HERMANN WEYL ON INTUITION AND THE CONTINUUM*

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Hermann Weyl (1885 – 1955) was one of the greatest and most versatile mathematicians of the 20th century. His work had a vast range, encompassing analysis, algebra, number theory, topology, differential geometry, relativity theory, quantum mechanics, and mathematical logic. He was also unusual among mathematicians in possessing acute literary and philosophical sensibilities — sensibilities to which he gave full expression in his writings. In this paper I shall use quotations from these writings to provide a sketch of Weyl's philosophical orientation, following which I attempt to elucidate his views on the mathematical continuum, bringing out the essential role he assigned to intuition.

Towards the end of his *Address on the Unity of Knowledge*, delivered at the 1954 Columbia University bicentennial celebrations, Weyl enumerates what he considers to be the essential constituents of knowledge. At the top of his list[1] comes

> *…intuition, mind's ordinary act of seeing what is given to it.*[2]

Throughout his life Weyl held to the view that intuition, or *insight*—rather than *proof*—furnishes the ultimate foundation of *mathematical* knowledge. Thus in his *Das Kontinuum* of 1918 he says:

> *In the Preface to Dedekind (1888) we read that "In science, whatever is provable must not be believed without proof." This remark is certainly characteristic of the way most mathematicians think. Nevertheless, it is a preposterous principle. As if such an indirect concatenation of grounds, call it a proof though we may, can awaken any "belief" apart from assuring ourselves through immediate insight that each individual step is correct. In all cases, this process of confirmation—and not the proof—remains the ultimate source from which knowledge derives its authority; it is the "experience of truth".*[3]

In his short philosophical autobiography of 1954, *Insight and Reflection,* Weyl tells of the impact made on him as a schoolboy by a commentary on Kant's "Critique of Pure Reason." He was especially taken with Kant's doctrine that space and time are not inherent in the objects of the world, existing as such and independently of our awareness, but are, rather, *conceptual forms* or *intuitions* based in our intellects. And he goes on to quote Fichte—whose language he describes as "always a bit eccentric", but whose philosophic idealism was nevertheless to exert a considerable intellectual influence on him—

> *Transparent penetrable space, the purest image of my knowing, cannot be inspected but must be seen intuitively, and within it my inspecting itself is so seen. The light is not*

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[1] The others, in order, are: *understanding and expression; thinking the possible;* and finally, in science, *the construction of symbols or measuring devices.*

[2] Weyl [1954], 629.

[3] Weyl [1987], 119.

> *outside of me, but rather in me.*[4]

Although Weyl was soon to abandon the greater part of Kant's doctrines, he cleaved always to the idea of the primacy of intuition that he had first learned from Kant.

On entering Göttingen University in 1904, Weyl read Hilbert's *Foundations of Geometry.* This *tour-de-force* of the axiomatic method, in comparison to which Kant's "bondage to Euclidean geometry" had come to seem naïve, greatly impressed Weyl, so much so that, as a result,

> *under this overwhelming blow, the structure of Kantian philosophy, to which I had clung with faithful heart, crumbled into ruins.*[5]

After this phi;osophical debâcle Weyl lapsed into an indifferent positivism for a while. In 1912–13 his interest in philosophy was rekindled by his coming to learn of Husserl's phenomenology, to which he had been introduced by his wife, a student of Husserl's. It was also at about this time that Fichtean metaphysical idealism came to "capture his imagination."

Athirst for philosophy, Weyl cannot have been reluctant to abandon the aridities of positivism for the potential springs of phenomenology, whose goal he describes as

> *… to capture phenomena in their essential being—purely as they yield themselves apart from all genetical and other theories in the encounter with our consciousness.*[6]

Nevertheless, captivated by idealism as he was, Weyl seems not to have come to full acceptance of its central thesis that the external world exists, in the final analysis, only as an object of consciousness. More than once in his writings Weyl draws attention to

> *the gap between immanent consciousness … and the concrete man that I am, who was born of a mother and will die.*[7]

In Weyl's eyes, the concrete individual, the bearer of consciousness, has as much claim to real existence as does consciousness itself.

In *The Open World* (1932), Weyl provides an eloquent formulation of his philosophical outlook, which quickly moves beyond its initial echoes of Schopenhauer:

> *The beginning of all philosophical thought is the realization that the perceptual world is but an image, a vision, a phenomenon of our consciousness; our consciousness does not directly grasp a transcendental real world which is as it appears. The tension between subject and object is no doubt reflected in our conscious acts, for example, in sense perceptions. Nevertheless, from the purely epistemological point of view, no objection can be made to a phenomenalism which would like to limit science to the description of what is "immediately given to consciousness". The postulation of the real ego, of the thou and of the world, is a metaphysical matter, not judgment, but an act of ackowledgment and belief. But this belief is after all the soul of all knowledge. It was an error of idealism to assume that the phenomena of consciousness guarantee the reality of the ego in an essentially different and somehow more certain way than the reality of the external world; in the transition from consciousness to reality the ego, the thou and the world rise into existence indissolubly connected and, as it were, at one stroke.* [8]

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[4] Quoted in Weyl [1969], 282.
[5] *Ibid.,* 283.
[6] *Ibid.,* 288.
[7] *Ibid.*, 294
[8] Weyl [1932], 26-27.

Here the *transcendental real world* is the "objective" realm beyond immediate consciousness, the world with which theoretical physics, for example, is concerned. Since this domain is inscrutable to intuition, it can only be charted *indirectly,* through the medium of what Weyl calls *symbolic construction*, or *theoretical creation.* He concludes his *Current Epistemological Situation in Mathematics* of 1925 with a passage in which knowledge obtained in this indirect fashion is contrasted with that given purely in intuition:

> *Theories permit consciousness to "jump over its own shadow", to leave behind the given, to represent the transcendent, yet, as is self-evident, only in symbols. It never leads, I believe, to a final result, like phenomenal knowledge, which, although subject to human error, is nevertheless by its nature immutable.*[9]

Although Weyl held that the roots of mathematics lay in the intuitively given as opposed to the transcendent, he recognized at the same time that it would be unreasonable to require all mathematical knowledge to possess intuitive immediacy. In *Das Kontinuum*, for example, he says:

> *The states of affairs with which mathematics deals are, apart from the very simplest ones, so complicated that it is practically impossible to bring them into full givenness in consciousness and in this way to grasp them completely.*[10]

But Weyl did not think that this fact furnished justification for extending the bounds of mathematics to embrace notions which cannot be given fully in intuition *even in principle* (e.g., the actual infinite). He held, rather, that this extension of mathematics into the transcendent is necessitated by the fact that mathematics plays an indispensable role in the physical sciences, in which intuitive evidence is *necessarily* transcended. As he says in *The Open World*::

> *... if mathematics is taken by itself, one should restrict oneself with Brouwer to the intuitively cognizable truths ... nothing compels us to go farther. But in the natural sciences we are in contact with a sphere which is impervious to intuitive evidence; here cognition necessarily becomes symbolical construction. Hence we need no longer demand that when mathematics is taken into the process of theoretical construction in physics it should be possible to set apart the mathematical element as a special domain in which all judgements are intuitively certain; from this higher standpoint which makes the whole of science appear as one unit, I consider Hilbert to be right.* [11]

In *Consistency in Mathematics* (1929), Weyl characterized the mathematical method as

> *the* a priori *construction of the possible in opposition to the* a posteriori *description of what is actually given.*[12]

The problem of identifying the limits on constructing "the possible" in this sense occupied Weyl a great deal. He was particularly concerned with the concept of the mathematical *infinite*, which he believed to elude "construction" in the idealized sense of set theory. Again to quote a passage from *Das Kontinuum:*

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[9] Weyl [1998a], 140.
[10] Weyl [1987], 17.
[11] Weyl [1932], 82.
[12] Weyl [1929], 249.

> *No one can describe an infinite set other than by indicating properties characteristic of the elements of the set. ... The notion that a set is a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; "inexhaustibility" is essential to the infinite.*[13]

But the necessity of injecting mathematics into external reality compels it to embody a conception of the actual infinite, as Weyl attests towards the end of *The Open World:*

> *The infinite is accessible to the mind intuitively in the form of a field of possibilities open to infinity, analogous to the sequence of numbers which can be continued indefinitely, but the completed, the actual infinite as a closed realm of actual existence is forever beyond its reach. Yet the demand for totality and the metaphysical belief in reality inevitably compel the mind to represent the infinite as closed being by symbolical construction.*[14]

Another mathematical "possible" to which Weyl gave a great deal of thought is the idea of the *continuum.* During the period 1918–1921 he wrestled with the problem of providing it with an exact mathematical formulation free of the taint of the actual infinite. As he saw it in 1918, there is an unbridgeable gap between intuitively given continua (e.g. those of space, time and motion) on the one hand, and the "discrete" exact concepts of mathematics (e.g. that of real number) on the other. For Weyl the presence of this split meant that the construction of the mathematical continuum could not simply be read off" from intuition. Rather, he believed at this time that the mathematical continuum must be treated as if it were an element of the transcendent realm, and so, in the end, justified in the same way as a physical theory. In Weyl's view, it was not enough that the mathematical theory be *consistent*; it must also be *reasonable.*

Das Kontinuum embodies Weyl's attempt at formulating a theory of the continuum which satisfies the first, and, as far as possible, the second, of these requirements. In the following passages from this work he acknowledges the difficulty of the task:

> ... *the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd.*[15]

> *... the continuity given to us immediately by intuition (in the flow of time and of motion) has yet to be grasped mathematically as a totality of discrete "stages" in accordance with that part of its content which can be conceptualized in an exact way.*[16]

> *Exact time- or space-points are not the ultimate, underlying atomic elements of the duration or extension given to us in experience. On the contrary, only reason, which thoroughly penetrates what is experientially given, is able to grasp these exact ideas. And only in the arithmetico-analytic concept of the real number belonging to the purely formal sphere do these ideas crystallize into full definiteness.* [17]

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[13] Weyl [1987], 23.
[14] Weyl [1932], 83.
[15] Weyl [1987], 108.
[16] *Ibid.,* 24. In this connection it is of interest to note that Brentano, in his *On What is Continuous* of 1914, had drawn the similar conclusion that the continuum concept is derived from primitive sensible intuition and indeed that "all our sensible intuitions present us with that which is continuous." This led him to regard the constructions of the continuum of Dedekind, Cantor, and their successors as "fictions".
[17] *Ibid.,* 94.

> *When our experience has turned into a real process in a real world and our phenomenal time has spread itself out over this world and assumed a cosmic dimension, we are not satisfied with replacing the continuum by the exact concept of the real number, in spite of the essential and undeniable inexactness arising from what is given.*[18]

However much he may have wished to do so, in *Das Kontinuum* Weyl did not aim to provide a mathematical formulation of the continuum as it is presented to intuition, which, as the quotations above show, he regarded as an impossibility (at that time at least). Rather, his goal was first to achieve *consistency* by putting the *arithmetical* notion of real number on a firm logical basis, and then to show that the resulting theory is *reasonable* by employing it as the foundation for a plausible account of continuous process in the objective physical world.[19]

As a practicing mathematician, Weyl had come to believe that, the work of Cauchy, Weierstrass, Dedekind and Cantor notwithstanding, mathematical analysis at the beginning of the 20th century would not bear logical scrutiny, for its essential concepts and procedures involved vicious circles to such an extent that, as he says, "every cell (so to speak) of this mighty organism is permeated by contradiction." In *Das Kontinuum* he tries to overcome this by providing analysis with a *predicative* formulation—not, as Russell and Whitehead had attempted, by introducing a hierarchy of logically ramified types, which Weyl seems to have regarded as too complicated—but rather by confining the comprehension principle to formulas whose bound variables range over just the initial given entities (numbers). Thus he restricts analysis to what can be done in terms of natural numbers with the aid of three basic logical operations, together with the operation of substitution and the process of "iteration", i.e., primitive recursion. Weyl recognized that the effect of this restriction would be to render unprovable many of the central results of classical analysis—e.g., Dirichlet's principle that any bounded set of real numbers has a least upper bound[20]—but he was prepared to accept this as part of the price that must be paid for the security of mathematics.

In §6 of *Das Kontinuum* Weyl presents his conclusions as to the relationship between the intuitive and mathematical continua. He poses the question: Does the mathematical framework he has erected provide an adequate representation of physical or temporal continuity as it is *actually experienced*? He begins his investigation by noting that, according to his theory, if one asks whether a given function is continuous, the answer is not fixed once and for all, but is, rather, dependent on the extent of the domain of real numbers which have been defined up to the point at which the question is posed. Thus the continuity of a function must always remain *provisional*; the possibility always exists that a function deemed continuous *now* may, with the emergence of "new" real numbers, turn out to be discontinuous *in the future*.[21]

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[18] *Ibid.,* 93.

[19] The connection between mathematics and physics was of course of paramount importance for Weyl. His seminal work on relativity theory, *Space-Time-Matter,* was published in the same year (1918) as *Das Kontinuum*; the two works show subtle affinities.

[20] In this connection it is of interest to note that on 9 February 1918 Weyl and George Pólya made a bet in Zürich in the presence of twelve witnesses (all of whom were mathematicians) that "within 20 years, Pólya, or a majority of leading mathematicians, will come to recognize the falsity of the least upper bound property." When the bet was eventually called, everyone—with the single exception of Gödel—agreed that Pólya had won.

[21] This fact would seem to indicate that in Weyl's theory the domain of definition of a function is not unambiguously determined by the function, so that the continuity of such a "function" may vary with its domain of definition. (This would be a natural consequence of Weyl's definition of a function as a certain kind of relation.) A simple but striking example of this phenomenon is provided in classical analysis by the function f which takes value 1 at each rational number, and 0 at each irrational number. Considered as a function defined on the rational numbers, f is constant and so continuous; as a function defined on the real numbers,

To reveal the discrepancy between this formal account of continuity based on real numbers and the properties of an intuitively given continuum, Weyl next considers the experience of seeing a pencil lying on a table before him throughout a certain time interval. The position of the pencil during this interval may be taken as a function of the time, and Weyl takes it as a fact of observation that during the time interval in question this function is continuous and that its values fall within a definite range. And so, he says,

> *This observation entitles me to assert that during a certain period this pencil was on the table; and even if my right to do so is not absolute, it is nevertheless reasonable and well-grounded. It is obviously absurd to suppose that this right can be undermined by "an expansion of our principles of definition"—as if new moments of time, overlooked by my intuition could be added to this interval, moments in which the pencil was, perhaps, in the vicinity of Sirius or who knows where. If the temporal continuum can be represented by a variable which "ranges over" the real numbers, then it appears to be determined thereby how narrowly or widely we must understand the concept "real number" and the decision about this must not be entrusted to* logical *deliberations over principles of definition and the like.*[22]

To drive the point home, Weyl focuses attention on the fundamental continuum of *immediately given phenomenal time*, that is, as he characterizes it,

> *... to that constant form of my experiences of consciousness by virtue of which they appear to me to flow by successively.*(*By "experiences" I mean what I experience, exactly as I experience it. I do not mean real psychical or even physical processes which occur in a definite psychic-somatic individual, belong to a real world, and, perhaps, correspond to the direct experiences.*)[23]

In order to correlate mathematical concepts with phenomenal time in this sense Weyl grants the possibility of introducing a rigidly punctate "now" and of identifying and exhibiting the resulting temporal points. On the collection of these temporal points is defined the relation of *earlier than* as well as a congruence relation of *equality of temporal intervals,* the basic constituents of a simple mathematical theory of time. Now Weyl observes that the discrepancy between phenomenal time and the concept of real number would vanish if the following pair of conditions could be shown to be satisfied:

1. *The immediate expression of the intuitive finding that during a certain period I saw the pencil lying there were construed in such a way that the phrase "during a certain period" was replaced by "in every temporal point which falls within a certain time span OE.* [Weyl goes on to say parenthetically here that he admits "that this no longer reproduces what is intuitively present, but one will have to let it pass, *if it is really legitimate to dissolve a period into temporal points.*")

2. *If P is a temporal point, then the domain of rational numbers to which ℓ belongs if and only if there is a time point L earlier than P such that*

$$OL = \ell . OE$$

can be constructed arithmetically in pure number theory on the basis of our principles of

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f fails to be continuous anywhere.

[22] Weyl [1987], 88.

[23] *Ibid.,* 88

definition, and is therefore a real number in our sense.[24]

Condition 2 means that, if we take the time span *OE* as a unit, then each temporal point *P* is correlated with a definite real number. In an addendum Weyl also stipulates the converse.

But can temporal intuition itself provide evidence for the truth or falsity of these two conditions? Weyl thinks not. In fact, he states quite categorically that

> *... everything we are demanding here is obvious* nonsense: *to* these *questions, the intuition of time provides no answer—just as a man makes no reply to questions which clearly are addressed to him by mistake and, therefore, are unintelligible when addressed to him.*[25]

The grounds for this assertion are by no means immediately evident, but one gathers from the passages following it that Weyl regards the experienced *continuous flow* of phenomenal time as constituting an insuperable barrier to the whole enterprise of representing this continuum in terms of individual points, and even to the characterization of "individual temporal point" itself. As he says,

> *The view of a flow consisting of points and, therefore, also dissolving into points turns out to be mistaken: precisely what eludes us is the nature of the continuity, the flowing from point to point; in other words, the secret of how the continually enduring present can continually slip away into the receding past.*
>
> *Each one of us, at every moment, directly experiences the true character of this temporal continuity. But, because of the genuine primitiveness of phenomenal time, we cannot put our experiences into words. So we shall content ourselves with the following description. What I am conscious of is for me both a being-now and, in its essence, something which, with its temporal position, slips away. In this way there arises the persisting factual extent, something ever new which endures and changes in consciousness.* [26]

Weyl sums up what he thinks can be affirmed about "objectively presented time"—by which I take it he means "phenomenal time described in an objective manner"—in the following two assertions, which he claims apply equally, *mutatis mutandis,* to every intuitively given continuum, in particular, to the continuum of spatial extension:

> 1. *An individual point in it is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a "point of transition" (which, of course, can in no way be understood mathematically);*
>
> 2. *it is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible.*[27]

The fact that single points in a true continuum "cannot be exhibited" arises, Weyl continues, from the fact that they are not genuine individuals and so cannot be characterized by their properties. In the physical world they are never defined absolutely, but only in terms of a *coordinate system,* which, in an arresting metaphor, Weyl describes as "the unavoidable residue of the eradication of the ego." This metaphor, which Weyl was to employ more than

[24] *Ibid.,* 89.
[25] *Ibid.,* 90.
[26] *Ibid.,* 91-92.
[27] *Ibid.,* 92.

once (e.g. in Weyl [1950], 8 and [1963], 123) reflects the continuing influence of phenomenological doctrine: in this case, the thesis that the existent is given in the first instance as the contents of a consciousness. Many years later, in *Insight and Reflection*, Weyl expanded this metaphor into a full-fledged analogy: this is described in the Appendix to the present paper.

By 1919 Weyl had come to embrace Brouwer's views on the intuitive continuum. The latter's influence looms large in Weyl's next paper on the subject, *On the New Foundational Crisis of Mathematics,* written in 1920. Here Weyl identifies two distinct views of the continuum: "atomistic" or "discrete"; and "continuous". In the first of these the continuum is composed of individual real numbers which are well-defined and can be sharply distinguished. Weyl describes his earlier attempt at reconstructing analysis in *Das Kontinuum* as atomistic in this sense:

> *Existential questions concerning real numbers only become meaningful if we analyze the concept of real number in this extensionally determining and delimiting manner. Through this conceptual restriction, an ensemble of individual points is, so to speak, picked out from the fluid paste of the continuum. The continuum is broken up into isolated elements, and the flowing-into-each other of its parts is replaced by certain conceptual relations between these elements, based on the "larger-smaller" relationship. This is why I speak of the* atomistic *conception of the continuum.*[28]

By this time Weyl had come to repudiate atomistic theories of the continuum, including that of *Das Kontinuum*. While intuitive considerations, together with Brouwer's influence, must certainly have fuelled Weyl's rejection of such theories, it also had a *logical* basis. For Weyl had come to regard as meaningless the formal procedure—employed in *Das Kontinuum*—of negating universal and existential statements concerning real numbers conceived as developing sequences or as sets of rationals. This had the effect of undermining the whole basis on which his theory had been erected, and at the same time rendered impossible the very formulation of a "law of excluded middle" for such statements. Thus Weyl found himself espousing a position[29] considerably more radical than that of Brouwer, for whom negations of quantified statements had a perfectly clear constructive meaning, under which the law of excluded middle is simply not generally affirmable.

Of existential statements Weyl says:

> *An existential statement—e.g., "there is an even number"—is not a judgement in the proper sense at all, which asserts a state of affairs; existential states of affairs are the empty invention of logicians.*[30]

Weyl termed such pseudostatements "judgement abstracts", likening them, with his unfailing flair, to "a piece of paper which announces the presence of a treasure, without divulging its location." Universal statements, although possessing greater substance than existential ones, are still mere intimations of judgements, "judgement instructions", for which Weyl provides the following metaphorical description:

> *If knowledge be compared to a fruit and the realization of that knowledge to the consumption of the fruit, then a universal statement is to be compared to a hard shell filled with fruit. It is, obviously, of some value, however, not as a shell by itself, but only for its*

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[28] Weyl [1998], 91.

[29] Weyl's contention is strikingly similar to (and may have had an influence on) Hilbert's later assertion that "contentual" statements are, from the finitist standpoint, incapable of being negated. See, e.g., Hilbert [1926], 378.

[30] Weyl [1998], 97.

content of fruit. It is of no use to me as long as I do not open it and actually take out a fruit and eat it.[31]

Above and beyond the claims of logic, Weyl welcomed Brouwer's construction of the continuum by means of sequences generated by free acts of choice, thus identifying it as a "medium of free Becoming" which "does not dissolve into a set of real numbers as finished entities". Weyl felt that Brouwer, through his doctrine of Intuitionism[32], had come closer than anyone else to bridging that "unbridgeable chasm" between the intuitive and mathematical continua. In particular, he found compelling the fact that the Brouwerian continuum is not the union of two disjoint nonempty parts—that it is, in a word, *indecomposable*. "A genuine continuum," Weyl says, "cannot be divided into separate fragments." In later publications he expresses this more colourfully by quoting Anaxagoras to the effect that a continuum "defies the chopping off of its parts with a hatchet."

Weyl also agrees with Brouwer that all functions everywhere defined on a continuum are continuous, but here certain subtle differences of viewpoint emerge. Weyl contends that what mathematicians had taken to be discontinuous functions actually consist of several continuous functions defined on separated continua. (For example, the "discontinuous" function defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$ in fact consists of the two functions with constant values 0 and 1 respectively defined on the separated continua $\{x: x < 0\}$ and $\{x: x \geq 0\}$. The union of these two continua fails to be the whole of the real continuum because of the failure of the law of excluded middle: it is not the case that, for be any real number x, either $x < 0$ or $x \geq 0$.) Brouwer, on the other hand, had not dismissed the possibility that discontinuous functions could be defined on proper parts of a continuum, and still seems to have been searching for an appropriate way of formulating this idea.[33] In particular, at that time Brouwer would probably have been inclined to regard the above function f as a genuinely discontinuous function defined on a *proper part* of the real continuum. For Weyl, it seems to have been a self-evident fact that all functions defined on a continuum are continuous, but this is because Weyl confines attention to functions which turn out to be continuous by definition. Brouwer's concept of function is less restrictive than Weyl's and it is by no means immediately evident that such functions must always be continuous.

Weyl defined real functions as mappings correlating each interval in the choice sequence determining the argument with an interval in the choice sequence determining the value "interval by interval" as it were, the idea being that approximations to the input of the function should lead effectively to corresponding approximations to the input. Such functions are continuous by definition. Brouwer, on the other hand, considers real functions as correlating choice sequences with choice sequences, and the continuity of these is by no means obvious. The fact that Weyl refused to grant (free) choice sequences—whose identity is in no way predetermined—sufficient individuality to admit them as arguments of functions perhaps betokens a commitment to the conception of the continuum as a "medium of free Becoming" even deeper than that of Brouwer.

There thus being only minor differences between Weyl's and Brouwer's accounts of the continuum, Weyl accordingly abandoned his earlier attempt at the reconstruction of analysis and "joined Brouwer." At the same time, however, Weyl recognized that the resulting gain in intuitive clarity had been bought at a considerable price, as witnessed by his remark in the 1927 edition of *Philosophy of Mathematics and Natural Science:*

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[31] *Ibid.,* 98.

[32] For my remarks on Weyl's relationship with Intuitionism I have drawn on the illuminating paper van Dalen [1995].

[33] Brouwer established the continuity of functions fully defined on a continuum in 1904, but did not publish a definitive account until 1927. In that account he also considers the possibility of partially defined functions.

Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the greater part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes.[34]

Although he later practiced intuitionistic mathematics very rarely, Weyl remained an admirer of intuitionism. And the "riddle of the continuum" retained its fascination for him: in one of his last papers, *Axiomatic and Constructive Procedures in Mathematics,* written in 1954, we find the observation that

> *... the constructive transition to the continuum of real numbers is a serious affair... and I am bold enough to say that not even to this day are the logical issues involved in that constructive concept completely clarified and settled.*[35]

It seems to me a great pity that Weyl did not live to see the emergence in the 1970s of *smooth infinitesimal analysis* [36], a mathematical framework within which his vision of a true continuum, not "synthesized" from discrete elements, is realized. Although the underlying logic of smooth infinitesimal analysis is intuitionistic—the law of excluded middle not being generally affirmable—mathematics developed within avoids the "unbearable awkwardness" to which Weyl refers above. And while it unquestionably falls within the compass of "symbolic construction", as opposed to immediate intuition, I believe that the simple and elegant way in which it gives expression to the indecomposability of the continuum and to the automatic continuity of functions defined thereon would have been recognized by Weyl as creating a natural bridge between the continua of intuition and formal mathematics. Moreover, smooth infinitesimal analysis embodies a notion of *infinitesimal quantity* making possible the development a truly infinitesimal *physics*, something which Weyl would surely have welcomed.[37]

It seems appropriate to conclude with the passage, from a review paper of 1946, in which Weyl summarizes the effect that the problem of foundations had had on his own work:

This history should make one thing clear: we are less certain than ever work about the ultimate foundations of (logic and) mathematics; like everybody and everything in the world today, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on my enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavours mean in the contexts of man's whole caring and knowing, suffering and creative existence in the world. [38]

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[34] Weyl [1963], 54.
[35] Weyl [1985], 17.
[36] See, e.g., Bell [1998].
[37] Support for this claim may be derived from Weyl's remark on p.92 of *Space-Time Matter* that

> *The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as it is in Riemann's geometry, and, indeed, the mainspring of all the eminent work of Riemann.*

[38] Weyl [1946], 13.

Appendix.

Weyl's Analogy between Egos and Coordinate Systems.

In Weyl [1969], an analogy is presented between coordinate systems and egos, which Weyl here refers to as "subjects". In this analogy objects, subjects, and the appearance of an object to a subject are correlated respectively with *points on a plane, (barycentric) coordinate systems in the plane,* and *coordinates of a point with respect to a such a coordinate system.*

In Weyl's analogy, a coordinate system S consists of the vertices of a fixed nondegenerate triangle T; each point p in the plane determined by T is assigned a triple of numbers summing to 1—its *barycentric coordinates* relative to S—representing the magnitudes of masses of total weight 1 which, placed at the vertices of T, have centre of gravity at p. Thus objects, i.e. points, and subjects i.e., coordinate systems or triples of points belong to the same "sphere of reality." On the other hand, the *appearances* of an object to a subject, i.e., triples of numbers, lie, Weyl asserts, in a different sphere, that of *numbers*. These *number-appearances,* as Weyl calls them, correspond to the experiences of a subject, or of pure consciousness.

From the standpoint of naïve realism the points (objects) simply exist as such, but Weyl indicates the possibility of constructing geometry (which under the analogy corresponds to external reality) solely in terms of number-appearances, so representing the world in terms of the experiences of pure consciousness, that is, from the standpoint of idealism. Thus suppose that we are given a coordinate system S. Regarded as a subject or "consciousness", from its point of view a point or object now corresponds to what was originally an appearance of an object, that is, a triple of numbers summing to 1; and, analogously, any coordinate system S' (that is, another subject or "consciousness") corresponds to three such triples determined by the vertices of a nondegenerate triangle. Each point or object p may now be *identified* with its coordinates relative to S. The coordinates of p relative to any other coordinate system S' can be determined by a straightforward algebraic transformation: these coordinates represent the *appearance* of the object corresponding to p to the subject represented by S'. Now these coordinates will, in general, *differ* from those assigned to p by our given coordinate system S, and will in fact coincide for all p if and only if S' is what is termed by Weyl the *absolute* coordinate system consisting of the three triples (1,0,0), (0,1,0), (0,0,1), that is, the coordinate system which corresponds to S itself. Thus, for this coordinate system, "object" and "appearance" coincide, which leads Weyl to term it the *Absolute I.* [39]

Weyl points out that this argument takes place entirely within the realm of numbers, that is, for the purposes of the analogy, the *immanent consciousness.* In order to do justice to the claim of objectivity that all "I"s are equivalent, he suggests that only such numerical relations are to be declared of interest as remain unchanged under passage from an "absolute" to an arbitrary coordinate system, that is, those which are invariant under arbitrary linear coordinate transformations. When this scheme is given a purely axiomatic formulation, Weyl sees a third viewpoint emerging in addition to that of realism and idealism, namely, a *transcendentalism* which "postulates a transcendental reality but is satisfied with modelling it in symbols."[40]

[39] This phrase Weyl derives from Fichte, whom he quotes as follows:

> *The I demands that it comprise all reality and fill up infinity. This demand is based, as a matter of necessity, on the idea of the infinite I; this is the absolute I (which is not the I given in real awareness.*

[40] But Weyl, ever-sensitive to the claims of subjectivity, goes on to say that this scheme does not resolve the enigma of selfhood. In this connection he refers to Leibniz's attempt to resolve the conflict between human freedom and divine predestination by having God select for

Interestingly, by the time this was written, Weyl seems to have moved away somewhat from the phenomenology that originally suggested the geometric analogy. For he asserts that a number of Husserl's theses become "demonstratively false" when translated into the context of the analogy, "something which," he opines, "gives serious cause for suspecting them." Unfortunately, he does not specify which of Husserl's theses he has in mind.

References

BELL, JOHN L. [1998] *A Primer of Infinitesimal Analysis*. Cambridge University Press.

HILBERT, D. [1926] 'On the Infinite.' Translated from German original in *Mathematicsche Annalen* **95** (1926). In Jean van Heijenoort, ed., *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. Harvard University Press, 1981.

MANCOSU, P. [1998] *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the1920s*. Oxford: Clarendon Press.

WEYL, H. [1929] 'Consistency in Mathematics". Rice Institute Pamphlet 16, 245-265. Reprinted in Weyl [1968] II, 150-170.

_________[1932] *The Open World: Three Lectures on the Metaphysical Implications of Science.* Yale University Press.

_________[1940] 'The Ghost of Modality'. Philosophical Essays in Memory of Edmund Husserl, Harvard University Press. Reprinted in Weyl [1968] III, 684-709.

_________[1946] 'Mathematics and Logic: A brief survey serving as a preface to a review of *The Philosophy of Bertrand Russell.*' *American mathematical Monthly* **53,** 2-13.

__________[1950] *Space-Time-Matter*, tr. Henry L. Brose. New York: Dover, 1950. (English translation of *Raum, Zeit, Materie,* Berlin: Springer Verlag, 1918.)

__________[1954] 'Address on the Unity of Knowledge.' Columbia University Bicentennial Celebration, 1954. Reprinted in Weyl [1968] IV, 623 - 630.

__________[1963] *Philosophy of Mathematics and Natural Science.* New York: Atheneum. (An expanded Engish version of *Philosophie der Mathematik und Naturwissenschaft,* München: Leibniz Verlag, 1927.)

__________[1968] *Gesammelte Abhandlungen,* I-IV, K. Chandrasehharan, ed. Berlin: Springer-Verlag.

__________[1969] "Insight and Reflection". (Lecture delivered at the University of Lausanne, Switzerland, May 1954. Translated from German original in *Studia Philosophica,* **15**, 1955.) In T.L. Saaty and F.J. Weyl, eds., *The Spirit and Uses of the Mathematical Sciences,* 281-301. New York: McGraw-Hill.

__________[1985] 'Axiomatic versus Constructive Procedures in Mathematics.' , ed. T. Tonietti. *Mathematical Intelligencer* **7** , no. 4, 10-17, 38.

__________[1987] *The Continuum: A Critical Examination of the Foundation of Analysis,* tr. S. Pollard and T. Bole. Kirksville, Mo.: Thomas Jefferson University Press. (English translation of *Das Kontinuum,* Leipzig: Veit, 1918.)

__________[1998] 'On the New Foundational Crisis in Mathematics'. (English Translation of 'Über der neue Grundslagenkrise der Mathematik,' *Mathematische Zeitschrift* **10**, 1921, 37-79.)

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existence, on the grounds of sufficient reason, certain beings, such as Judas and St. Peter, whose nature thereafter determines their entire history. Concerning this solution Weyl remarks

> *[it] may be objectively adequate, but it is shattered by the desperate cry of Judas: Why did* I *have to be Judas! The impossibility of an objective formulation to this question strikes home, and no answer in the form of an objective insight can be given. Knowledge cannot bring the light that is I into coincidence with the murky, erring human being that is cast out into an individual fate.* (Weyl [1969], 297.)

In Mancosu [1998], 86 - 122.
__________[1998a] 'On the Current Epistemological Situation in Mathematics.' English translation of 'Die Heutige Erkenntnislage in der Mathematik, *Symposion* **1**, 1925-27, 1-32.) In Mancosu [1998], 123-142.
VAN DALEN, D. [1995] "Hermann Weyl's Intuitionistic Mathematics', *Bulletin of Symbolic Logic* **1**, no. 2, 145-169.