Introduction

Continuous as the stars that shine
And twinkle on the milky way,
They stretched in never-ending line
Along the margin of a bay:
Ten thousand saw I at a glance,
Tossing their heads in sprightly dance.

William Wordsworth

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

William Blake

The homeland, friends, is a continuous act
As the world is continuous.

Jorge Luis Borges

We are all familiar with the idea of *continuity*. To be continuous[1] is to constitute an unbroken or uninterrupted whole, like the ocean or the sky. A continuous entity—a *continuum*[2]—has no "gaps". Opposed to continuity is *discreteness*: to be discrete[3] is to be separated, like the scattered pebbles on a beach or the leaves on a tree. Continuity connotes unity; discreteness, plurality.

The realm of the continuous is traditionally associated with intuition, that of the discrete, with reason. The discrete is a model of tidiness in which quality is reduced to quantity and over which the concept of *number* reigns supreme. Populated by units lacking intrinsic qualities and so wholly indistinguishable from one another, in the dominion of the discrete difference is manifested through plurality alone. The simplicity of the principles governing discreteness has recommended it as a paragon of intelligibility, a realm within which reason can be realized to its fullest extent. By contrast, the continuous is a jungle, a labyrinth. It teems with such exotic and intractable entities as incommensurable lines, horn angles, space curves, one-sided surfaces. The taming of this jungle by reduction to the discrete has been a principal task, if not the principal task, of mathematics.

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[1] The word "continuous" derives from a Latin root meaning "to hang together" or "to cohere"; this same root gives us the nouns "continent"—an expanse of land unbroken by sea—and "continence"—self-restraint in the sense of "holding oneself together". Synonyms for "continuous" include: connected, entire, unbroken, uninterrupted.

[2] The term "continuum" has become a buzzword, especially popular with science fiction writers (e,g,. Ballard 1993). This was surely the result of (continued) exposure to the phrase "space-time continuum" with which popular accounts of relativity theory have been peppered.

[3] The word "discrete" derives from a Latin root meaning "to separate". This same root yields the verb "discern"—to recognize as distinct or separate—and the cognate "discreet"—to show discernment, hence "well-behaved". It is a curious fact that, while "continuity" and "discreteness" are antonyms, "continence" and "discreetness" are synonyms. Synonyms for "discrete" include separate, distinct, detached, disjunct.

While the constituency of the continuous harbours many complex entities, it is ultimately grounded in such apparently simple notions as space, time, and extension, continua at the very core of our experience. And certain philosophers have maintained that all natural processes occur continuously: witness, for example, Leibniz's famous apothegm *natura non facit saltus*—"nature makes no jump".

In mathematics the word "continuous" is used in a sense close to its ordinary meaning, but has come to be furnished with increasingly precise definitions. So, for instance, in the later 18th century continuity of a function was taken to mean that infinitesimal changes in the value of the argument induce infinitesimal changes in the value of the function. With the abandonment of infinitesimals in the 19th century this definition gave way to one employing the more precise concept of *limit*.

While it is the fundamental nature of a continuum to be *undivided*, it is nevertheless generally (although not invariably) held that any continuum admits of repeated or successive *division without limit.* This means that the process of dividing it into ever smaller parts will never terminate in an *indivisible* or an *atom*—that is, a part which, lacking proper parts itself, cannot be further divided. In a word, continua are *divisible without limit* or *infinitely divisible.* The unity of a continuum thus conceals a potentially infinite plurality. In antiquity this claim met with the objection that, were one to carry out completely—if only in imagination—the process of dividing an extended magnitude, such as a continuous line, then the magnitude would be reduced to a multitude of atoms—in this case, extensionless points—or even, possibly, to nothing at all. But then, it was held, no matter how many such points there may be—even if infinitely many—they cannot be "reassembled" to form the original magnitude, for surely a sum of extensionless elements still lacks extension[4]. Moreover, if indeed (as seems unavoidable) infinitely many points remain after the division, then, following Zeno, the magnitude may be taken to be a (finite) motion, leading to the seemingly absurd conclusion that infinitely many points can be "touched" in a finite time.

Such difficulties attended the birth, in the 5th century B.C., of the school of *atomism.* The founders of this school, Leucippus and Democritus, claimed that matter, and, more generally, extension, is not infinitely divisible. Not only would the successive division of matter ultimately terminate in atoms, that is, in discrete particles incapable of being further divided, but matter had *in actuality* to be conceived as being compounded from such atoms. In attacking infinite divisibility the atomists were at the same time mounting a claim that the continuous is ultimately reducible to the discrete, whether it be at the physical, theoretical, or perceptual level. Atomism was to flower into a general doctrine of the reducibility of the complex to the simple[5]: in addition to the physical atomism of the ancients, one can identify *epistemological* atomism, or the doctrine of units of perception; *linguistic* atomism, the alphabetic principle[6]; *logical* atomism, the positing of atomic or elementary propositions; and *biological* atomism, the postulation of discrete organic units such as cells or genes. A version of atomism can also be found in *mathematics*, namely, the doctrine—originating with the Pythagoreans of the 6th century B.C.—that all mathematical concepts are ultimately reducible to numbers.

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[4] Of course, this presupposes that there are no "gaps" between the elements or points, which is implicit in the assumption that the points have been obtained by complete division of a continuum.

[5] Whyte (1961), p. 12.

[6] It has been suggested that the emergence of atomism is connected with the alphabetic principle on which the great majority of natural (written) languages rest. In Needham (1954–), vol. 4, 26(b), the parallel is noted between the limitless variety of words formable from the relatively few letters of the alphabet, and the idea that a very small number of "elementary" particles could, in a multitude of combinations, engender the limitless variety of material bodies. But in China atomism never really took root (see below); in this connection Needham observes, "the Chinese written character is an organic whole, a Gestalt, and minds accustomed to an ideographic language would perhaps hardly have been so open to the idea of an atomic constitution of matter." As Needham points out, however, the Chinese recognized the function of the atomic principle in numerous contexts, for example the reduction of written characters to radicals, the composition of melodies from the notes of the pentatonic scale, and the representation of Nature through the permutations and combinations of the broken and unbroken lines in the hexagrams of their ancient work of divination the *I Ching.*

The eventual triumph of the atomic theory in physics and chemistry in the 19th century paved the way for the idea of "atomism", as applying to matter, at least, to become widely familiar: it might well be said, to adapt Sir William Harcourt's famous observation in respect of the socialists of his day, "We are all atomists now." Nevertheless, only a minority of philosophers of the past espoused atomism at a metaphysical level, a fact which may explain why the analogous doctrine upholding continuity lacks a familiar name: that which is unconsciously acknowledged requires no name. Peirce, that great wordsmith, coined the term *synechism* (from Greek *syneche,* "continuous") for his own philosophy—a philosophy permeated by the idea of continuity in its sense of "being connected"[7]. In what follows I shall appropriate Peirce's term and use it in a sense shorn of its Peircean overtones, simply as a contrary to atomism. I shall also use the term "divisionism" for the more specific doctrine that continua are infinitely divisible.

Closely associated with the concept of a continuum is that of *infinitesimal*[8]. An *infinitesimal magnitude* has been somewhat hazily conceived as a continuum "viewed in the small", an "ultimate part" of a continuum. In something like the same sense as a discrete entity is made up of its individual units, its "indivisibles", so, it was maintained, a continuum is "composed" of infinitesimal magnitudes, its ultimate parts. (It is in this sense, for example, that mathematicians of the 17th century held that continuous curves are "composed" of infinitesimal straight lines.) Now the "coherence" of a continuum entails that each of its (connected) parts is also a continuum, and, accordingly, divisible. Since points are indivisible, it follows that no point can be part of a continuum. Infinitesimal magnitudes, as parts of continua, cannot, of necessity, be points: they are, in a word, *nonpunctiform.*

Magnitudes are normally taken as being *extensive* quantities, like mass or volume, which are defined over extended regions of space. By contrast, infinitesimal magnitudes have been construed as *intensive* magnitudes resembling locally defined intensive quantities such as temperature or density. The effect of "distributing" or "integrating" an intensive quantity over such an intensive magnitude is to convert the former into an infinitesimal extensive quantity: thus temperature is transformed into infinitesimal heat and density into infinitesimal mass. When the continuum is the trace of a motion, the associated infinitesimal/intensive magnitudes have been identified as *potential* magnitudes—entities which, while not possessing true magnitude themselves, possess a *tendency* to generate magnitude through motion, so manifesting "becoming" as opposed to "being".

An infinitesimal *number* has one which, while not coinciding with zero, is in some sense smaller than any finite number. In "practical" approaches to the differential calculus an infinitesimal is a number so small that its square and all higher powers can be "neglected". In the theory of limits the term "infinitesimal" is sometimes applied to any sequence whose limit is zero.

The concept of an *indivisible* is closely allied to, but to be distinguished from, that of an infinitesimal. An indivisible is, by definition, something that cannot be divided, which is usually understood to mean that it has no proper parts. Now a partless, or indivisible entity does not necessarily have to be infinitesimal: souls, individual consciousnesses, and Leibnizian monads all supposedly lack parts but are surely not infinitesimal. But these have in common the feature of being unextended; extended

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[7] It should also be mentioned that the German philosopher Johann Friedrich Herbart (1776-1841) introduced the term *synechology* for the part of his philosophical system concerned with the continuity of the real.

[8] According to th*e Oxford English Dictionary* the term *infinitesimal* was originally

> *an* ordinal, *viz. the "infinitieth" in order; but, like other ordinals, also used to name fractions, thus* infinitesimal part *or* infinitesimal *came to mean unity divided by infinity* ($\frac{1}{\infty}$), *and thus an infinitely small part or quantity.*

entities such as lines, surfaces, and volumes prove a much richer source of "indivisibles". Indeed, if the process of dividing such entities were to terminate, as the atomists maintained, it would necessarily issue in indivisibles of a qualitatively different nature. In the case of a straight line, such indivisibles would, plausibly, be points; in the case of a circle, straight lines; and in the case of a cylinder divided by sections parallel to its base, circles**.** In each case the indivisible in question is infinitesimal in the sense of *possessing one fewer dimension than its generating figure*. In the 16th and 17th centuries indivisibles in this sense were used in the calculation of areas and volumes of curvilinear figures, a surface being thought of as the sum of linear indivisibles and a volume as the sum of planar indivisibles.

The concept of infinitesimal was beset by controversy from its beginnings. The idea makes an early appearance in the mathematics of the Greek atomist philosopher Democritus c. 450 B.C., only to be banished c. 350 B.C. by Eudoxus in what was to become official "Euclidean" mathematics. We have noted their reappearance as indivisibles in the sixteenth and seventeenth centuries: in this form they were systematically employed by Kepler, Galileo's student Cavalieri, the Bernoulli clan, and a number of other mathematicians. In the guise of the beguilingly named "linelets" and "timelets", infinitesimals played an essential role in Barrow's "method for finding tangents by calculation", which appears in his *Lectiones Geometricae* of 1670. As "evanescent quantities" infinitesimals were instrumental (although later abandoned) in Newton's development of the calculus, and, as "inassignable quantities", in Leibniz's. The Marquis de l'Hôpital, who in 1696 published the first treatise on the differential calculus (entitled *Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes*), invokes the concept in postulating that "a curved line may be regarded as being made up of infinitely small straight line segments," and that "one can take as equal two quantities differing by an infinitely small quantity."

However useful it may have been in practice, the concept of infinitesimal could scarcely withstand logical scrutiny. Derided by Berkeley in the 18th century as "ghosts of departed quantities", in the 19th century execrated by Cantor as "cholera-bacilli" infecting mathematics, and in the 20th roundly condemned by Bertrand Russell as "unnecessary, erroneous, and self-contradictory", these useful, but logically dubious entities were believed to have been finally supplanted in the foundations of analysis by the limit concept which took rigorous and final form in the latter half of the 19th century. By the beginning of the 20th century, the concept of infinitesimal had become, in analysis at least, a virtual "unconcept".

Nevertheless the proscription of infinitesimals did not succeed in extirpating them; they were, rather, driven further underground. Physicists and engineers, for example, never abandoned their use as a heuristic device for the derivation of correct results in the application of the calculus to physical problems. Differential geometers of the stature of Lie and Cartan relied on their use in the formulation of concepts which would later be put on a "rigorous" footing. And, in a technical sense, they lived on in the algebraists' investigations of nonarchimedean fields.

A new phase in the long contest between the continuous and the discrete has opened in the past few decades with the refounding of the concept of infinitesimal on a solid basis. This has been achieved in two essentially different ways, the one providing a rigorous formulation of the idea of infinitesimal *number*, the other of infinitesimal *magnitude.*

First, in the nineteen sixties Abraham Robinson, using methods of mathematical logic, created *nonstandard analysis,* an extension of mathematical analysis embracing both "infinitely large" and infinitesimal numbers in which the usual laws of the arithmetic of real numbers continue to hold, an idea which, in essence, goes back to Leibniz. Here by an infinitely large number is meant one which exceeds every positive integer; the reciprocal of any one of these is infinitesimal in the sense that, while being nonzero, it is smaller than every positive fraction $1/n$. Much of the usefulness of nonstandard analysis stems from the fact that within it every statement of ordinary analysis involving limits has a succinct and highly intuitive translation into the language of infinitesimals.

The second development in the refounding of the concept of infinitesimal is the emergence in the nineteen seventies of *synthetic differential geometry,* also known as *smooth infinitesimal analysis*. Smooth infinitesimal analysis provides an image of the world in which the continuous is an autonomous notion, not explicable in terms of the discrete. Founded on the methods of category theory, it is a rigorous framework of mathematical analysis in which every function between spaces is smooth (i.e., differentiable arbitrarily many times, and so in particular continuous) and in which the use of limits in defining the basic notions of the calculus is replaced by *nilpotent infinitesimals*, that is, of quantities so small (but not actually zero) that some power—most usefully, the square—vanishes. Smooth infinitesimal analysis embodies a concept of intensive magnitude in the form of *infinitesimal tangent vectors* to curves. A tangent vector to a curve at a point *p* on it is a short straight line segment ℓ passing through the point and pointing along the curve. In fact we may take ℓ actually to be an infinitesimal *part* of the curve. Curves in smooth infinitesimal analysis are "locally straight" and accordingly may be conceived as being "composed of" infinitesimal straight lines in de l'Hôpital's sense, or as being "generated" by an infinitesimal tangent vector.

The development of nonstandard and smooth infinitesimal analysis has breathed new life into the concept of infinitesimal, and—especially in connection with smooth infinitesimal analysis—supplied novel insights into the nature of the continuum.

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In the first part of this book we trace the evolution of the concepts of the continuous (along with its opposite, the discrete), and the infinitesimal. The second part is devoted to an exposition of the various ways in which these topics are treated in today's mathematics.