marking the end of one episode and the beginning of another may be said to bound doubly. In the case of a spatial continuum there are numerous additional possibilities: here a boundary may bound in all the directions of which it is capable of bounding, or it may bound in only some of these directions. In the former case, the boundary is said to exist in *full plerosis*; in the latter, in *partial plerosis.* Brentano writes:

> *…the spatial nature of a point differs according to whether it serves as a limit in all or only in some directions. Thus a point located inside a physical thing serves as a limit in all directions, but a point on a surface or an edge or a vertex serves as a limit in only some direction. And the point in a vertex will differ in accordance with the directions of the edges that meet at the vertex… I call these specific distinctions differences of plerosis. Like any manifold variation, plerosis admits of a more and a less. The plerosis of the centre of a cone is more complete than that of a point on its surface; the plerosis of a point on its surface is more complete than that of a point on its edge, or that of its vertex. Even the plerosis of the vertex is the more complete the less the cone is pointed.*[8]

Brentano believed that the concept of plerosis enabled sense to be made of the idea that a boundary possesses "parts", even when the boundary lacks dimensions altogether, as in the case of a point. Thus, even though the present or "now" is, according to Brentano, temporally unextended and exists only as a boundary between past and future, it still possesses two "parts" or aspects: it is both the end of the past and the beginning of the future. It is worth mentioning that for Brentano it was not just the "now" that existed only as a boundary; since he held that "existence" in the strict sense means "existence *now*", it necessarily followed that existing things exist only as boundaries of what has existed or of what will exist, or both.

Brentano ascribes particular importance to the fact that points in a continuum can *coincide.* On this matter he writes:

> *Various other thorough studies could be* made [on the continuum concept] *such as a study of the impossibility of adjacent points and the* possibility of coincident points, *which have, despite their coincidence, distinctness and full relative independence.* [This] *has been and is misunderstood in many ways. It is commonly believed that if four different-coloured quadrants of a circular area touch each other at its centre, the centre belongs to only one of the coloured surfaces and must be that colour only. Galileo's judgment on the matter was more correct; he expressed his interpretation by saying paradoxically that the centre of*

[8] Quoted in *ibid.,* p. xvii.

> *the circle has as many parts as its periphery. Here we will only give some indication of these studies by commenting that everything which arises in this connection* follows from the point's relativity as involves a continuum and the fact that it is essential for it to belong to a continuum. *Just as the possibility of the coincidence of different points is connected with that fact, so is the existence of a point in diverse or more or less perfect* plerosis. *All of this is overlooked even today by those who understand the continuum to be an actual infinite multiplicity and who believe that we get the concept not by abstraction from spatial and temporal intuitions but from the combination of fractions between numbers, such as between 0 and 1.*[9]

Brentano's doctrines of plerosis and coincidence of points are well illustrated by applying them to the traditional philosophical problem of the initiation of motion: if a thing begins to move, is there a last moment of its being at rest or a first moment of its being in motion? The usual objection to the claim that both moments exist is that, if they did, there would be a time between the two moments, and at that time the thing could be said neither to be at rest nor to be in motion—in violation of the law of excluded middle. Brentano's response would be to say that both moments do exist, but that they *coincide*, so that there are no times between them; the violation of the law of excluded middle is thereby avoided. More exactly, Brentano would assert that the temporal boundary of the thing's being at rest—the end of its being at rest—is the same as the temporal boundary of the thing's being in motion—the beginning of its being in motion—, but the boundary is twofold in respect of its plerosis. The boundary is, in fact, in half plerosis at rest and in half plerosis in motion.

As we have seen, Brentano's analysis of the continuum centred on its phenomenological and qualitative aspects, which are by their very nature incapable of reduction to the discrete. Brentano's rejection of the mathematicians' attempts to "arithmetize" the continuum—to represent it in discrete terms—is thus hardly surprising. He might, on the other hand, have been more sympathetic to the accounts of the continuum put forward by the other members of our sextet—all of whom, as I have observed, took the continuous as something necessarily transcending the discrete.

This brings me to Peirce. Peirce's view of the continuum was, in a sense, intermediate between that of Brentano and the arithmetizers. Like Brentano, he held that the cohesiveness of a continuum rules out the possibility of it being a mere collection of discrete individuals, or points, in the usual sense. But he also held that any continuum harbours an *unboundedly large* collection of points—in his colourful terminology, a *supermultitudinous* collection—what we would today call a proper class. Peirce held that if "enough" points were to be crowded together by

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[9] Brentano (1974), p. 357.

carrying insertion of new points between old to its ultimate limit they would—through a *logical* "transformation of quantity into quality"—lose their individual identity and become fused into a true continuum. Here are his observations on the matter:

> *It is substantially proved by Euclid that there is but one assignable quantity which is the limit of a convergent series. That is, if there is an increasing convergent series,* A *say, and a decreasing convergent series,* B *say, of which every approximation exceeds every approximation of* A*, and if there is no rational quantity which is at once greater than every approximation of* A *and less than every approximation of* B*, then there is but one surd quantity so intermediate…There is one surd quantity and only one for each convergent series, calling two series the same if their approximations all agree after a sufficient number of terms, or if their difference approximates toward zero. But this is only to say that the multitude of surds equals the multitude of denumerable sets of rational numbers which is… the* primipostnumeral[10] *multitude.*
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> *…We remark that there is plenty of room to insert a* secundipostnumeral *multitude of quantities between [a] convergent series and its limit. Any one of those quantities may likewise be separated from its neighbours, and we thus see that between it and its nearest neighbours there is ample room for a* tertiopostnumeral *multitude of other quantities, and so on through the whole denumerable series of postnumeral quantities.*
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> *But if we suppose that* all *such orders of systems of quantities have been inserted, there is no longer any room for inserting any more. For to do so we must select some quantity to be thus isolated in our representation. Now whatever one we take, there will always be quantities of higher order filling up the spaces on the two sides.*
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> *We therefore see that such a supermultitudinous collection sticks together by logical necessity. Its constituent individuals are no longer distinct and independent subjects. They have no existence—no hypothetical existence—except in their relations to one another. They are not subjects, but phrases expressive of the properties of the continuum.*
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> *…Supposing a line to be a supermultitudinous collection of points, … to sever a line in the middle is to disrupt the logical identity of the point there, and make it two points. It is impossible to sever a continuum by separating the connections of the points, for the points only exist by virtue of those connections. The only way to sever a continuum is to burst it, that is, to convert what was one into two.* [11]

There is some resemblance between Peirce's conception of a continuous line and John Conway's system of *surreal numbers*. Conway's system may be characterized as being an η_α-field for every ordinal α, that is, a real-closed ordered field *S* which satisfies the condition that, for any pair of subsets *X, Y* for which every member of *X* is less than every

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[10] Peirce assumed what amounts to the generalized continuum hypothesis in supposing that each possible infinite set has one of the cardinalities $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$. These he termed *denumerable, primipostnumeral, secundipostnumeral,* etc.

[11] Peirce (1976), p. 95.

member of *Y*, there is an element of *S* strictly between *X* and *Y*.[12] It is not hard to show that, between any pair of members of *S* there is a *proper class*[13] of members of *S*—in Peirce's terminology, a supermultitudinous collection. Nevertheless, *S* is still discrete: its elements, while supermultitudinous, remain distinct and unfused (were it not for this fact, Conway would scarcely be justified in calling the members of *S* "numbers"). On the face of it the discreteness of *S* would seem to imply that the presence of superabundant quantity in Peirce's sense is not enough to ensure continuity. (Of course, Brentano would have dismissed this idea[14] altogether.)

Peirce's conception of the number continuum is also notable for the presence in it of an abundance of *infinitesimals*, a feature it shares with du Bois-Reymond's and Veronese's "non-Archimedean" number systems (I do not know whether Peirce was aware of their work). Peirce championed the retention of the infinitesimal concept in the foundations of the calculus, both because of what he saw as the efficiency of infinitesimal methods, and because he regarded infinitesimals as furnishing the "glue" that caused points on a continuous line to lose their individual identity: indeed, he writes

> *The very word* continuity *implies that the instants of time or the points of a line are everywhere welded together.*

In defending infinitesimals, he remarks that

> *It is singular that nobody objects to* $\sqrt{-1}$ *as involving any contradiction, nor, since Cantor, are infinitely great quantities much objected to, but still the antique prejudice against infinitely small quantities remains.*[15]

Peirce actually held the view that the conception of infinitesimal is suggested by introspection—that the *specious present* is in fact an infinitesimal:

> *It is difficult to explain the fact of memory and our apparently perceiving the flow of time, unless we suppose immediate consciousness to extend beyond a single instant. Yet if we make such a supposition we fall into grave difficulties, unless*

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[12] In their Introduction to Peirce [1992], Ketner and Putnam characterize Peirce's conception of the continuum as "a possibility of repeated division which can never be exhausted in any possible world, not even in a possible world in which one can complete [nondenumerably] infinite processes. This description would seem to apply equally well to Conway's conception.

[13] To be precise, one can define, by transfinite recursion, an injection of the proper class of ordinal numbers into any given open interval of *S*..

[14] On several occasions Brentano criticized Poincaré's construction of the continuum by repeated insertion of points.

[15] Peirce (1976), p. 123

> *we suppose the time of which we are immediately conscious to be strictly infinitesimal.*[16]
>
> *We are conscious of the present time, which is an instant, if there be any such thing as an instant. But in the present we are conscious of the flow of time. There is no flow in an instant. Hence, the present is not an instant.*[17]

In concluding my remarks on Peirce I think it *apropos* on this occasion to quote from a letter addressed by Peirce in 1900 to the editor of *Science* in which he defends his views on infinitesimals against the strictures of Josiah Royce:

> *Professor Royce remarks that my opinion that differentials may quite logically be considered as true infinitesimals, if we like, is shared by no mathematician* "outside of Italy". *As a logician, I am more comforted by corroboration in the clear mental atmosphere of Italy than I could be by any seconding from a tobacco-clouded and bemused land (if any such there be) where no philosophical eccentricity misses its champion, but where sane logic has not found favor.*

I come now to *Weyl.*

Weyl's philosophical outlook was influenced by Brentano, but above all by Husserl (who was, of course, a student of Brentano). Weyl accepted the principal tenet of Husserlian phenomenology —that the only things which are directly given to us, that we can know completely, are objects of consciousness, and it is these with which philosophy, and all knowledge, must begin. This is acknowledged by Weyl in the introduction to *Space-Time-Matter* (1919), his famous book on the theory of relativity. Modestly describing his remarks as "a few reflections of a philosophical character," he observes that the objective world "constructed" by mathematical physics cannot of necessity coincide with the subjective world of qualities given through perception:

> *Expressed as a general principle, this means that the real world, and every one of its constituents, are, and can only be, given as intentional objects of acts of consciousness. The immediate data which I receive are the experiences of consciousness in just the form in which I receive them ... we may say that in a sensation an object, for example, is actually physically present for me—to whom that sensation relates—in a manner known to everyone, yet, since it is characteristic, it cannot be described more fully.*[18]

His phenomenological orientation is proclaimed still more emphatically when he goes on to say:

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[16] *Ibid.,* p. 124.
[17] *Ibid.,* p. 925.
[18] Weyl [1950], p. 4.

> *...the datum of consciousness is the starting point at which we must place ourselves if we are to understand the absolute meaning of, as well as the right to, the supposition of reality ... "Pure consciousness" is the seat of what is philosophically a priori.* [19]

Towards the end of his *Address on the Unity of Knowledge*, delivered at the 1954 Columbia University bicentennial celebrations, Weyl enumerates what he considers to be the essential constituents of knowledge. At the top of his list[20] comes

> *...intuition, mind's ordinary act of seeing what is given to it.*[21]

Throughout his life Weyl held to the view that intuition, or *insight*—rather than *proof*— furnishes the ultimate foundation of *mathematical* knowledge. Thus in his *Das Kontinuum* of 1918 he says:

> *In the Preface to Dedekind (1888) we read that "In science, whatever is provable must not be believed without proof." This remark is certainly characteristic of the way most mathematicians think. Nevertheless, it is a preposterous principle. As if such an indirect concatenation of grounds, call it a proof though we may, can awaken any "belief" apart from assuring ourselves through immediate insight that each individual step is correct. In all cases, this process of confirmation—and not the proof—remains the ultimate source from which knowledge derives its authority; it is the "experience of truth".*[22]

Although Weyl held that the roots of mathematics lay in the intuitively given—the "inner"—as opposed to the transcendent—the "outer"—, he recognized at the same time that it would be unreasonable to require all mathematical knowledge to possess intuitive immediacy. In *Das Kontinuum*, for example, he says:

> *The states of affairs with which mathematics deals are, apart from the very simplest ones, so complicated that it is practically impossible to bring them into full givenness in consciousness and in this way to grasp them completely.*[23]

But Weyl did not think that this fact furnished justification for extending the bounds of mathematics to embrace notions which cannot be given fully in intuition *even in principle* (e.g., the actual infinite). He held, rather, that this extension of mathematics into the transcendent is necessitated by the fact that mathematics plays an indispensable role in the physical sciences, in which intuitive evidence is *necessarily* transcended. As he says in *The Open World*:

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[19] *Ibid.*, p. 5.
[20] The others, in order, are: *understanding and expression; thinking the possible;* and finally, in science, *the construction of symbols or measuring devices.*
[21] Weyl [1954], 629.
[22] Weyl [1987], 119.
[23] Weyl [1987], 17.

> *... if mathematics is taken by itself, one should restrict oneself with Brouwer to the intuitively cognizable truths ... nothing compels us to go farther. But in the natural sciences we are in contact with a sphere which is impervious to intuitive evidence; here cognition necessarily becomes symbolical construction. Hence we need no longer demand that when mathematics is taken into the process of theoretical construction in physics it should be possible to set apart the mathematical element as a special domain in which all judgements are intuitively certain; from this higher standpoint which makes the whole of science appear as one unit, I consider Hilbert to be right.* [24]

In *Consistency in Mathematics* (1929), Weyl characterized the mathematical method as

> *the* a priori *construction of the possible in opposition to the* a posteriori *description of what is actually given.*[25]

The problem of identifying the limits on constructing "the possible" in this sense occupied Weyl a great deal. He was particularly concerned with the concept of the mathematical *infinite*, which he believed to elude "construction" in the idealized sense of set theory. Again to quote a passage from *Das Kontinuum:*

> *No one can describe an infinite set other than by indicating properties characteristic of the elements of the set. ... The notion that a set is a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; "inexhaustibility" is essential to the infinite.*[26]

But the necessity of injecting mathematics into external reality compels it to embody a conception of the actual infinite, as Weyl attests towards the end of *The Open World:*

> *The infinite is accessible to the mind intuitively in the form of a field of possibilities open to infinity, analogous to the sequence of numbers which can be continued indefinitely, but the completed, the actual infinite as a closed realm of actual existence is forever beyond its reach. Yet the demand for totality and the metaphysical belief in reality inevitably compel the mind to represent the infinite as closed being by symbolical construction.*[27]

Weyl gave a great deal of thought to the concept of the continuum. During the period 1918–1921 he wrestled with the problem of providing it with an exact mathematical formulation free of the taint of the actual infinite. As he saw it in 1918, there is an unbridgeable gap between intuitively given continua (e.g. those of space, time and motion) on the one hand, and the "discrete" exact concepts of mathematics (e.g. that of

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[24] Weyl [1932], 82.
[25] Weyl [1929], 249.
[26] Weyl [1987], 23.
[27] Weyl [1932], 83.

real number) on the other. For Weyl the presence of this split meant that the construction of the mathematical continuum could not simply be "read off" from intuition. Rather, he believed at this time that the mathematical continuum must be treated as if it were an element of the transcendent realm, and so, in the end, justified in the same way as a physical theory. In Weyl's view, it was not enough that the mathematical theory be *consistent*; it must also be *reasonable.*

Das Kontinuum (1918) embodies Weyl's attempt at formulating a theory of the continuum which satisfies the first, and, as far as possible, the second, of these requirements. In the following passages from this work he acknowledges the difficulty of the task:

> *... the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd.*[28]

> *... the continuity given to us immediately by intuition (in the flow of time and of motion) has yet to be grasped mathematically as a totality of discrete "stages" in accordance with that part of its content which can be conceptualized in an exact way.*[29]

> *Exact time- or space-points are not the ultimate, underlying atomic elements of the duration or extension given to us in experience. On the contrary, only reason, which thoroughly penetrates what is experientially given, is able to grasp these exact ideas. And only in the arithmetico-analytic concept of the real number belonging to the purely formal sphere do these ideas crystallize into full definiteness.* [30]

> *When our experience has turned into a real process in a real world and our phenomenal time has spread itself out over this world and assumed a cosmic dimension, we are not satisfied with replacing the continuum by the exact concept of the real number, in spite of the essential and undeniable inexactness arising from what is given.*[31]

However much he may have wished to do so, in *Das Kontinuum* Weyl did not aim to provide a mathematical formulation of the continuum as it is presented to intuition, which, as the quotations above show, he regarded as an impossibility (at that time at least). Rather, his goal was first to achieve *consistency* by putting the *arithmetical* notion of real number on a firm logical basis, and then to show that the resulting theory is *reasonable* by employing it as the foundation for a plausible account of continuous process in the objective physical world.[32]

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[28] Weyl [1987], 108.
[29] *Ibid.,* 24.
[30] *Ibid.,* 94.
[31] *Ibid.,* 93.
[32] The connection between mathematics and physics was of course of paramount importance for Weyl. His seminal work on relativity theory, *Space-Time-Matter,* was

As a practicing mathematician, Weyl had come to believe that, the work of Cauchy, Weierstrass, Dedekind and Cantor notwithstanding, mathematical analysis at the beginning of the 20th century would not bear logical scrutiny, for its essential concepts and procedures involved vicious circles to such an extent that, as he says, "every cell (so to speak) of this mighty organism is permeated by contradiction." In *Das Kontinuum* he tries to overcome this by providing analysis with a *predicative* formulation—not, as Russell and Whitehead had attempted, by introducing a hierarchy of logically ramified types, which Weyl seems to have regarded as too complicated—but rather by confining the comprehension principle to formulas whose bound variables range over just the initial given entities (numbers). Thus he restricts analysis to what can be done in terms of natural numbers with the aid of three basic logical operations, together with the operation of substitution and the process of "iteration", i.e., primitive recursion. Weyl recognized that the effect of this restriction would be to render unprovable many of the central results of classical analysis—e.g., Dirichlet's principle that any bounded set of real numbers has a least upper bound[33]—but he was prepared to accept this as part of the price that must be paid for the security of mathematics.

In §6 of *Das Kontinuum* Weyl presents his conclusions as to the relationship between the intuitive and mathematical continua. He poses the question: Does the mathematical framework he has erected provide an adequate representation of physical or temporal continuity as it is *actually experienced*? He begins his investigation by noting that, according to his theory, if one asks whether a given function is continuous, the answer is not fixed once and for all, but is, rather, dependent on the extent of the domain of real numbers which have been defined up to the point at which the question is posed. Thus the continuity of a function must always remain *provisional*; the possibility always exists that a function deemed continuous *now* may, with the emergence of "new" real numbers, turn out to be discontinuous *in the future*. [34]

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published in the same year (1918) as *Das Kontinuum*; the two works show subtle affinities.

[33] In this connection it is of interest to note that on 9 February 1918 Weyl and George Pólya made a bet in Zürich in the presence of twelve witnesses (all of whom were mathematicians) that "within 20 years, Pólya, or a majority of leading mathematicians, will come to recognize the falsity of the least upper bound property." When the bet was eventually called, everyone—with the single exception of Gödel—agreed that Pólya had won.

[34] This fact would seem to indicate that in Weyl's theory the domain of definition of a function is not unambiguously determined by the function, so that the continuity of such a "function" may vary with its domain of definition. (This would be a natural consequence of Weyl's definition of a function as a certain kind of relation.) A simple but striking example of this phenomenon is provided in classical analysis by the function f which takes value 1 at each rational number, and 0 at each irrational

To reveal the discrepancy between this formal account of continuity based on real numbers and the properties of an intuitively given continuum, Weyl next considers the experience of seeing a pencil lying on a table before him throughout a certain time interval. The position of the pencil during this interval may be taken as a function of the time, and Weyl takes it as a fact of observation that during the time interval in question this function is continuous and that its values fall within a definite range. And so, he says,

> *This observation entitles me to assert that during a certain period this pencil was on the table; and even if my right to do so is not absolute, it is nevertheless reasonable and well-grounded. It is obviously absurd to suppose that this right can be undermined by "an expansion of our principles of definition"—as if new moments of time, overlooked by my intuition could be added to this interval, moments in which the pencil was, perhaps, in the vicinity of Sirius or who knows where. If the temporal continuum can be represented by a variable which "ranges over" the real numbers, then it appears to be determined thereby how narrowly or widely we must understand the concept "real number" and the decision about this must not be entrusted to* logical *deliberations over principles of definition and the like.*[35]

To drive the point home, Weyl focuses attention on the fundamental continuum of *immediately given phenomenal time*, that is, as he characterizes it,

> *... to that constant form of my experiences of consciousness by virtue of which they appear to me to flow by successively.*(*By "experiences" I mean what I experience, exactly as I experience it. I do not mean real psychical or even physical processes which occur in a definite psychic-somatic individual, belong to a real world, and, perhaps, correspond to the direct experiences.*)[36]

In order to correlate mathematical concepts with phenomenal time in this sense Weyl grants the possibility of introducing a rigidly punctate "now" and of identifying and exhibiting the resulting temporal points. On the collection of these temporal points is defined the relation of *earlier than* as well as a congruence relation of *equality of temporal intervals,* the basic constituents of a simple mathematical theory of time. Now Weyl observes that the discrepancy between phenomenal time and the concept of real number would vanish if the following pair of conditions could be shown to be satisfied:

> 1. *The immediate expression of the intuitive finding that during a certain period I saw the pencil lying there were construed in such a way that the phrase "during a*

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number. Considered as a function defined on the rational numbers, *f* is constant and so continuous; as a function defined on the real numbers, *f* fails to be continuous anywhere.

[35] Weyl [1987], 88.

[36] *Ibid.,* 88

> *certain period" was replaced by "in every temporal point which falls within a certain time span OE.* [Weyl goes on to say parenthetically here that he admits "that this no longer reproduces what is intuitively present, but one will have to let it pass, *if it is really legitimate to dissolve a period into temporal points.*")
>
> 2. *If P is a temporal point, then the domain of rational numbers to which ℓ belongs if and only if there is a time point L earlier than P such that OL = ℓ.OE can be constructed arithmetically in pure number theory on the basis of our principles of definition, and is therefore a real number in our sense.*[37]

Condition 2 means that, if we take the time span *OE* as a unit, then each temporal point *P* is correlated with a definite real number. In an addendum Weyl also stipulates the converse.

But can temporal intuition itself provide evidence for the truth or falsity of these two conditions? Weyl thinks not. In fact, he states quite categorically that

> *... everything we are demanding here is obvious* nonsense: *to* these *questions, the intuition of time provides no answer—just as a man makes no reply to questions which clearly are addressed to him by mistake and, therefore, are unintelligible when addressed to him.*[38]

The grounds for this assertion are by no means immediately evident, but one gathers from the passages following it that Weyl regards the experienced *continuous flow* of phenomenal time as constituting an insuperable barrier to the whole enterprise of representing this continuum in terms of individual points, and even to the characterization of "individual temporal point" itself. As he says,

> *The view of a flow consisting of points and, therefore, also dissolving into points turns out to be mistaken: precisely what eludes us is the nature of the continuity, the flowing from point to point; in other words, the secret of how the continually enduring present can continually slip away into the receding past.*
>
> *Each one of us, at every moment, directly experiences the true character of this temporal continuity. But, because of the genuine primitiveness of phenomenal time, we cannot put our experiences into words. So we shall content ourselves with the following description. What I am conscious of is for me both a being-now and, in its essence, something which, with its temporal position, slips away. In this way there arises the persisting factual extent, something ever new which endures and changes in consciousness.* [39]

Weyl sums up what he thinks can be affirmed about "objectively presented time"—by which I take it he means "phenomenal time described in an objective manner"—in the following two assertions, which he claims apply equally, *mutatis mutandis,* to every intuitively given

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[37] *Ibid.,* 89.
[38] *Ibid.,* 90.
[39] *Ibid.,* 91-92.

continuum, in particular, to the continuum of spatial extension:

> 1. *An individual point in it is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a "point of transition" (which, of course, can in no way be understood mathematically);*
>
> 2. *it is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible.*[40]

The fact that single points in a true continuum "cannot be exhibited" arises, Weyl continues, from the fact that they are not genuine individuals and so cannot be characterized by their properties. In the physical world they are never defined absolutely, but only in terms of a *coordinate system,* which, in an arresting metaphor, Weyl describes as "the unavoidable residue of the eradication of the ego." This metaphor, which Weyl was to employ more than once (e.g. in Weyl [1950], 8 and [1963], 123) reflects the continuing influence of phenomenological doctrine: in this case, the thesis that the existent is given in the first instance as the contents of a consciousness.

By 1919 Weyl had come to embrace Brouwer's views on the intuitive continuum. The latter's influence looms large in Weyl's next paper on the subject, *On the New Foundational Crisis of Mathematics,* written in 1920. Here Weyl identifies two distinct views of the continuum: "atomistic" or "discrete"; and "continuous". In the first of these the continuum is composed of individual real numbers which are well-defined and can be sharply distinguished. Weyl describes his earlier attempt at reconstructing analysis in *Das Kontinuum* as atomistic in this sense:

> *Existential questions concerning real numbers only become meaningful if we analyze the concept of real number in this extensionally determining and delimiting manner. Through this conceptual restriction, an ensemble of individual points is, so to speak, picked out from the fluid paste of the continuum. The continuum is broken up into isolated elements, and the flowing-into-each other of its parts is replaced by certain conceptual relations between these elements, based on the "larger-smaller" relationship. This is why I speak of the* atomistic *conception of the continuum.*[41]

By this time Weyl had indeed come to repudiate atomistic theories of the continuum, including that of *Das Kontinuum.* He writes:

> *In traditional analysis, the continuum appeared as the set of its points; it was considered merely as a special case of the basic logical relationship of* element and set. *Who would not have already noticed that, up to now, there was no place in mathematics for the equally fundamental relationship of* part and whole? *The*

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[40] *Ibid.,* 92.
[41] Weyl [1998], 91.

> *fact, however,* that it has parts, *is a fundamental property of the continuum; and so (in harmony with intuition, so drastically offended against by today's "atomism") this relationship is taken as the mathematical basis for the continuum by Brouwer's theory. This is the real reason why the method used in delimiting subcontinua and in forming continuous functions starts out from* intervals *and not* points *as the primary elements of construction. Admittedly a set also has parts. Yet what distinguishes the parts of sets in the realm of the "divisible" is the existence of "elements" in the set-theoretical sense, that is, the existence of* parts that themselves do not contain any further parts. *And indeed, every part contains at least one "element". In contrast, it is inherent in the nature of the continuum that* every part of it can be further divided without limitation. *The concept of a point must be seen as an idea of a limit, "point" is the idea of a* limit *of a division extending* in infinitum. *To represent the continuous connection of the points, traditional analysis, given its shattering of the continuum into isolated points, had to have recourse to the concept of a* neighbourhood. *Yet, because the concept of continuous function remained mathematically sterile in the resulting generality, it became necessary to introduce the possibility of "triangulation" as a restrictive condition.*

Like Brentano, Weyl knew that to "shatter a continuum into isolated points" would be to eradicate the very feature which characterizes a continuum—the fact that its cohesiveness is inherited by every one of its parts.

Weyl accordingly welcomed Brouwer's construction of the continuum by means of sequences generated by free acts of choice, thus identifying it as a "medium of free Becoming" which "does not dissolve into a set of real numbers as finished entities". Weyl felt that Brouwer, through his doctrine of Intuitionism[42], had come closer than anyone else to bridging that "unbridgeable chasm" between the intuitive and mathematical continua. In particular, he found compelling the fact that the Brouwerian continuum is not the union of two disjoint nonempty parts—that it is, in a word, *indecomposable. "*A genuine continuum," declares Weyl, "cannot be divided into separate fragments." In later publications he expresses this more colourfully by quoting Anaxagoras to the effect that a continuum "defies the chopping off of its parts with a hatchet."

Weyl also agrees with Brouwer that all functions everywhere defined on a continuum are continuous, but here certain subtle differences of viewpoint emerge. Weyl contends that what mathematicians had taken to be discontinuous functions actually consist of several continuous functions defined on separated continua. (For example, the "discontinuous" function defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$ in fact consists of the two functions with constant values 0 and 1 respectively defined on the separated continua $\{x: x < 0\}$ and $\{x: x \geq 0\}$. The union of these two continua fails to be the whole of

[42] For my remarks on Weyl's relationship with Intuitionism I have drawn on the illuminating paper van Dalen [1995].

the real continuum because of the failure of the law of excluded middle: it is not the case that, for be any real number x, either $x < 0$ or $x \geq 0$.) Brouwer, on the other hand, had not dismissed the possibility that discontinuous functions could be defined on proper parts of a continuum, and still seems to have been searching for an appropriate way of formulating this idea.[43] In particular, at that time Brouwer would probably have been inclined to regard the above function f as a genuinely discontinuous function defined on a *proper part* of the real continuum. For Weyl, it seems to have been a self-evident fact that all functions defined on a continuum are continuous, but this is because Weyl confines attention to functions which turn out to be continuous by definition. Brouwer's concept of function is less restrictive than Weyl's and it is by no means immediately evident that such functions must always be continuous.

Weyl defined real functions as mappings correlating each interval in the choice sequence determining the argument with an interval in the choice sequence determining the value "interval by interval" as it were, the idea being that approximations to the input of the function should lead effectively to corresponding approximations to the input. Such functions are continuous by definition. Brouwer, on the other hand, considers real functions as correlating choice sequences with choice sequences, and the continuity of these is by no means obvious. The fact that Weyl refused to grant (free) choice sequences—whose identity is in no way predetermined—sufficient individuality to admit them as arguments of functions perhaps betokens a commitment to the conception of the continuum as a "medium of free Becoming" even deeper than that of Brouwer.

There thus being only minor differences between Weyl's and Brouwer's accounts of the continuum, Weyl accordingly abandoned his earlier attempt at the reconstruction of analysis and "joined Brouwer." At the same time, however, Weyl recognized that the resulting gain in intuitive clarity had been bought at a considerable price, as witnessed by his remark in the 1927 edition of *Philosophy of Mathematics and Natural Science:*

> *Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the greater part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes.*[44]

[43] Brouwer established the continuity of functions fully defined on a continuum in 1904, but did not publish a definitive account until 1927. In that account he also considers the possibility of partially defined functions.

[44] Weyl [1963], 54.

Although he later practiced intuitionistic mathematics very rarely, Weyl remained an admirer of intuitionism. And the "riddle of the continuum" retained its fascination for him: in one of his last papers, *Axiomatic and Constructive Procedures in Mathematics,* written in 1954, we find the observation that

> *... the constructive transition to the continuum of real numbers is a serious affair... and I am bold enough to say that not even to this day are the logical issues involved in that constructive concept completely clarified and settled.*[45]

This brings me finally to that most dissident of voices, Brouwer. In Brouwer's philosophy the *temporal continuum* played a predominant role. Indeed it is the awakening of awareness of the temporal continuum in the subject, an event termed by Brouwer "The Primordial Happening" or "The Primordial Intuition of Time", that engenders the fundamental concepts and methods of mathematics. In "Mathematics, Science and Language" (1929), he describes how the notion of number—the discrete—emerges from the awareness of the continuous:

> Mathematical Attention *as an act of the will serves the instinct for self-preservation of individual man; it comes into being in two phases;* time awareness *and* causal attention. *The first phase is nothing but the fundamental intellectual phenomenon of the falling apart of a moment of life into two qualitatively different things of which one is experienced as giving away to the other and yet is retained by an act of memory. At the same time this split moment of life is separated from the Ego and moved into a world of its own, the world of perception. Temporal twoity, born from this time awareness, or the two-membered sequence of time phenomena, can itself again be taken as one of the elements of a new twoity, so creating temporal threeity, and so on. In this way, by means of the self-unfolding of the fundamental phenomenon of the intellect, a* time sequence of phenomena is created of arbitrary multiplicity. [46]

But in his doctoral dissertation of 1907 he regards continuity and discreteness as complementary notions, neither of which is reducible to the other:

> *...We shall go further into the basic intuition of mathematics (and of every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a "between", which is never exhausted by the insertion of new entities. Since continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear, it is impossible to avoid [regarding each one of them as a primitive entity... Having recognized that the intuition of continuity, of "fluidity" is as primitive as that of several things conceived as forming together a unit, the latter being at the basis of every*

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[45] Weyl [1985], 17.

[46] Manocosu [1998], p. 45.

> *mathematical construction, we are able to state properties of the continuum as "a matrix of points to be thought of as a whole".*[47]

In that work Brouwer states quite categorically that the continuum is not constructible from discrete points:

> *…The* continuum as a whole [*is*] *given to us by intuition; a construction for it, an action which would create from the mathematical intuition 'all' its points as individuals, is inconceivable and impossible.*[48]

Later Brouwer was to modify this doctrine. In his mature thought, he radically transformed the concept of "point", endowing points with sufficient fluidity to enable them to serve as generators of a "true" continuum. This fluidity was achieved by admitting as "points", not only fully defined discrete numbers such as $\sqrt{2}$, π, *e*, and the like—which have, so to speak, already achieved "being"—but also "numbers" which are in a perpetual state of "becoming" in that their the entries in their decimal (or dyadic) expansions are the result of free acts of choice by a subject operating throughout an indefinitely extended time. The resulting *choice sequences* cannot be conceived as finished, completed objects: at any moment only an initial segment is known. In this way Brouwer obtained the mathematical continuum in a way compatible with his belief in the primordial intuition of time—that is, as an unfinished, indeed unfinishable entity in a perpetual state of growth. Weyl had every reason to be impressed with Brouwer's achievement!

In Brouwer's later conception, the mathematical continuum is indeed "constructed", not, however, by initially shattering, as did Cantor and Dedekind, an intuitive continuum into isolated points, but rather by assembling it from a complex of continually changing overlapping parts. I do not doubt that Brentano would have found Brouwer's account of the continuum considerably more congenial than that of the arithmetizers.

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