

ON THE STRENGTH OF THE SIKORSKI  
EXTENSION THEOREM FOR BOOLEAN ALGEBRAS

J.L. BELL

**§1. Introduction.** The *Sikorski Extension Theorem* [6] states that, for any Boolean algebra  $A$  and any complete Boolean algebra  $B$ , any homomorphism of a subalgebra of  $A$  into  $B$  can be extended to the whole of  $A$ . That is,

**Inj:** *Any complete Boolean algebra is injective (in the category of Boolean algebras).*

The proof of **Inj** uses the axiom of choice (**AC**); thus the implication **AC**  $\rightarrow$  **Inj** can be proved in Zermelo-Fraenkel set theory (**ZF**). On the other hand, the *Boolean prime ideal theorem*

**BPI:** *Every Boolean algebra contains a prime ideal (or, equivalently, an ultrafilter)*

may be equivalently stated as:

*The two element Boolean algebra 2 is injective,*

and so the implication **Inj**  $\rightarrow$  **BPI** can be proved in **ZF**.

In [3], Luxemburg surmises that this last implication cannot be reversed in **ZF**. It is the main purpose of this paper to show that this surmise is correct. We shall do this by showing that **Inj** implies that **BPI** holds in every Boolean extension of the universe of sets, and then invoking a recent result of Monro [5] to the effect that **BPI** does *not* yield this conclusion.

**§2. Preliminaries.** We work in **ZF**; thus the axiom of choice is *not* assumed. We shall suppose some familiarity with Boolean-valued models of set theory as presented, e.g. in [1]. We employ the standard notations. If  $B$  is a complete Boolean algebra,  $V^{(B)}$  is the Boolean-valued universe constructed from  $B$ . There is a canonical embedding  $x \mapsto \hat{x}$  from the real universe  $V$  of sets into  $V^{(B)}$ . If  $\sigma$  is a sentence of the language of set theory (possibly containing names for elements of  $V^{(B)}$ ), we write  $\llbracket \sigma \rrbracket^B$  (or just  $\llbracket \sigma \rrbracket$ ) for the Boolean value of  $\sigma$  calculated in  $V^{(B)}$ , and  $V^{(B)} \models \sigma$  for  $\llbracket \sigma \rrbracket^B = 1_B$ , the top element of  $B$ . The object  $U_B \in V^{(B)}$  defined by  $U_B = \{ \langle \hat{x}, x \rangle : x \in B \}$  is called the *canonical (generic) ultrafilter* in  $\hat{B}$ ; as is well known, we have

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$V^{(B)} \models U_B$  is an ultrafilter in the Boolean algebra  $\hat{B}$ .

Note also that, if  $A$  is any Boolean algebra, then  $V^{(B)} \models \hat{A}$  is a Boolean algebra.

Since we are not assuming the axiom of choice, we must now consider a number of delicate points about  $V^{(B)}$  which can be proved to hold without its use. Chief among these is the following special case of the Maximum Principle (1.27 of [1]).

2.1. LEMMA (proved in ZF). *If  $V^{(B)} \models \exists! x\phi(x)$ , then there is  $u \in V^{(B)}$  such that  $V^{(B)} \models \phi(u)$ . (Here  $\phi(x)$  has just the free variable  $x$ , but may contain names for elements of  $V^{(B)}$ .)*

PROOF. First note that since  $V^{(B)} \models \exists! x\phi(x)$ , we have

$$(*) \quad [\phi(u)] \wedge [\phi(v)] \leq [u = v]$$

for any  $u, v \in V^{(B)}$ . Now we have

$$1 = [\exists x\phi(x)] = \bigvee_{x \in V^{(B)}} [\phi(x)]$$

and using the axioms of replacement and regularity we can find a set  $\{u_i: i \in I\} \subseteq V^{(B)}$  such that

$$1 = \bigvee_{x \in V^{(B)}} [\phi(x)] = \bigvee_{i \in I} [\phi(u_i)].$$

If we now define  $u \in V^{(B)}$  by  $\text{dom}(u) = \bigcup_{i \in I} \text{dom}(u_i)$ , and for  $z \in \text{dom}(u)$ ,

$$u(z) = \bigvee_{i \in I} [\phi(u_i) \wedge z \in u_i],$$

then, using (\*), we easily show, as in the proof of 1.25 of [1], that  $[\phi(u_i)] \leq [u = u_i]$  for each  $i \in I$ . It follows that

$$[\phi(u)] \geq \bigvee_{i \in I} [\phi(u_i)] \wedge [u = u_i] = \bigvee_{i \in I} [\phi(u_i)] = 1. \quad \blacksquare$$

The next point is that without loss of generality we may assume that  $V^{(B)}$  is *separated*, i.e. for any  $x, y \in V^{(B)}$  we have  $V^{(B)} \models x = y$  iff  $x = y$ . To justify this, we define the equivalence relation  $\sim$  on  $V^{(B)}$  by  $x \sim y$  iff  $V^{(B)} \models x = y$ , and then employ "Scott's trick" of replacing each  $x \in V^{(B)}$  by the set of objects  $y$  in  $V^{(B)}$  of lowest rank such that  $x \sim y$ . This procedure turns  $V^{(B)}$  into a separated structure.

Finally, we shall need the following ideas derived from [7]. If  $V^{(B)} \models \langle A, \leq_A \rangle$  is a Boolean algebra, define

$$A \otimes B = \{x \in V^{(B)}: [x \in A]^B = 1_B\}.$$

(Since  $V^{(B)}$  is now separated,  $A \otimes B$  is easily shown to be a set.) Define  $\leq$  (sometimes written  $\leq_{A \otimes B}$ ) on  $A \otimes B$  by

$$x \leq y \leftrightarrow [x \leq_A y]^B = 1_B$$

for  $x, y \in A \otimes B$ . Using Lemma 2.1, it is readily shown (in ZF) that  $\langle A \otimes B, \leq \rangle$  is a Boolean algebra in which, for  $x, y \in A \otimes B$ ,

$$x \wedge y \text{ is the unique } z \in A \otimes B \text{ such that} \\ [z = \inf\{x, y\} \text{ in } A]^B = 1_B,$$

$x \vee y$  is the unique  $z \in A \otimes B$  such that

$$[[z = \sup \{x, y\} \text{ in } A]]^B = 1_B,$$

$x^*$  is the unique  $z \in A \otimes B$  such that

$$[[z = x^* \text{ in } A]]^B = 1_B.$$

(Here  $x^*$  denotes the Boolean complement of  $x$ .) Moreover,  $B$  is embeddable in  $A \otimes B$  via the map  $e$  defined by setting, for each  $b \in B$ ,  $e(b) =$  unique  $x \in A \otimes B$  for which

$$(2.2) \quad [[x = 1_A]]^B = b, \quad [[x = 0_A]]^B = b^*.$$

(This definition uses the mixing lemma in  $V^{(B)}$  (1.25 of [1]), whose proof does not require AC.) We shall use the embedding  $e$  to identify  $B$  with its image in  $A \otimes B$ , so that  $B$  becomes a subalgebra of  $A \otimes B$ . From (2.2) it follows that, for  $b \in B$ ,

$$(2.3) \quad [[b = 1_A]]^B = b, \quad [[b = 0_A]]^B = b^*$$

and

$$(2.4) \quad [[b = 0_A \vee b = 1_A]]^B = 1_B.$$

**§3. The main result.** Given Boolean algebras  $A$  and  $B$ , we write  $B \preceq A$  for  $B$  is a subalgebra of  $A$ .  $B$  is called an *absolute retract* if for any Boolean algebra  $A$ , whenever  $B \preceq A$  there is an (epi) morphism  $h: A \rightarrow B$  which is the identity on  $B$ . We can now prove the following result.

**3.1. THEOREM.** *Let  $B$  be a complete Boolean algebra. Then the following conditions are provably equivalent in ZF.*

- (i)  $B$  is injective;
- (ii)  $B$  is an absolute retract;
- (iii) for any Boolean algebra  $A$  such that  $B \preceq A$ , there is  $U \in V^{(B)}$  such that  $V^{(B)} \models U$  is an ultrafilter in  $\hat{A}$  and  $U_B \subseteq U$ ;
- (iv) for any  $C \in V^{(B)}$  such that  $V^{(B)} \models C$  is a Boolean algebra, there is  $U \in V^{(B)}$  such that  $V^{(B)} \models U$  is an ultrafilter in  $C$ .

**PROOF.** (i)  $\rightarrow$  (ii) is obvious.

(ii)  $\rightarrow$  (iii). Assume (ii), let  $B \preceq A$ , and let  $h: A \rightarrow B$  be a homomorphism which is the identity on  $B$ . If we put  $U = \{\langle \hat{a}, h(a) \rangle : a \in A\}$  then it is easily verified that  $V^{(B)} \models U$  is an ultrafilter in  $\hat{A}$ . Moreover, we have, for  $b \in B$ ,  $[[\hat{b} \in U_B]] = b = h(b) = [[\hat{b} \in U]]$  whence  $V^{(B)} \models U_B \subseteq U$ . Hence (iii).

(iii)  $\rightarrow$  (iv). Assume (iii) and let  $C \in V^{(B)}$  satisfy  $V^{(B)} \models C$  is a Boolean algebra. Then  $C \otimes B$  is a Boolean algebra and  $B \preceq C \otimes B$ . It follows that  $V^{(B)} \models \hat{B}$  and  $(C \otimes B)^\wedge$  are Boolean algebras and  $\hat{B} \preceq (C \otimes B)^\wedge$ . Now, working in  $V^{(B)}$ , let  $F$  be the filter in  $(C \otimes B)^\wedge$  generated by the canonical ultrafilter  $U_B$ ; that is, in  $V^{(B)}$ ,  $F = \{x \in (C \otimes B)^\wedge : \exists y \in U_B \cdot y \leq_{(C \otimes B)^\wedge} x\}$ . We claim that in  $V^{(B)}$ ,  $C$  is isomorphic to the quotient algebra  $(C \otimes B)^\wedge / F$ . To see this, define  $h \in V^{(B)}$  by  $h = \{\langle \hat{x}, x \rangle^{(B)} : x \in C \otimes B\} \times \{1_B\}$ . It is easy to verify that, in  $V^{(B)}$ ,  $h$  is a homomorphism of  $(C \otimes B)^\wedge$  onto  $C$ . To show that  $C \cong (C \otimes B)^\wedge / F$  in  $V^{(B)}$ , it suffices to show that  $V^{(B)} \models F = h^{-1}(1_C)$ . To prove this, we observe that, for  $x \in C \otimes B$ ,

$$\begin{aligned}
[\hat{x} \in F] &= [\exists y \in U_B \cdot y \leq_{(C \otimes B)^{\wedge}} \hat{x}] \\
&= \bigvee_{b \in B} b \wedge [\hat{b} \leq_{(C \otimes B)^{\wedge}} \hat{x}] \\
&= \bigvee \{b \in B: b \leq_{C \otimes B} x\}.
\end{aligned}$$

Also, for  $b \in B$ , we have

$$\begin{aligned}
b \leq [x = 1_C] &\leftrightarrow [b = 1_C] \leq [x = 1_C] \quad (\text{by 2.3}) \\
&\leftrightarrow V^{(B)} \models b = 1_C \rightarrow x = 1_C \\
&\leftrightarrow V^{(B)} \models b \leq_C x \quad (\text{by 2.4}) \\
&\leftrightarrow b \leq_{C \otimes B} x.
\end{aligned}$$

Hence  $[h(\hat{x}) = 1_C] = [x = 1_C] = \bigvee \{b \in B: b \leq_{C \otimes B} x\} = [\hat{x} \in F]$  by the above, proving the claim.

Now by (iii) there is  $U \in V^{(B)}$  such that  $V^{(B)} \models U$  is an ultrafilter in  $(C \otimes B)^{\wedge}$  containing  $U_B$ . Then  $V^{(B)} \models F \subseteq U$  and so  $V^{(B)} \models h[U]$  is an ultrafilter in  $(C \otimes B)^{\wedge}/F \cong C$ . This gives (iv).

(iv)  $\rightarrow$  (i). Assume (iv) and let  $h$  be a homomorphism of a subalgebra  $C$  of a Boolean algebra  $A$  into  $B$ . Put  $U = \{\langle \hat{x}, h(x) \rangle: x \in C\}$ ; then, as before,  $V^{(B)} \models U$  is an ultrafilter in  $\hat{C}$ . Working in  $V^{(B)}$ , let  $F$  be the filter in  $\hat{A}$  generated by  $U$ , and let  $h$  be the canonical epimorphism of  $\hat{A}$  onto  $\hat{A}/F$ . Using (iv), let  $U \in V^{(B)}$  be an ultrafilter in  $\hat{A}/F$ . Then, in  $V^{(B)}$ ,  $U' = h^{-1}[U]$  is an ultrafilter in  $\hat{A}$  extending  $F$ . (Note that, in claiming  $h^{-1}[U]$  as an explicit object of  $V^{(B)}$ , we are tacitly using Lemma 1.1.) If we now define  $g: A \rightarrow B$  by  $g(a) = [\hat{a} \in U']^B$ , then it is readily verified that  $g$  is a homomorphism of  $A$  into  $B$  extending  $h$ . ■

REMARKS. (1) The equivalence of (i) and (ii) was originally proved in [3] by a method entirely different from the one employed here, and (iv)  $\rightarrow$  (i) is essentially proved in [4]. It is (i)  $\rightarrow$  (iv) which appears to be new; as we shall see, it is crucial for our purposes.

(2) Notice that, since the full Maximum Principle is not available, condition (iv) of 3.1 is ostensibly *stronger* than the condition  $V^{(B)} \models \mathbf{BPI}$ . In this connection one may ask whether **Inj** is equivalent to the statement: "For all complete Boolean algebras  $B$ ,  $V^{(B)} \models \mathbf{BPI}$ ". I do not know the answer to this question.

We have as a consequence the main result of the paper.

3.2. COROLLARY. **Inj** is not provable from **BPI** in **ZF** (assuming the consistency of the latter).

PROOF. We first employ 3.1 to show that, if  $B$  is injective, then  $V^{(B)} \models \mathbf{BPI}$ . For suppose  $C \in V^{(B)}$ ; let  $b = [C \text{ is a Boolean algebra}]^B$  and, using the mixing lemma in  $V^{(B)}$ , let  $C' \in V^{(B)}$  be such that

$$[C = C']^B = b, \quad [C' = \hat{2}]^B = b^*.$$

Then  $V^{(B)} \models C'$  is a Boolean algebra and so, by (i)  $\rightarrow$  (iv) of 3.1, there is  $U \in V^{(B)}$  such that  $V^{(B)} \models U$  is an ultrafilter in  $C'$ . Clearly

$$\begin{aligned}
b &= [C' = C]^B \leq [U \text{ is an ultrafilter in } C']^B \wedge [C' = C]^B \\
&\leq [U \text{ is an ultrafilter in } C]^B \\
&\leq [\exists X. X \text{ is an ultrafilter in } C]^B.
\end{aligned}$$

It follows that  $V^{(B)} \models \mathbf{BPI}$ .

Now let  $M$  be the Halpern-Levy model of ZF in which  $\mathbf{BPI}$  holds but  $\mathbf{AC}$  fails (see, e.g., [2]). Monro [5] has constructed a complete Boolean algebra  $B$  in  $M$  such that, in  $M$ ,  $[\mathbf{BPI}]^B = 0_B$ . It follows from the above that, in  $M$ ,  $B$  is not injective, and so  $\mathbf{Inj}$  fails in  $M$ . The result follows. ■

**§4. Some final observations.** Let us call a sentence  $\sigma$  of the language of set theory *persistent* if we can prove in ZF that, if  $\sigma$  holds, it continues to hold in every Boolean extension of  $V$ . Of course,  $\mathbf{AC}$  is persistent. Monro [5] shows, on the other hand, that several consequences of  $\mathbf{AC}$ , in particular  $\mathbf{BPI}$  and the ordering principle, are not persistent. In contrast, we have

4.1. THEOREM (ZF).  $\mathbf{Inj}$  is persistent.

PROOF. Suppose  $\mathbf{Inj}$  holds, and let  $B$  be a complete Boolean algebra. We need to show that  $V^{(B)} \models \mathbf{Inj}$ , and by 3.1 it suffices to show that, in  $V^{(B)}$ , every complete Boolean algebra is an absolute subretract. And for this to be the case it suffices to show that, if  $A, C$  are any elements of  $V^{(B)}$  such that  $V^{(B)} \models A$  and  $C$  are Boolean algebras,  $C$  is complete, and  $C \leq A$ , then there is  $h \in V^{(B)}$  such that  $V^{(B)} \models h$  is a homomorphism of  $A$  onto  $C$  which is the identity on  $C$ .

Now  $B \leq C \otimes B \leq A \otimes B$ , and, by 5.2.1 of [7] (whose proof does not require  $\mathbf{AC}$ ),  $C \otimes B$  is complete. Since  $\mathbf{Inj}$  is assumed to hold, there is a homomorphism  $g: A \otimes B \rightarrow C \otimes B$  which is the identity on  $C \otimes B$ , and hence also on  $B$ . We have

$$V^{(B)} \models \hat{B} \leq (C \otimes B)^\wedge \leq (A \otimes B)^\wedge \text{ and } \hat{g} \text{ is a homomorphism} \\ \text{of } (A \otimes B)^\wedge \text{ onto } (C \otimes B)^\wedge.$$

Also, if  $F, F'$  are the filters generated by the canonical ultrafilter  $U_B$  in  $(C \otimes B)^\wedge$ ,  $(A \otimes B)^\wedge$  respectively, then, by the proof of (iii)  $\rightarrow$  (iv) of 3.1 we have

$$V^{(B)} \models C \cong (C \otimes B)^\wedge / F \text{ and } A \cong (A \otimes B)^\wedge / F.$$

It now follows easily from this and the fact that  $g$  is the identity on  $C \otimes B$  that in  $V^{(B)}$ ,  $\hat{g}$  induces a homomorphism of  $A$  onto  $C$  which is the identity on  $C$ . This completes the proof. ■

It is tempting to conjecture on the basis of this result and 3.2 that  $\mathbf{Inj}$  is actually equivalent to  $\mathbf{AC}$ . I have not, however, been able to settle this question.

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