INFINITESIMALS

Traditionally, an *infinitesimal quantity* is one which, while not necessarily coinciding with zero, is in some sense smaller than any finite quantity. In "practical" approaches to the differential calculus an infinitesimal is a quantity so small that its square and all higher powers can be "neglected". In the theory of limits the term "infinitesimal" is sometimes applied to any sequence whose limit is zero. An *infinitesimal magnitude* may be regarded as what remains after a continuum (*v.* the article "Continuity") has been subjected to an exhaustive analysis, in other words, as a continuum "viewed in the small". It is in this sense that continuous curves have sometimes been held to be "composed" of infinitesimal straight lines.

Infinitesimals have a long and colourful history. They make an early appearance in the mathematics of the Greek atomist philosopher Democritus (c.450 B.C.), only to be banished by the mathematician Eudoxus (c.350 B.C.) in what was to become official "Euclidean" mathematics. Taking the somewhat obscure form of "indivisibles", they reappear in the mathematics of the late middle ages and later played an important role in the development of the calculus. Their doubtful logical status led in the nineteenth century to their abandonment and replacement by the limit concept.

In recent years, however, the concept of infinitesimal has been refounded on a rigorous basis. First, in the 1960s, Abraham Robinson, using methods of mathematical logic, created *nonstandard analysis* (NSA, *q.v.*), in which infinitesimals are introduced as infinitely small numbers satisfying the usual laws of arithmetic. And in the 1970s new developments in *category theory* (**NSA**) (*q.v.*) have made it possible to construct a consistent mathematical framework incorporating nonzero infinitesimals whose squares (and higher powers) literally vanish. This framework, known as *synthetic differential geometry* (**SDG**), is a consistent formulation of the *smoothly continuous world*: in it all functions or correlations between mathematical objects are smooth, that is, continuous and differentiable arbitrarily many times. In **SDG** the part *D* of the real line consisting of all infinitesimals plays a special role in that it can only be subjected to translations and rotations: while being larger than zero, it is still too small to "bend" or "break". As a consequence, any curve may be regarded as being "composed" of copies of *D,* that is, of infinitesimally small straight lines. These facts make it possible in **SDG** to develop differential geometry and the calculus in a direct intuitive manner, avoiding the use of limits.

The infinitesimals of **SDG** are to be contrasted with those of **NSA**. In **NSA**, the field ℝ of real numbers is "enlarged" to a field ℝ* containing "infinitely large" elements in such a way as to preserve the usual algebraic properties of ℝ: infinitesimals are then obtained as multiplicative inverses of these infinitely large elements. The infinitesimals of **SDG**, on the other hand, do not possess multiplicative inverses and cannot be obtained in this way. In **NSA**, infinitesimals are not so in an absolute sense, but only in relation to the elements of ℝ: that is, speaking metaphorically, an "observer" situated within a model of **NSA** would be unable to detect the presence of infinitesimals in ℝ*. By contrast, the characteristic property of having a vanishing square, possessed by the infinitesimals of **SDG**, is an intrinsic property, perfectly detectable within any of its models.

Bibliography

Bell, J.L., *Infinitesimals.* Synthese 75 (1988), 285-315.

Bell, J.L., *Infinitesimals and the Continuum.* Mathematical Intelligencer 17,2 (1995), 55-57.

Kock, A., *Synthetic Differential Geometry.* Cambridge: Cambridge University Press, 1981.

Moerdijk, I. and Reyes, G.E., *Models for Smooth Infinitesimal Analysis.* New york: Springer- Verlag, 1991.

Robinson, A., *Non-standard Analysis.* Amsterdam: North-Holland, 1966.