Hilbert's $\varepsilon$-Operator in Intuitionistic Type Theories

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Abstract

We investigate Hilbert's $\varepsilon$-calculus in the context of intuitionistic type theories, that is, within certain systems of intuitionistic higher-order logic. We determine the additional deductive strength conferred on an intuitionistic type theory by the adjunction of closed $\varepsilon$-terms. We extend the usual topos semantics for type theories to the $\varepsilon$-operator and prove a completeness theorem. The paper also contains a discussion of the concept of "partially defined" $\varepsilon$-term.

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1. Introduction

We investigate Hilbert's $\varepsilon$-calculus ([1], [6]) in the context of intuitionistic type theories, that is, within certain systems of intuitionistic higher-order logic. The paper is organized as follows: Section 2, which is chiefly expository in nature, contains a compressed account (following [2]) of the basic framework of type theory we shall employ, and of its semantics. In Section 3 we introduce the $\varepsilon$-operator. Noting that, in contrast with the classical case, the $\varepsilon$-calculus is in general not conservative over intuitionistic systems (see, e.g., [3]), we determine the additional deductive strength conferred on an intuitionistic type theory by the adjunction of (closed) $\varepsilon$-terms. In Section 4 we introduce and develop the concept of "partially defined" $\varepsilon$-terms. In Section 5 we extend the topos semantics for type theories to the $\varepsilon$-operator and prove a completeness theorem. Section 6 contains some examples establishing the independence of various concepts introduced in the paper. Finally, in Section 7 we explain why we have confined attention to closed $\varepsilon$-terms.

2. A framework for intuitionistic type theories

We summarize briefly the system presented in [2]. A language $L$ for intuitionistic type theory, or a language for short, has the following ingredients:

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1) I am grateful to J.R. Lambek whose stimulating correspondence and paper [5] exerted a strong influence on the present work.
Basic symbols:

1 (unit type),

\( \Omega \) (trash value type),

\( \mathcal{T}, \mathcal{U}, \ldots \) (ground type).

\( f, g, h, \ldots \) (function symbols).

Types. These are members of the smallest class containing \( 1, \Omega \) and the ground types and closed under products and powers; here the product of two types \( A \) and \( B \) is denoted by \( A \times B \) and the power of a type \( A \) is denoted by \( PA \).

Signatures. Each function symbol \( f \) is assigned a pair of types called its signature. Notation: \( f : A \rightarrow B \).

Terms and their associated types are specified as follows:

(i) \( # \) is a term of type \( 1 \), and for each type \( A \) we are given a list of variables \( x, y, z, \ldots \) of that type.

(ii) terms are closed under the following operations (where \( \tau : A \) indicates that the term \( \tau \) has type \( A \)):

\( (e, \tau) : A \times B \) for \( e : A \) and \( \tau : B \).

\( f(\tau) : B \) for \( f : A \rightarrow B \).

\( [x : \alpha] : \text{PA} \) for \( x : A \) and \( \alpha : \Omega \).

\( \sigma = \tau : A \) for \( \sigma, \tau : A \).

\( \sigma \in \tau : A \) for \( x : A \) and \( \tau : \text{PA} \).

Formulas. These are the terms of type \( \Omega \).

Sentences are, as usual, formulas without free variables, i.e., variables \( x \) not appearing in a context of the form \( \{x : \alpha\} \).

We use the letters \( \alpha, \beta, \gamma \) to denote formulas and write \( \alpha(x/y) \) for the result of substituting \( x \) for \( y \) at each of the latter’s free occurrences in \( \alpha \).

Axioms and rules of inference. We adopt a sequent notation, writing \( \Gamma \vdash \alpha \) for the sequent composed of a finite set \( \Gamma \) of formulas and a formula \( \alpha \), and \( \vdash \) for \( \emptyset \). The axioms for \( \vdash \) are, writing \( \alpha \rightarrow \beta \) for \( \alpha, \beta \):

\( x = # \) (with \( x : 1 \)),

\( x = y, \alpha(x/y) \equiv \alpha(y/x) \) (with \( x, y \) free for \( x \) in \( \alpha \)),

\( \{x, y\} = \{x', y'\} \equiv \{x', y'\} = \{x, y\} \) (with \( x, y \) free for \( x \) in \( \alpha \)),

\( x \in \{x : \alpha\} \equiv \alpha \).

The rules of inference are:

\( \Gamma, \alpha, \Gamma \vdash \beta \) (provided all free variables of \( \alpha \) appear free in the conclusion),

\( \vdash \alpha \),

\( \Gamma, \sigma, \Gamma \vdash \tau \)

(\( \Gamma )\) (with \( \tau \) free for \( x \) in \( \alpha \) and all the members of \( \Gamma \)),

\( \vdash \sigma = \tau \)

(\( \vdash \tau \) provided \( \tau \) is not free in conclusion),

\( \alpha, \beta, \Gamma \vdash \Gamma \)

(\( \alpha = \beta \)).

These axioms and rules of inference yield a system of natural deduction in \( \vdash \) if \( \emptyset \) is any collection of sentences in \( \Gamma \), we say that the sequent \( \Gamma \vdash \alpha \) is derivable from \( \Gamma \), and write \( \Gamma \vdash \alpha \), provided there is a derivation of \( \Gamma \vdash \alpha \) using the basic axioms \( \vdash \) and the rules of inference. For \( \Gamma \vdash \alpha \) we write \( \Gamma \vdash \alpha \), and for \( \vdash \) we write \( \vdash \).

A theory in \( \vdash \) is a collection of sentences closed under derivability.

A theorem in \( \vdash \) is any sentence derivable in \( \vdash \).

A theory in \( \vdash \) is called a type theory if \( \vdash \) is any collection of sentences in \( \emptyset \), and \( \vdash \) and the rules of inference. For \( \vdash \) we write \( \vdash \) and, for \( \vdash \), we write \( \vdash \).

A theory in \( \vdash \) is said to be conservative extension of \( \vdash \), if for any sequent \( \Gamma \vdash \alpha \), we have \( \Gamma \vdash \alpha \).

Logical operators in \( \vdash \) are defined as follows:

\( \text{true} \equiv \# \not\equiv \# \).

\( \alpha \land \beta \equiv (\alpha, \beta) \equiv (\text{true}, \text{true}) \).

\( \alpha \rightarrow \beta \equiv (\alpha, \beta) \equiv (\text{true}, \text{false}) \).

\( \forall x \alpha \equiv \{x : \alpha\} = \{x : \text{true}\} \).

\( \text{false} \equiv \forall x u. u = \text{true} \) (with \( x : \Omega \)),

\( \lnot \alpha \equiv \alpha \rightarrow \text{false} \).

\( \alpha \lor \beta \equiv \forall x (\alpha \rightarrow x \land \beta \rightarrow x) \rightarrow u \) (with \( x, u : \Omega \not\equiv \alpha, \beta \)).

Other logical operators such as \( \exists x \) and set-theoretic terms such as \( \{x : \alpha\} \) can be introduced in the usual way. It can then be shown that the theorems of (free) higher-order intuitionistic logic are derivable in \( \vdash \) (see [2], Ch. 3).

Convention. Whenever a formula is introduced as \( \alpha(x, y, \ldots) \), we shall suppose that all the free variables of \( \alpha \) occur among \( x, y, \ldots \).

The natural domain of interpretation for type theories are the toposes (see Ch. 2). These are categories possessing a terminal object \( 1 \), a subobject classifier finite products, and exponentials of the form \( \text{P}^\alpha \) (which we shall write as \( \Gamma^\alpha \)).

The interpretation \( I \) of \( \Gamma \) in a topos \( E \) is an assignment.
to each type \( A \) of an object \( a_i \) in such a way that \( \text{for } i = 1, \ldots, n \) \( \pi_i: \rho_i = \Omega \chi \). Let \( B \) be a B of type \( C \), then \( B = B \).

In [2], Ch. 3, it is shown how to extend any interpretation \( \Gamma \) to arbitrary terms in a way that, if \( \tau \) is a term, \( \exists \alpha \cdot \exists \beta \cdot \exists \gamma \) is assigned to the E-premise \( A \rightarrow B \). Then \( \gamma \) is a formula leads to a notion of validity of \( \Gamma \) under \( f \), written \( \Gamma \models \alpha \). It is said to be a model of \( \mathcal{S} \) if \( \Gamma \models \alpha \) whenever \( \Gamma \models \beta \).

Any type theory \( S \) determines a type \( C(S) \), the toposes of \( S \)-sets and maps. The sets of \( C(S) \) are all types of the form \( \mathcal{P}(A) \) and the arrows are all \( S \)-provable functional relations. There is a natural interpretation \( C(S) \) of the \( C(S) \) which is also a model of \( S \) in fact we have \( \Gamma \models \alpha \) iff \( \Gamma \models \alpha \). This fact leads to the Completeness Theorem for Type Theories, namely:

\[
\Gamma \vdash \alpha \quad \text{iff} \quad \Gamma \models \alpha \quad \text{for every model } \Gamma \text{ of } S.
\]

The procedure of associating a topos with a type theory can be reversed. Given a topos \( \mathcal{C} \), we can associate with it a language \( L(\mathcal{C}) \) called its internal language. Roughly speaking, \( L(\mathcal{C}) \) has the objects of \( \mathcal{C} \) as types and the arrows of \( \mathcal{C} \) as functional relations. There is a natural interpretation \( L(\mathcal{C}) \) of \( \mathcal{C}(\mathcal{L}) \) in \( \mathcal{L}(\mathcal{C}) \), the theory of \( \mathcal{C}(\mathcal{L}) \), in \( L(\mathcal{C}) \) which is called \( L(\mathcal{C}) \). It is the case that \( \Gamma \vdash \alpha \) iff \( \Gamma \models \alpha \). Any topos \( \mathcal{C} \) can be shown to be isomorphic, i.e., \( \Gamma \vdash \alpha \Rightarrow \Gamma \models \alpha \) for some (closed) term \( \tau \).

We shall describe the following fundamental result (proved in [1] for a somewhat similar system of type theory than the present one). Call a type theory \( T \) well-behaved for any formula \( \alpha(x) \). \( \Gamma \vdash \alpha \) implies \( \Gamma \vdash \alpha(x) \) for some (closed) term \( \tau \).

Theorem 2.1. Any type theory has a conservative extension.

Proof. We first perform the following general construction. Let \( \Sigma \) be a set of axioms of \( L \) such that each \( \alpha \in \Sigma \) has at most one free variable \( x \). We shall write the subscripts "\( \alpha \)" from \( \Sigma \) as well as another "\( \alpha \)" for a "\( \alpha \)" or "\( \alpha \)", etc. Let \( \Sigma \) the language obtained from \( L \) by adding for each \( \alpha \in \Sigma \) a new function symbol \( \alpha \). Let \( \Gamma \) be the theory in \( L(\mathcal{C}) \) whose terms are those of \( \mathcal{C}(\mathcal{L}) \) together with all the sequents of the form \( \Gamma \vdash \alpha(x_1, \ldots, x_n) \), where \( x_1, \ldots, x_n \in \Gamma \). We remark that \( \Gamma \vdash \alpha(x_1, \ldots, x_n) \), where \( \alpha \in \Sigma \) for any \( \alpha \in \Sigma \).

Claim. \( \Gamma \vdash \alpha(x) \) for all \( \alpha \in \Sigma \), \( \text{if } S(\Sigma) \text{ is a conservative extension of } S \).

We sketch a proof of this claim. If \( \Gamma \vdash \alpha \) for \( \alpha \in \Sigma \), then \( \exists \alpha \cdot \exists \beta \cdot \exists \gamma \) is assigned to \( \alpha \rightarrow \beta \). Then \( \gamma \) is a formula leads to a notion of validity of \( \Gamma \) under \( f \), written \( \Gamma \models \alpha \). It is said to be a model of \( \mathcal{S} \) if \( \Gamma \models \alpha \) whenever \( \Gamma \models \beta \).

3. Type theories with a c-operator

Let \( S \) be a type theory in a language \( L \). A type \( A \) of \( L \) is said to be \( S \)-inhabited if \( \exists x. x = x \) for all \( x \). A type \( A \) is \( S \)-inhabited if \( \exists x. x = x \) for all \( x \). A type \( A \) is \( S \)-inhabited if \( \exists x. x = x \) for all \( x \).

The type \( L_1(S) \) over \( S \) is obtained by adding to \( L \) the symbol \( \epsilon \) and the new term forming clause

\[
\text{for any formula } \alpha(x) \text{ with } x \text{ of } S \text{-inhabited type, } \epsilon(x) \text{ is a (closed) term of the same type as } x.
\]

Of course, all occurrences of \( x \) in \( \epsilon(x) \) are regarded as being bound occurrences. Note also that directed terms may be formed in \( L_1(S) \). For example, \( L_1(S) \) contains the term \( \epsilon(x, y) \) provided \( x \) and \( y \) have \( S \)-inhabited types.

The extension \( S \) of the theory in \( L_1(S) \) obtained from \( S \) by adding as axioms the sequents

\[
\exists y. \epsilon(y,a)
\]

for any formula \( \alpha(x) \) with \( x \) of \( S \)-inhabited type.

We are going to determine the deductive strength of \( S \), relative to \( S \). Let \( S^* \) be the theory in \( L \) obtained from \( S \) by adding as axioms the sequents

\[
\text{for all formulas } \alpha(x) \text{ with } x \text{ of } S \text{-inhabited type.}
\]

Theorem 2.1. For any sequent \( \Sigma \) of \( L \), \( \Gamma \vdash \beta \) iff \( \Gamma^\Sigma \vdash \beta \).

Proof. For the "\( \Gamma \vdash \beta \)" direction, we note that \( \exists x. \epsilon(x,a) \), whence \( \exists x. \epsilon(x,a) \), \( \omega(x,a \in a) \), \( \exists x. \epsilon(x,a) \), \( \omega(x,a \in a) \), \( \omega(x,a \in a) \) by extensional generalization.

Conversely, let \( \Gamma \) be a conservative extension of \( S \) as provided by Theorem 2.1. Then for any \( S \)-formula \( \alpha(x) \) with \( x \) of \( S \)-inhabited type we have \( \Gamma \vdash \alpha(x) \), for some closed term \( \tau \) of the language \( L(\mathcal{C}) \) of \( \Gamma \). For each \( S \)-formula \( \alpha(x) \) choose such \( \tau \) and denote it by \( \tau_\alpha \). Then the sequent \( \exists x. \epsilon(x,\tau_\alpha) \) is then derivable from \( \Gamma \). We now define a type preserving map \( \tau \rightarrow \tau \) recursively as follows:

\[
\text{where } \exists x. \epsilon(x,\tau_\alpha) \text{ is an axiom of } S(\Sigma), \text{ i.e., where } a_1, \ldots, a_n \in \tau_\alpha \beta. \text{ It follows that } \Gamma^\Sigma \vdash \alpha(x,\tau_\alpha) \rightarrow \alpha(x,\tau_\alpha). \text{ Then } \exists x. \epsilon(x,\tau_\alpha) \rightarrow \exists x. \epsilon(x,\tau_\alpha). \text{ Then } \Gamma^\Sigma \vdash \tau \rightarrow \tau \text{ as required.}
\]
It is readily checked (by induction) that this definition is coherent, type preserving, and commutes with logical operations (i.e., \((\alpha \land \beta)^* = \alpha^* \land \beta^*\), etc.).

We now claim that, for any sequent \(\Gamma \vdash \beta\) in \(L(S)\),

\[(\star) \quad \Gamma \vdash \beta = \Gamma^* \vdash \beta^*,\]

where \(\Gamma^* = \{\sigma^* : \sigma \in \Gamma\}\) and \(\beta^* = \{\alpha^* : \alpha \in \beta\}\). To prove \((\star)\), start with a derivation \(P\) of \(\Gamma \vdash \beta\) in \(L(S)\). Replace each sequent \(\Delta \vdash \phi\) in \(P\) by the sequent \(\Delta^* \vdash \phi^*\), thus obtaining a new \(\Gamma^*\) of sequents in \(L(T)\). This process carries basic axioms to basic axioms, axioms of \(S\) to axioms of \(S^*\), and application of a role of inference to an application of the same role of inference, and any \(S\)-axiom \(\sigma \vdash \alpha / \tau / \theta\) to the sequent \(\sigma^* \vdash \alpha^* / \tau^* / \theta\), which, as we have observed above, is derivable in \(T\). Therefore \(\Gamma^*\) is a derivation of \(\Gamma^* \vdash \beta^*\) in \(T\), which establishes \((\star)\).

Finally, if \(\Gamma \vdash \beta\) is a sequent of \(T\), \(\Gamma^* \vdash \beta^*\) is the same as \(\Gamma \vdash \beta\), so that \(\Gamma \vdash \beta\) implies \(\Gamma^* \vdash \beta^*\) by \((\star)\). And \(\Gamma \vdash \beta\) by the fact that \(\Gamma^*\) is a conservative extension of \(S\).

This result has an illuminating corollary, to which we require some definitions.

A type theory \(T\) is a language \(L(T)\) (possibly containing \(e\)-terms) in which we can deduce \(\tilde{r} \vdash \alpha / \tau / \theta\) with \(\tilde{r}\) a \(T\)-inhabited type. \(\tilde{r}\) is \(T\)-inhabited if, for any \(\tau^*\) of \(T\)-inhabited type, there is a closed \(T\)-term \(r\) of the same type as \(x\) such that \(\tilde{x} \vdash \tilde{r} \vdash \alpha / \tau / \theta\).

It is readily shown that any classical theory is stable, and a theory is \(\tilde{x}\)-inhabited if and only if it is witnessed and stable. We shall show in Section 6 that the first implication cannot be reversed, and that the \(\tilde{x}\)-inhabited property is genuinely stronger than that of being witnessed and of being stable.

We can now state and prove our promised

Corollary 3.2: The following are equivalent for any type theory \(S\):

(i) \(S\) is stable;
(ii) \(S\) has a \(T\)-inhabited extension of \(S\);
(iii) \(S\) has a \(T\)-inhabited extension of \(S\).

Proof: (i) \(\Rightarrow\) (ii) is an immediate consequence of Theorem 3.1, since it is evident that \(S\) is stable if \(S \vdash \tilde{r}\).

\[\Gamma \vdash \alpha / \tau / \theta \quad \text{in} \quad L(S)\]

is a \(T\)-inhabited type. The proof of (iii) \(\Rightarrow\) (i) is left as an easy exercise to the reader.

Finally, for (iii) \(\Rightarrow\) (iii), suppose that \(S\) is a \(T\)-inhabited extension of \(S\). Then any \(S\)-inhabited type is \(T\)-inhabited, since no new types are added in the passage from \(L(S)\) to \(L(S^*)\). So for any \(\alpha \vdash \tilde{r} \vdash \tau / \theta\), \(\tilde{r}\) is of \(S\)-inhabited type. It is also \(T\)-inhabited, since \(\tau^\pi \vdash \tilde{r}^\pi \vdash \alpha^\pi / \tau^\pi / \theta^\pi\)

and it follows that \(S\) is a \(T\)-inhabited extension of \(S\). Hence (iii) \(\Rightarrow\) (i).

Remarks. (1) It follows in particular that any classical theory satisfies (ii) in (iii) of the Corollary; for first-order theories this fact is well-known (see [6]).

(2) The implication (i) \(\Rightarrow\) (iii) of the Corollary is asserted in [3] for a somewhat different system of type theory.

4. The partial \(e\)-cooperator

In intensional type theories there is a straightforward way of formulating a notion of "\(\tilde{x}\)-inhabited" term which, as we shall see, leads naturally to the concept of \(e\)-cooperator.

Now although there is no explicit provision for \(\tilde{x}\)-inhabited terms in our present framework, we can produce an acceptable surrogate as follows. Suppose \(A\) were a \(\tilde{x}\)-inhabited" term of some type \(A\) in a type theory \(S\). Consider the extension \(U\) of the property of being equal to \(\tilde{x}\). Then \(U\) would be a closed term of type \(\mathcal{P}\) satisfying the condition

\[(a) \quad \exists \tilde{x} \forall \tilde{y} \in U \tilde{x} \equiv \tilde{y}\]

(b) for any formula \(\phi(x)\), there is a \(\tilde{x}\)-inhabited term \(\tilde{x}\) such that \(\exists \tilde{x} \forall \tilde{y} \in U \tilde{x} = \tilde{y}\).

(Note that we were able to drop the condition that the type of \(\tilde{x}\) be inhabited since we are now only concerned with terms that are partially defined.) But \(\phi(x) = \exists \tilde{x} \forall \tilde{y} \in U \tilde{x} = \tilde{y}\) would be the singleton arising as the extension of the property of being equal to \(\tilde{x}\), so that, if it were \(\tilde{x} = \tilde{x}\), we have \(\exists \tilde{x} \forall \tilde{y} \in U \tilde{x} = \tilde{y}\) to \(U\). So our condition (b) becomes

\[(b') \quad \text{for any formula } \phi(x) \text{ with } x \cdot A, \text{ there is a } \tilde{x}\text{-inhabited } \tilde{a} \text{ such that } \exists \tilde{x} \forall \tilde{y} \in U \tilde{x} = \tilde{y}\text{.}

A type theory \(S\) satisfying \((b')\) for all types \(A\) will be called partially \(\tilde{x}\)-inhabited.

Turning now to the concept of stability, we introduce the defined predicate

\[\text{sing}(\tilde{x}) \equiv \exists \tilde{x} \forall \tilde{y} \in U \tilde{x} = \tilde{y}, \tilde{u} \equiv \tilde{y}\]

where \(U\) is a closed term.

Note that a closed term \(U\) in \(\mathcal{P}\) is then an \(\tilde{x}\)-inhabited term if \(\exists \tilde{x} \text{ sing}(\tilde{x})\). A type theory \(S\) is now said to be partially stable if for any formula \(\phi(x)\), we have

\[\exists \tilde{x} \forall \tilde{y} \in U \tilde{x} \equiv \tilde{y}\]
Proof. We prove (ii); the proof of (iii) is similar and that of (i) easy. Suppose that $S$ is Hilbertian and let $a(x)$ be a formula with $x : A$. Define $\beta[y] : 3 \exists u \equiv (\exists x)[u(x) \land u(x)]$ with $x : \mathcal{P}A$. There is a closed term $V : \mathcal{P}A$ such that $3\beta[xY] \equiv [\beta[xV]]$. Let $U \equiv (x : [x \equiv \{x\}])$; then $U$ is a $\mathcal{A}$-singleton and we have

\[
\exists \alpha(x) \forall x \exists \beta[yV]
\]

\[
\begin{align*}
\vdash & x \exists uU \equiv \{x\} \\
\vdash & \exists xU \equiv \{x\} \land u(x) \\
\vdash & \exists xU \equiv \{x\} \land u(x)
\end{align*}
\]

as required. □

We can now introduce, by analogy with $\varepsilon$, the partial $\varepsilon$-operator, which we shall denote by $\pi$.

The partial $\varepsilon$-language $\mathcal{L}_\varepsilon(S)$ over $S$ is obtained by adding to the language $\mathcal{L}$ of $S$ the symbol $\pi$ and the term-forming clause

- for any formula $a(x)$ with $x : A$, $\pi a$ is a (closed) term of type $\mathcal{P}A$.

The partial $\varepsilon$-extension $S_\varepsilon$ of $S$ is the theory in $\mathcal{L}_\varepsilon(S)$ obtained from $S$ by adding axioms of the sequents

\[
\text{[\text{sing}(x, y)] \text{ and } \exists \alpha(x) \forall y \exists z(y) \equiv \{x\}}
\]

for all $\mathcal{L}_\varepsilon(S)$-formulas $a(x)$. Clearly $S_\varepsilon$ is partially Hilbertian.

Now let $S^\pi$ be the theory in $\mathcal{L}$ obtained by adding to $S$ all sequents

\[
\text{[\text{sing}(u, y)] \land \exists \alpha(y) \equiv \{u, x\}}
\]

for all $a(x)$. Clearly $S^\pi$ is partially stable, and $S^\pi$ is an extension of $S^\pi$.

In essentially the same way as we proved Theorem 3.1, we obtain the

**Theorem 4.2.** For any sequent $\Gamma \vdash \beta$ of $\mathcal{L}$,

\[
\Gamma \vdash \pi \beta \iff \Gamma \vdash \beta.
\]

**Proof.** We give a sketch. Let $T$ be a conservative witnessed extension of $S^\pi$ as guaranteed by Theorem 2.1. For each $a(x)$ choose a closed term $t_{a(x)} : \mathcal{P}A$ in $T$. Hence,

\[
\vdash \text{sing}(x, y) \land \exists \alpha(y) \equiv \{t_{a(x)}, y\}.
\]

Define a translation $\tau \tau' \tau''$ of the terms of $\mathcal{L}_\varepsilon(S)$ to those of $\mathcal{L}(T)$ as in the proof of Theorem 3.1, except that now $(\pi a, y) = \{t_{a(x)}, y\}$. The proof now proceeds like that of Theorem 3.1. □

**Corollary 4.3.** The following are equivalent for any type theory $S$:

(i) $S$ is partially stable.

(ii) $S_\varepsilon$ is a conservative extension of $S^\pi$.

(iii) $S$ has a conservative partially Hilbertian extension. □

5. **Interpreting the $\varepsilon$-operator in a topos**

We shall call a topos $E$ Hilbertian if any diagram of the form $X \rightarrow A \rightarrow 1$ in $E$ can be expanded to a commutative diagram of the form

\[
\begin{array}{ccc}
X & \rightarrow & U \\
\downarrow & & \downarrow \\
X & \rightarrow & 1
\end{array}
\]

Hilbertian toposes will turn out to be the appropriate structures for interpreting $\varepsilon$-operators in type theories.

Note first that, in a Hilbertian topos, every subobject of 1 is projective. For this it suffices to show that for the canonical epi-mono factorization $X \rightarrow U \rightarrow 1$ of any $X \rightarrow 1$ there is an arrow $U \rightarrow X$. Now if $X^\ast$ is the partial map classifier of $X$ (see [4, Ch. 1]), we have a diagram $X \rightarrow X^\ast \rightarrow 1$. Using the Hilbertian property, we expand this to

\[
\begin{array}{ccc}
X & \rightarrow & U \\
\downarrow & & \downarrow \\
X & \rightarrow & 1
\end{array}
\]

and the assertion follows.

The proof of the following proposition is routine and, accordingly, omitted (see [2, 4.32]).

**Proposition 5.1.** Let $E$ be a topos. Then we have:

(i) $\text{Th}(E)$ is witnessed iff 1 is projective in $E$.

(ii) $\text{Th}(E)$ is partially Hilbertian iff every subobject of 1 is projective in $E$.

(iii) $\text{Th}(E)$ is Hilbertian iff $E$ is Hilbertian.

Let $E$ be a Hilbertian topos. For each object $X$ let

\[
\begin{array}{ccc}
X & \rightarrow & UX \\
\downarrow & & \downarrow \\
X & \rightarrow & 1
\end{array}
\]

be the canonical epi-mono factorization of $X \rightarrow 1$. For any $A \rightarrow 1$, choose for each monic $m : X \rightarrow A$ arrows $\varepsilon(m) : 1 \rightarrow A$ and $\pi(m) : UX \rightarrow X$ such that the following diagram commutes:
We shall call $\xi(m)$ the canonical element of $A$ determined by the nonce $m$. We shall assume that each Hilbertian topos has been assigned canonical elements $\xi(m)$ for all nonces $m$.

Now let $S$ be a type theory in a language $L$ and let $I$ be an interpretation of $L$ in a Hilbertian topos $E$. We extend $I$ to an interpretation of the $\xi$-language $L(\xi)$ as follows. For any arrow $\alpha : A \to D$, let $\Xi$ be the monic (unique up to isomorphism) to $A$ classified by $\alpha$, i.e. such that the following diagram is a pullback:

\[
\begin{array}{c}
\Xi \\
\downarrow \downarrow \\
A & \\
\alpha \\
\end{array}
\]

Then for any formula $\alpha(x)$ with the type $A$ of $x$ $S$-inhabited we define

\[
(\xi_m; \alpha) : (\xi(S)) : 1 \to A.
\]

**Proposition 5.2.** If $I$ is a model of $S$, then $I$, so extended, is a model of $S^\xi$.

**Proof.** It suffices to show that the sequent $\exists\alpha(\xi_m; \alpha)$ is valid under $I$ and for this it suffices (by, e.g., [2], 214) to show that there is an arrow $y$ making the following diagram commute:

\[
\begin{array}{c}
\exists \alpha \\
\downarrow \downarrow \\
A & \\
\xi_m \\
\end{array}
\]

But the commutativity of the diagram (*) above shows that we may take $y$ to be $\xi(\alpha)$.

In general, interpretations of $L(\xi)$ in Hilbertian toposes validate more than just the theorems of $S$. For example, let us call two formulas $\alpha(x)$, $\beta(y)$ $(S^\xi)$-similar if $(\alpha \equiv \beta)$ and $x$ and $y$ are the same $S$-inhabited type; $(\alpha \equiv \beta)$ is free for $y$ in $\beta(y)$; $x$ is free for $x$ in $\alpha$ and $(\alpha \to \beta) \ni \alpha(x)$ is free for $y$ in $\beta(y)$. It is then easy to see that any interpretation in a Hilbertian topos assigns the same value to $\xi_m\alpha$ and $\xi_m\beta$ whenever $\alpha(x)$ and $\beta(y)$ are similar. This being the case, what is the theory common to all Hilbertian toposes? We proceed to identify it.

Given a theory $T$ in some language $L$, let $T^\xi$ be obtained by adding to $L$ all $\xi$-terms $\xi_m\alpha$ not already present in $L$ for formulas $\alpha(x)$ with $x$ of $T$-inhabited type. Let $T^\xi$ be the theory in $\xi(S^\xi)$ obtained from $T$ by adding to axioms the sequents

\[
\exists\alpha(\xi_m; \alpha) \text{ for } \xi \in \Sigma,
\]

\[
\xi_m \alpha = \xi_m \beta \quad \text{for all } (T^\xi)-similar formulas } \alpha(x), \beta(y).
\]

Now define recursively $T_0 = S$, $T_{n+1} = (T_n)^{\xi}$. Let $\Sigma^\xi = \bigcup_{n\in\mathbb{N}} T_n$. Let $\xi(S^\xi) = \bigcup_{n\in\mathbb{N}} T_n$. We call $\Sigma^\xi$ the strong $\xi$-extension of $S$.

**Lemma 5.3.** $\Sigma^\xi$ is Hilbertian and $\Sigma^\xi \sim_{\xi} = \xi_0$ for any $\xi$-similar formulas $\alpha(x)$, $\beta(y)$.

**Proof.** If $\alpha(x)$ is a formula of $\xi(S)$ with $x$ of $S^\xi$-inhabited type, then $\nu_\alpha$ is a formula of $\xi(S)$ with $x$ of $S^\xi$-inhabited type, then $\nu_\alpha = \xi_m \alpha$ for some $m$; let $n_0$ be the least such $m$. Then $\exists\alpha(\xi_m; \alpha)$ is an axiom of $T_{n_0}$ so that $\exists\alpha(\xi_m; \alpha)$ is an axiom of $T_{n_0}$. Therefore $\Sigma^\xi$ is Hilbertian. Now let $\alpha(x)$, $\beta(y)$ be $\Sigma^\xi$-similar formulas. Then there is a least $n$ such that $\alpha(x)$ and $\beta(y)$ are $T_n$-similar. So $\xi_m \alpha = \xi_m \beta$ is an axiom of $T_{n+1}$ and hence derivable in $\Sigma^\xi$.

Any model $I$ of $S$ in a Hilbertian topos can be extended to a model (also denoted by $I$) of $\Sigma^\xi$ by iterating the procedure of extending $I$ to models of $S_1$ we have the details to the reader.

All this leads to

**Theorem 5.4 (Existential Completeness Theorem).** For any sequent $\exists\alpha$ of $\xi(S^\xi)$ and hence, a formula, of $\xi(L(S))$, $I \vdash \exists\alpha$ if and only if $I \vdash\alpha$ for every Hilbertian model $I$ of $S$.

**Proof.** We give just a sketch, as the details are tedious but routine.

Show by induction on $n$ that $\exists\alpha \vdash \nu_\alpha$ implies $I \vdash \nu_\alpha$ for every Hilbertian model $I$ of $S$.

Consider the topos $C(\Sigma^\xi)$. It is Hilbertian, for if $\alpha$ is an $\Sigma^\xi$-inhabited type and $X = \{x : \alpha\}$ with $x : A$ and $\xi_\alpha$ is the insertion resp.

\[
\begin{array}{c}
\exists \in : \mathbb{N} \\
\downarrow \downarrow \\
\alpha \\
\end{array}
\]

Now choose canonical elements in $C(\Sigma^\xi)$ in such a way that $\xi_\alpha \alpha$ is always the map $1 \to \alpha$ with value $\xi_\alpha$. Lemma 5.3 implies that this definition is coherent (it would not be if we had not replaced $S_1$ with $\Sigma^\xi$). One can now show (along the lines of the proof of 3.2 of [2]) that $I \vdash \exists\alpha$ if and only if $I \vdash \exists\alpha$. Since $C(\Sigma^\xi)$ is a Hilbertian model of $S$, the implication follows. Q.E.D.
Remark. It is possible to establish similar results relating partial ω-theories and 'partially Hilbertian' toposes, i.e., those in which every subobject of 1 is projective. We omit the details.

6. Mostly counterexamples

We establish the nontriviality of various properties of type theories we have introduced. First, we require the following

Proposition 6.1. Let S be a stable theory in a language with a natural number system (N; 0, S) (see [2], Ch. 7). Then for any sentences β, γ we have

\[ \gamma \implies (\beta \implies \gamma) \implies (\gamma \implies \beta). \]

Proof. Write \( \mathcal{L} \) for \( \mathcal{L} \) and let \( x, y \) be variables of type \( N \). It is well known (see [3], 7.8) that \( N \) is decidable, i.e., \( \mathcal{L} \models \forall x \forall y (x = y \lor x \neq y) \). Now define

\[ a(x) = (x = 0 \lor x = 1). \]

Then \( \mathcal{L} \models \forall x \forall y \forall z (a(y) \lor z = y) \implies (z = y \lor z = x) \lor z = x \implies \gamma \), so that

\[ \mathcal{L} \models \forall x \forall y \forall z (a(y) \lor z = y) \implies (z = y \lor z = x) \lor z = x \implies \gamma. \]

Therefore

\[ \mathcal{L} \models \forall x \forall y \forall z (a(y) \lor z = y) \implies (z = y \lor z = x) \lor z = x \implies \gamma. \]

So if \( S \) is stable then the required conclusion follows immediately. □

Corollary 6.2. Any stable theory \( S \) in a language with a natural number system satisfies (an equivalent of) Morley's law, viz., for any sentence \( \gamma \), \( \beta \rightarrow \gamma \rightarrow \neg \neg \beta \).

Remark. The proof of Proposition 6.1 is easily adapted to yield the following strengthening of a result of [2], viz.,

Let \( T \) be a theory in the first-order intuitionistic \( \lambda \)-calculus such that, for some constants \( 0, 1 \), we have \( \forall x \forall y (x = y \lor x \neq y) \implies \beta \implies \gamma \implies \beta \). Then for any sentences \( \beta, \gamma \), \( \beta \implies (\gamma \implies \beta) \implies (\beta \implies \gamma) \implies (\gamma \implies \beta). \)

Now we can establish the following diagram relating the various properties we have introduced, where the arrows indicate irreversible implications.

It suffices, in view of what we already know, to establish the following non-implications.

I. Hilbertian \( \varnothing \), classical. For this it suffices to produce a Hilbertian topos which is not classical. To this end, let \( M \) be a monad consisting of two elements \( a, b \) such that \( a \neq b \). Then since \( M \) is not a group, the topology \( \text{Set}^M \) of \( M \)-sets is not classical. But it is Hilbertian. For let \( X \) be a non-empty sub-\( M \)-set of an \( M \)-set \( A \). Choose \( a \in X \) and let \( b : a \in X \). Then the diagram

\[ \begin{array}{ccc}
X & \rightarrow & 1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array} \]

commutes, yielding the desired conclusion.

II. Classical \( \varnothing \), witnessed. For this we need only exhibit a Boolean topos which is not witnessed. An example is the topos of CoC, where \( \mathbb{C} \) is a group with at least 2 elements.

III. Witnessed \( \varnothing \), partially Hilbertian. It suffices to produce a topos in which \( 1 \) is, but not all of its subobjects are, projective. To this end, let \( P \) be the ordered set \( (\omega, \leq) \) augmented by a least element \( \ast \). Consider the topos \( \mathbb{E} = \text{Set}^P \) of sets varying over \( P \). Since \( P \) has a least element, \( \ast \), it is projective in \( \mathbb{E} \). But not every subobject of \( \ast \) is projective in \( \mathbb{E} \). For consider the \( E \)-object \( X \) defined by \( X(x) = \{ x \in \omega : \exists n \in \omega \text{ such that } x(n) = \ast \} \). Then the evident arrow \( X \rightarrow U \) is epic but there is no \( E \)-arrow \( U \rightarrow X \).

IV. Partially Hilbertian \( \varnothing \), stable. It suffices, by Corollary 6.2, to produce a topos \( \mathbb{E} \) in which every subobject of \( 1 \) is projective but in which the law

\[ \neg \gamma \lor \neg \neg \beta \]

for some \( \beta \) is not valid. Such a topos can be constructed from \([4]\). In Exercise 5d of that volume, it is asserted that, if \( X \) is a topological space which is separable and zero-dimensional, then every subobject of \( 1 \) is projective in the topos \( \text{Sh}(X) \) of sheaves on \( X \). In particular, this is the case for \( \text{Sh}(\mathbb{N}^\ast) \), where \( \mathbb{N}^\ast \) is the one-point compactification of the discrete space of integers. But it is well known that \( X \) is valid in \( \text{Sh}(X) \) if \( X \) is extremally disconnected, which \( \mathbb{N}^\ast \) manifestly is not. So \( \ast \) is not valid in \( \text{Sh}(\mathbb{N}^\ast) \), as claimed.
Remark. Recall that a topos is said to be Boolean if its associated theory is classical. Now while it is easy enough to find Hilbertian toposes which are Boolean — for example, any topos in which the axiom of choice holds (cf. Section 7) — non-Boolean examples such as the one presented in 1 above are harder to procure. A natural place to start looking is among the toposes of the form $Set^X$ for a partially ordered set $X$. However, we are doomed to disappointment here in view of the following observation.

Suppose the object $P$ in $Set^X$ has the property that every arrow $U \to F$ with $U \to 1$ can be extended to an arrow $1 \to F$ (which would be the case if $Set^X$ were Hilbertian and $F \to 1$ surjective). Then every transition map $F_{xy}$ with $x \leq y$ in $P$ is surjective. For suppose any $q \in P$ and $x \in F(q)$ (if $F(q) = \emptyset$, then $F$ is automatically surjective for any $p \leq q$). Define $U$ in $Set^X$ by $U(t) = F_x(q)$ if $q \leq t$, $U(t) = \emptyset$ if $q \not\leq t$. Then $U$ is a subobject of $F$ and $U \to 1$ in $Set^X$. Suppose that there is $x : 1 \to F$ such that $U \leq x$ in $Set^X$. Given $p \leq q$, let $k : x_p \to y_k(0) \in \mathbb{F}(q)$. Then $x = x(p) \leq x(q) = (x(q))(F_p(0))$. So $x = F_{xy}(b)$ and $F_{xy}$ is surjective as claimed.

It follows from this that if $Set^P$ is Hilbertian, $P$ is discrete and $Set^P$ Boolean. (We note in passing that this leads to many more examples of non-Hilbertian toposes in which every subobject of 1 is projective: for instance, any topos $Set^X$ with $X$ an ordinal $\geq 2$.) On the other hand, we might obtain non-Boolean Hilbertian toposes by turning our attention to toposes of sheaves (as opposed to presheaves). For example, is there a topological space whose category of sheaves meets these requirements? I have not been able to answer this question.

7. The $\varepsilon$-operator and the axiom of choice

We have only allowed the $\varepsilon$-operator to act on formulas with at most one free variable, thereby admitting only closed $\varepsilon$-terms. What happens if we relax this restriction? Suppose, for example, that in forming $\varepsilon x(S)$ and $S_0$ we drop the restriction that $\varepsilon x$ can only be formed when $x$ is at most $y$ free and now allow $x$ to contain an additional free variable $z$. Then we would have

\[ \varepsilon y \varepsilon z \varepsilon x(S, y, z, y_0, p_0) \]

where the term $\varepsilon x$ now has a free variable $z$. So if $f$ is in the map $z \to \varepsilon x$ we have

\[ \forall z \exists y \varepsilon z \varepsilon x(S, y, z, y_0, p_0) \]

Thus, if $g : \mathbb{F}(y, x)$, then $g(S, y, z, y_0, p_0)$. That is, under these conditions we have derived the axiom of choice in $S_0$ (cf. [2], 4.29).

Therefore, the admission of $\varepsilon$-term with even one free variable to an intuitionistic type theory makes the axiom of choice to be derived. But it is well known (see [2], 4.23(e) of [2], for example) that any intuitionistic type theory in which the axioms of choice can be derived is not classical. So the admission of $\varepsilon$-terms with one free variable collapses an intuitionistic type theory to a classical theory. Since the calculus has already been much investigated in a classical (albeit first order) context, the justice of confining our attention to closed $\varepsilon$-terms should now be evident.

In conclusion, we note that a type theory containing only closed $\varepsilon$-terms can be forced to become classical, albeit in a weakened sense, by adding as axioms the sequents asserting that the $\varepsilon$-terms satisfy an extensionality principle, viz.,

\[ \forall x, y \exists \alpha \varepsilon x = \varepsilon y \]

For then, following the proof of the Theorem in Section 5 of [4], we will be able to derive $\varepsilon y \forall \gamma \gamma \Rightarrow y$ for all sentences $\gamma$. For completeness we briefly indicate how this is done. Let $T$ be obtained by adding the above sequents to $S$. Let $\gamma$ be any sentence. Then define $o(x) : \varepsilon \gamma \Rightarrow \gamma$ by

\[ o(x) = \varepsilon \gamma \Rightarrow y \gamma \Rightarrow y \gamma \]

with $x \varepsilon y$. It is now easily shown that $\varepsilon y \gamma \Rightarrow y \gamma \Rightarrow y \gamma$ but extensionality now gives $\exists \gamma \gamma \Rightarrow y \gamma \Rightarrow y \gamma$, whereas $\gamma \gamma \Rightarrow y \gamma$ $\varepsilon y \Rightarrow y \gamma$. We conclude that $\varepsilon y \forall \gamma \Rightarrow y \gamma$ as claimed.

For more on extensionality in the first-order case, see [1].

Note added in proof (April 1993). Andreas Blas (private communication) has shown that for any Hilbertian topos $E$, the algebra $E((1, 1))$ of global elements of $E$ is Boolean. It follows from this that any local Hilbertian topos (in particular, any Hilbertian topos of sheaves) is Boolean, answering in the negative the question at the end of Section 5.

References


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