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FROM ABSOLUTE TO LOCAL MATHEMATICS

In this paper (a sequel to [1]) I put forward a "local" interpretation of mathematical concepts based on notions derived from *category theory*. The fundamental idea is to abandon the unique absolute universe of sets central to the orthodox set-theoretic account of the foundations of mathematics, replacing it by a plurality of local mathematical frameworks – *elementary toposes* – defined in category-theoretic terms. Such frameworks will serve as local surrogates for the classical universe of sets. In particular they will possess sufficiently rich internal structure to enable mathematical concepts and assertions to be interpreted within them. With the relinquishment of the absolute universe of sets, mathematical concepts will in general no longer possess absolute meaning, nor mathematical assertions absolute truth values, but will instead possess such meanings or truth values only *locally*, i.e., *relative* to local frameworks. There is an evident parallel between this approach to the interpretation of mathematical concepts and the interpretation of physical concepts within the *theory of relativity*: this we discuss in section 2. Section 3 examines the procedure of passing from one local framework to another, observing that it is an instance of the dialectical process of *negating constancy*. In particular, we show (following F. W. Lawvere) how the construction of models of Robinson's nonstandard analysis and the proofs of Cohen's independence results in set theory may be construed as instances of this procedure.

1. CATEGORY THEORY AND LOCAL MATHEMATICAL FRAMEWORKS

Category theory (cf. [2] or [14]) provides a general apparatus for dealing with mathematical structures and their mutual relations and transformations. Invented by Eilenberg and MacLane in the 1940's, it arose as a branch of algebra by way of topology, but quickly transcended its origins. Category theory may be said to bear the same relation to abstract algebra as the latter does to elementary algebra. For elementary algebra results from the replacement of *constant*

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quantities (i.e., numbers) by *variables*, keeping the operations on these quantities fixed. Abstract algebra, in its turn, carries this a stage further by allowing the *operations* to vary while ensuring that the resulting mathematical structures retain a certain prescribed *form* (groups, rings, or what have you). Finally, category theory allows even the form of the structures to vary, giving rise to a general theory of mathematical structure or form. Thus the genesis of category theory is an instance of the dialectical process of replacing the *constant* by the *variable*, a theme which will play an important role in what I have to say here.

In category theory the transformations (called *arrows*) between structures (called *objects*) play an autonomous role which is in no way subordinate to that played by the structures themselves. (Thus category theory is like a language in which the verbs are on an equal footing with the nouns.) In this respect category theory differs crucially from *set theory* where the corresponding notion of *function* is reduced to the concept of *set* (of points). As a consequence, the notion of transformation in category theory is vastly more general than the set-theoretic notion of function. In particular, for example, the concept of category-theoretic transformation will admit interpretations in which one variable quantity depends functionally on another but where the corresponding "function" is not describable as a set of (ordered pairs of) "points" (for instance when the functional dependence arises as the phenomenological description of the motion of a body).

The generality of category theory has enabled it to play an increasingly important role in the *foundations of mathematics*. Its emergence has had the effect of subtly undermining the prevailing doctrine that all mathematical concepts are to be referred to a fixed absolute universe of sets. Category theory, in contrast, suggests that mathematical concepts and assertions should be regarded as possessing meaning only *in relation to* a variety of more or less *local* frameworks. To indicate what I mean, let us follow MacLane [14] in considering the category-theoretic interpretation of the concept "group". From the set-theoretical point of view, the term "group" signifies a *set* (equipped with a couple of operations) satisfying certain elementary axioms expressed in terms of the elements of the set. Thus the set-theoretical interpretation of this concept is always referred to the *same* framework, the *universe of sets*. Now consider the category-theoretic account of the group concept. Here the reference to the "elements" of the group has been replaced by an "arrows only" formulation, thereby enabling the concept to become interpretable not merely in the universe (category) of sets, but in essentially *any* category. The possibility of varying the framework of interpretation offered by category theory confers on the group concept a truly protean generality. Indeed, the interpretation of the term "group" within the category of topological spaces is *topological group*, within the category of differentiable manifolds it is *Lie group* and within the category of sheaves over a topological space it is *sheaf of groups*.

We see that the category-theoretic meaning of a mathematical concept such as *group* is now determined only *in relation* to an ambient category, and this ambient category can *vary*. Thus the effect of casting a mathematical concept in categorical language is to confer a further degree of *ambiguity of reference* on the concept.

To some extent this ambiguity of reference of mathematical concepts is already present within classical set theory, since its axioms are formulated in *first-order* terms and therefore admit many essentially *different* interpretations. Indeed, as far back as 1922, Skolem [15] remarked that for this reason set-theoretical notions – in particular, infinite cardinalities – are *relative*. He concluded that axiomatized set theory "was not a satisfactory ultimate foundation for mathematics". Skolem's structures were largely ignored by mathematicians, but a new challenge to the absoluteness of the set-theoretical framework arose in 1963 when Paul Cohen constructed models of set theory (i.e., Zermelo-Fraenkel set theory ZF) in which important mathematical *propositions* such as the continuum hypothesis and the axiom of choice are falsified (Gödel having already in the late 1930's produced models in which the propositions are validated). The resulting ambiguity in the *truth values* of mathematical propositions was regarded by many set-theorists (and even by more "orthodox" mathematicians) as a much more serious matter than the "mere" ambiguity of reference of mathematical concepts already pointed out by Skolem. In fact, the techniques of Cohen and his successors have led to an enormous proliferation of models of set theory with essentially different mathematical properties, which in turn has engendered a disquieting uncertainty in the minds of set-theorists as to the identity of the "real" universe of sets, or at least as to precisely what mathematical properties it should possess. The upshot is that the set concept – in so far as it is capturable by first order axioms – has turned out to be *radically underdetermined*.

What I suggest is that we *accept* the radically underdetermined nature of the set concept and *abandon* the quest for the absolute universe of sets in the form proposed by classical set theory. It should be emphasized that this does *not* necessarily require that we espouse the extreme formalist/finitist view (or gospel of despair) that set-theoretical concepts (or those involving the infinite, at any rate) are meaningless. No, I think the answer lies in recognizing that the meaning (or reference) of these concepts is determined only *relative* to models of ZF, or, more generally, to the local frameworks of interpretation to be introduced in a moment. In this event, an assertion like the continuum hypothesis will no longer be regarded as the possessor of an absolute but unknown truth value, for the unique universe of sets which was presumed to furnish the said truth value will no longer exist. Note, however, that although the concept of *absolute truth* of set-theoretical assertions will have vanished from the scene, there will appear in its place the subtler concept of *invariance*, that is, *validity in all local frameworks*. Thus, e.g., whereas the theorems of constructive arithmetic *will* turn out to possess the property of invariance, set-theoretical assertions such as the axiom of choice, or the continuum hypothesis, will *not*, because they will hold true in some local frameworks but not in others.

Having reached the point where models of set theory are treated as local frameworks of interpretation for mathematical concepts, or in other words, having replaced the concept of the unique universe of sets by the concept of a varying framework of interpretation, it becomes natural to attempt to formulate these ideas within the conceptual language best equipped to handle variation: the language of *category theory*. Thus the first thing we require is a category-theoretic formulation of the notion of *model of set theory*. This we find in Lawvere and Tierney's concept of an (elementary) *topos* (see, e.g., [3], [9], [10]).

To arrive at the concept of a topos, we start with the familiar category **S** of sets whose objects are all sets (in a given model M of set theory) and whose arrows are all mappings (in M) between sets in M. We observe that **S** has the following properties.

(i) There is a 'terminal' object 1 such that, for any object X, there is a unique arrow $X \to 1$ (for 1 we may take any one-element set, in particular $\{0\}$).

(ii) Any pair of objects A, B has a Cartesian product $A \times B$.

(iii) For any pair of objects A, B one can form the 'exponential' object B^A of all mappings $A \to B$.

(iv) There is an 'object of truth values' Ω such that for each object X there is a natural correspondence between subobjects (subsets) of X and arrows $X \to \Omega$. (For Ω we may take the set $\{0, 1\}$; arrows $X \to \Omega$ are then *characteristic functions* on X, and the exponential object Ω^X corresponds to the *power set* of X.)

All four of the above conditions can be formulated in purely category-theoretic (arrows only) language: a (small) category satisfying them is called a *topos*.

The concept of a topos is in fact much more general than that of (the category of sets in) a model of set theory in the original sense. This is revealed by the fact that, in addition to **S**, all of the following are toposes: (1) the category $V^{(B)}$ of Boolean-valued sets and mappings within a Boolean extension (cf. [1]) of a model of set theory; (2) the category of sheaves (or presheaves) of sets on a topological space; (3) the category of all diagrams of mappings of sets

$$X_0 \to X_1 \to X_2 \to \cdots.$$

Evidently the objects of each of these categories may be regarded as sets which are *varying* in some manner: in case (1) over a *Boolean algebra*, in case (2) over a *topological space* and in case (3) over *discrete time*. (So, in this parlance, the category **S** itself is the category of sets "varying" over the one point set 1.) These examples suggest that a topos may be conceived of as a category of *variable sets*: the familiar category **S** we started with is the "limiting" case in which the variation of the objects has been reduced to zero. For this reason, **S** is called a *topos of constant sets*. Thus, the category-theoretic formulation of the set concept – the notion of topos – turns out to be, as does the notion of category itself, another consequence of the dialectical procedure of replacing the *constant* by the *variable*.

In any model of set theory one has natural "logical operations" $\wedge$

(conjunction), $\vee$ (disjunction), $\neg$ (negation), $\rightarrow$ (implication), defined on the object of truth values $2 = \{0, 1\}$ corresponding in the usual way to the set theoretical operations $\cap$, $\cup$, $-$, $\Rightarrow$ of intersection, union, complementation and residuation. The richness of a topos' internal structure enables this correspondence to be carried through there as well. Thus, in any topos $\mathbf{E}$ we get natural arrows defined on its object $\Omega_{\mathbf{E}}$ of truth values which may be thought of as internally defined "logical operations" in $\mathbf{E}$. Since these logical operations are defined entirely in terms of the internal mathematical structure of $\mathbf{E}$, a topos may be regarded as an apparatus for synthesizing *logic* from *mathematics*. The remarkable thing is that the logic so obtained is, in general, *intuitionistic*; in other words, the logical algebra $\boldsymbol{\Omega}_{\mathbf{E}} = (\Omega_{\mathbf{E}}, \wedge, \vee, \neg, \Rightarrow)$ is a *Heyting algebra* (cf. [9]). For example, when $\mathbf{E}$ is the topos $\mathbf{Shv}(X)$ of sheaves on a topological space X, $\boldsymbol{\Omega}_{\mathbf{E}}$ is the Heyting algebra of open sets in X. On the other hand, when $\mathbf{E}$ is a Boolean extension $V^{(B)}$, $\boldsymbol{\Omega}_{\mathbf{E}}$ is the Boolean algebra B, so in this case the internal logic of $\mathbf{E}$ is *classical*.

The fact that propositional logic is interpretable in any topos $\mathbf{E}$ is really only the beginning. For by employing the "internal completeness" of the truth value object Ω one can provide interpretations of the quantifiers $\forall$ and $\exists$ and so enable statements of first-order logic to become interpretable in $\mathbf{E}$. Moreover, by exploiting the presence, for any object A of $\mathbf{E}$, of the exponentials Ω^A, Ω^{Ω^A}, etc., which may be regarded as representing the collections of properties, properties of properties, etc., defined over A, we find that statements of higher-order logic become interpretable in $\mathbf{E}$, just as in an ordinary model of set theory. (And then the truth value object Ω of $\mathbf{E}$ represents the domain of possible "truth values" of such statements in $\mathbf{E}$.) In fact, it can be shown (cf. [7]) that toposes are precisely the models for theories formulated within a natural typed higher order language based on intuitionistic logic. Each topos $\mathbf{E}$ is associated with such a language whose types match the objects of $\mathbf{E}$ and whose function symbols match the arrows of $\mathbf{E}$. A theory in such a language is a set of sentences closed under intuitionistically valid deductions. Given such a theory T, a topos $\mathbf{E}_T$ can be constructed which is a model of T, and conversely, given a topos $\mathbf{E}$, we can form a theory $T_{\mathbf{E}}$ (the set of sentences "true" in $\mathbf{E}$) whose associated topos $\mathbf{E}_{T_{\mathbf{E}}}$ is categorically equivalent to $\mathbf{E}$.

Thus higher-order typed intuitionistic theories (which we shall call simply "theories") and toposes are essentially equivalent. Each topos **E** may be regarded as being *formally described* by its associated theory $T_{\mathbf{E}}$, and each theory T as being *concretely realized or embodied* by its associated topos $\mathbf{E}_T$. A theory T may be regarded as a *generalized set theory* and a topos which is a model of T as a *local universe of discourse* within which the mathematical assertions made by T are true and the constructions sanctioned by T can be carried out.

Two constructions of paramount importance in mathematics are those of the *natural numbers* and *real numbers*. In set theory the construction of the set of natural numbers is sanctioned by the axiom of infinity, and the set of real numbers then essentially obtained as the power set of the set of natural numbers. The procedure in topos theory is similar, except that the axiom of infinity is replaced by the so-called *Peano-Lawvere axiom* (cf. [9]) which asserts the existence of an *object of natural numbers* characterized by the universal possibility of defining functions on it by recursion. Given such an object N (which can be shown to be unique up to isomorphism) in a topos **E**, the object of real numbers (the counterpart within **E** of the set of reals) may then be defined in terms of the exponential object Ω^N by imitating the usual classical procedures (i.e., Cauchy sequences or Dedekind cuts: but note that, in contrast with the classical case, in a general topos these techniques may lead to non-isomorphic results!).

Henceforth I shall use the term *local* (mathematical) *framework* for *topos with an object of natural numbers*. These local frameworks now become the generalized models of set theory or local universes of discourse to which mathematical concepts are to be referred: thus arises the *local interpretation* of mathematical concepts. Analogously, the theories associated with these local frameworks are the generalized set theories in which mathematical constructions and assertions are to be codified.

Return for a moment to the case of the topos (now a local framework) $\mathbf{Shv}(X)$ of sheaves over a topological space X. Thinking of the Heyting algebra $\mathbf{O}(X)$ of open sets as a domain of truth values, it is natural to regard $\mathbf{Shv}(X)$ as the generalized model of set theory "generated" by this domain of truth values. (When X is the one-point space 1, $\mathbf{O}(X)$ is essentially the two element set $\{0, 1\}$ so, since $\mathbf{Shv}(1)$ is a topos **S** of constant sets, the latter is, as one would confidently expect, the model of set theory generated by the classical truth value domain $\{0, 1\}$.) It is also suggestive (and consonant with the origins of the concept of topos) to regard $\mathbf{Shv}(X)$ as the *generalized topological space* obtained from X by building a set-theoretical structure on the open sets of X. With this idea in mind, any topos (local framework!) may also be conceived as a *generalized space* as well as a generalized model of set theory. (This is the viewpoint suggested by algebraic geometry: the original source of the concept of topos.) The resulting interplay of topological and set-theoretical concepts is a key feature of topos theory.

I have emphasized that it is in the spirit of category theory to regard no framework as absolute. This tendency is realized by the possibility of *moving* from one category to another via the concept of *functor*. A functor may be regarded as a transformation between categories that preserves their basic categorical structure. Now when the categories concerned are local frameworks (toposes!), there is a stronger notion of transformation available, which we shall call *admissible* or *continuous* transformation. Formally, an admissible transformation $f: \mathbf{E} \to \mathbf{F}$ between a local framework $\mathbf{E}$ and a local framework $\mathbf{F}$ is a pair of functors $f^*: \mathbf{E} \to \mathbf{F}$, $f_*: \mathbf{F} \to \mathbf{E}$ where f_* is right adjoint to f^*, which in turn is left exact. (Topos theorists will note that this is the opposite of a geometric morphism.) In the "geometric" case where $\mathbf{E}$ and $\mathbf{F}$ are the toposes of sheaves over topological spaces X and Y and we think of $\mathbf{E}$ and $\mathbf{F}$ as generalized *spaces*, the admissible transformations between $\mathbf{E}$ and $\mathbf{F}$ correspond exactly to the *continuous* maps from Y to X: this is the source of the term "continuous". The functors f^*, f_* are called the *components* of f. If there is an admissible transformation between a framework $\mathbf{E}$ and a framework $\mathbf{F}$, then $\mathbf{F}$ is said to be *defined* over $\mathbf{E}$. This terminology is suggested by the fact that, if $\mathbf{S}$ is a framework of constant sets and $f: \mathbf{S} \to \mathbf{F}$ is any admissible transformation to a framework $\mathbf{F}$, then the components of f are given by (for I, X objects of $\mathbf{S}$, $\mathbf{F}$ respectively):

> $f^*(I) = I$-fold copower (disjoint union) of 1 in $\mathbf{F}$; $f_*(X) =$ set in $\mathbf{S}$ of "elements" of X, i.e., arrows $1 \to X$ in $\mathbf{F}$.

That is, we may think of $f^*(I)$ as the "representative" in $\mathbf{F}$ of the constant set I and $f_*(X)$ as the "extent" or "projection in $\mathbf{S}$" of the "variable" set X.

The possibility of shifting via admissible transformations from one local framework to another is central to the interpretation of mathematical concepts proposed here, and points up its essentially kinetic and relational character. In this connection one is struck by the evident analogy with the physical geometric notion of *change of coordinate system*. And indeed, just as in astronomy one effects a change of coordinate system to simplify the description of the motion of a planet, so also it proves possible to simplify the formulation of a mathematical concept by effecting a shift of local mathematical framework. Consider, for example, the concept "real-valued continuous function on a topological space X" (interpreted in a topos $\mathbf{S}$ of constant sets). Any such function may be regarded as a real number (or quantity) *varying continuously* over X. Now consider the topos $\mathbf{Shv}(X)$ of sheaves over X. Here *everything* is varying (continuously) over X, so shifting to $\mathbf{Shv}(X)$ from $\mathbf{S}$ essentially amounts to placing oneself in a framework which is, so to speak, itself "moving along" with the variation over X of the given variable real numbers. This causes the variation of any variable real number not to be "noticed" in $\mathbf{Shv}(X)$; it is accordingly there regarded as being a *constant* real number. In this way the concept "real-valued continuous function on X" is transformed into the concept "real number" when interpreted in $\mathbf{Shv}(X)$. (To be strictly precise, the objects in $\mathbf{Shv}(X)$ satisfying the condition of being (Dedekind) real numbers correspond, via the admissible transformation $\mathbf{S} \to \mathbf{Shv}(X)$, to the real-valued continuous functions on X.) Putting it the other way around, the concept "real number" interpreted in $\mathbf{Shv}(X)$ corresponds to the concept "real-valued continuous function on X" interpreted in $\mathbf{S}$. This observation provides the basis for various proofs of independence in intuitionistic analysis (analysis interpreted in $\mathbf{Shv}(X)$: cf. [8]).

We give two other examples of this procedure. Let B be a complete Boolean algebra of commuting projections on a Hilbert space H. Then the real numbers in $V^{(B)}$, the Boolean extension of V by $V^{(B)}$, correspond to those *self-adjoint operators* on H whose spectral components lie in B (cf. [16]). This provides the basis for Takeuti and Davis's approach ([6]) to the foundations of quantum mechanics in which they suggest that "quantizing" a statement of classical physics amounts to interpreting it in $V^{(B)}$. Finally, if B is the reduced measure algebra of a measure space Z, then the real numbers in $V^{(B)}$ corresponds to *measurable functions* on Z. This yields Takeuti's Boolean-valued analysis according to which real analysis interpreted in $V^{(B)}$ corresponds to the theory of measurable functions in $\mathbf{S}$ (cf. [16]).

We see, then, that varying the local framework of interpretation transforms the concept "real number" into the concepts "continuous function", "measurable function" or even "self-adjoint operator". This provides striking testimony that under the local interpretation a mathematical concept possessing a fixed *sense* now inevitably has a variable *reference*. Indeed, we may think of the *sense* of the concept of "domain of all real numbers" (i.e., the continuum) as being fixed by its definition within an appropriate theory, while, as we have seen, its *reference* varies with the local framework of interpretation. This resolves, or rather dissolves, the dilemma of classical set theory in which a concept such as "domain of all real numbers", although surely *intended* to possess a unique reference in fact *cannot* because of the first-order formulation of its definition within the language of set theory. The local interpretation not only accepts this variability of reference but welcomes it and assigns it a central position.

2. SOME ANALOGIES WITH THE THEORY OF RELATIVITY

The local interpretation of mathematical concepts, based as it is on category theory, has an essentially *relational* character. According to the local interpretation, the reference of a mathematical concept, insofar as it can be construed as an *entity*, is no longer to be regarded as being a thing in itself, whose nature is independent of other things, and whose characteristic properties are entirely intrinsic to it. On the contrary, the properties of a mathematical entity are now determined by, and indeed only have meaning in terms of, the totality of its relationships with other entities.

The recognition that properties originally held to be intrinsic must instead be treated as relational has arisen frequently in the history of thought. For example, Leibniz recognized that a state of rest or motion of a material body is not an intrinsic state of the body but only has meaning in relation to other bodies. One of the profoundest instances of this phenomenon arose in the transition from *classical* (Newtonian) to *relativistic physics*, when physical concepts such as simultaneity of events and mass of a body formerly ascribed an absolute meaning were seen to possess meaning only in relation to *local coordinate systems*.

There is an evident *analogy* between local mathematical frameworks and the local coordinate systems of relativity theory: each serve as the appropriate *reference frames* for fixing the meaning of mathematical, or physical concepts respectively. (And we have already mentioned the analogy between admissible transformations of frameworks and changes of coordinate system.) Pursuing this analogy suggests certain further parallels.

Thus, for example, consider the concept of *invariance*. In relativistic physics *invariant physical laws* are statements of mathematical physics (e.g., Maxwell's equations) which, suitably formulated, hold *universally*, i.e., in every local coordinate system. Analogously, *invariant mathematical laws* are mathematical assertions which again hold universally, i.e., in every local mathematical framework. These in fact turn out to be the theorems of higher order intuitionistic logic with a natural number system, which include, for instance, the theorems of intuitionistic arithmetic but not the axiom of choice or the law of excluded middle. Thus the invariant mathematical laws are those which are demonstrable *constructively*: this points up the significance of constructive reasoning for the local interpretation. Notice in this connection that a theorem of classical logic which is not constructively provable (e.g., the law of excluded middle) will not in general hold universally until it has been transformed into its intuitionistic correlate (which, e.g., in the case of the excluded middle $A \vee \neg A$ is $\neg\neg(A \vee \neg A)$). The procedure of translating classical into intuitionistic logic (see, e.g., [5]) is thus the counterpart of casting physical laws into invariant form.

The physical concept of *inertial coordinate system* also has its counterpart in the local interpretation of mathematics. An inertial coordinate system is one in which undisturbed bodies undergo no accelerations, i.e., in which *Newton's first law of motion* holds. Thus inertial coordinate systems act as surrogates for Newtonian absolute space. Analogously, a *classical* local mathematical framework is one in which objects undergo no variations (i.e., are "constant"), in other words, one which resembles the classical universe of constant sets as closely as possible. This resemblance can be ensured by the satisfaction of the *axiom of choice* suitably formulated in a strong categorical form (see, e.g., [3]). For the truth of the (strong) axiom of choice in a framework **E** implies not only that the internal logic of **E** is *classical* and *bivalent* (i.e., there are only two truth values "true" and "false") but also that the arrows of **E** resemble set-theoretic mappings in that they are determined by their action on "points" of their domains. These features may be taken as distinguishing frameworks of constant sets among arbitrary frameworks. Since the axiom of choice ensures the presence of these features, we may *define* a classical framework to be one in which this axiom is satisfied while at the same time observing that it is an accepted principle of classical set theory. Accordingly, *classical local frameworks correspond to inertial coordinate systems and the axiom of choice to Newton's first law of motion*.

These observations suggest the idea that the local interpretation of mathematical concepts bears the same relation to classical set theory as relativity theory does to classical physics.

3. THE NEGATION OF CONSTANCY

We have remarked that the transition from the notion of model of set theory to the notion of topos (local framework) is an instance of the dialectical process of replacing the *constant* by the *variable*. A particularly striking form of this process arises when the transition is made by an *admissible transformation*. Suppose that we are given a classical local framework **S** (i.e., one satisfying the axiom of choice: the objects of **S** may then be regarded as being "constant"), and a local framework **E** defined over **S**, i.e., for which there is an admissible transformation $\mathbf{S} \to \mathbf{E}$. We may regard **E** as a framework which results when the objects of **S** are allowed to *vary* in some manner. (For example, when **E** is $\mathbf{Shv}(X)$, the objects of **E** are those varying smoothly over the open sets of X.) Thus, in passing from **S** to **E** we dialectically *negate* the "constancy" of the objects in **S**, and introduce "variation" or "change" into the new objects of **E**. In passing from a classical framework to a framework defined over it we are, in short, *negating constancy*.

Now in certain important cases we can proceed in turn to dialectically negate the "variation" in **E** to obtain a new classical framework $\mathbf{S}^*$ in which constancy again prevails. $\mathbf{S}^*$ may be regarded as arising from **S** through the dialectical process of *negating negation*. In general, $\mathbf{S}^*$ is not equivalent to **S** and is therefore, according to a well-known result of topos theory (cf. [4] or [10]) not defined over **S**, so that the second "negation", i.e., the passage from **E** to $\mathbf{S}^*$, cannot be an admissible transformation (but it is a functor, in fact a "logical" functor). Thus the action of negating negation in this sense *transcends* admissibility. This is the price exacted for reinstating constancy in passing to $\mathbf{S}^*$.

To establish the importance of this process of negating negation we show that it underlies two key developments in the foundations of mathematics: *Robinson's nonstandard analysis* and *Cohen's independence proofs in set theory*. (The discussion here owes much to Lawvere's seminal article [12].)

Fix a classical framework **S**, which we shall think of as consisting of constant sets: we shall reserve the term "set" for "object of **S**".

Given a set I, each element $i \in I$ may be identified with the principal ultrafilter $U_i = \{A \subseteq I : i \in A\}$ over I. This identification suggests that we think of *arbitrary* ultrafilters over I as "generalized points" of I. The collection of generalized points of I forms a new set βI (the Stone-Čech compactification of I). Elements of I (now identified as a subset of βI) are called *standard* points of I, and elements of $\beta I - I$ *ideal* points of I. If I is infinite, ideal points always exist.

Now consider the local framework $\mathbf{E} = \mathbf{S}^I$ of sets varying over I. Objects of $\mathbf{S}^I$ (which we shall call *variable sets*) are I-indexed families of sets $X = \langle X_i : i \in I \rangle$. An *element* of an object X is an I-indexed family $\langle x_i : i \in I \rangle$ such that $x_i \in X_i$ for $i \in I$, i.e., a "choice function" on X. Thus the Cartesian product $\prod_{i \in I} X_i$ is the set of "elements" of the variable set X.

Each (constant) set A is associated with the variable set $\hat{A}$ given by the constant function with value A. The set of 'elements' of the variable set A is then A^I.

The framework $\mathbf{S}^I$ is defined over $\mathbf{S}$ via the admissible transformation $\mathbf{S} \to \mathbf{S}^I$ given by $f^*(A) = \hat{A}$, $f_*(X) = \prod_{i \in I} X_i$.

Given an element $i_0 \in I$, we can arrest the variation of any variable set X by evaluating X at i_0, i.e., by considering X_{i_0}. If we apply this in particular to the set A^I of "elements" of the variable set $\hat{A}$, that is, if we evaluate each such 'element' at i_0, we just retrieve A. So in this case, if we negate the constancy of (the elements of) A by passing to the set A^I of (variable) "elements" of A, and then negate the variation of these "elements" by evaluating at a *standard* point of I, we come full circle. This instance of "negation of negation" is, accordingly, *trivial*. The situation is decidedly different, however, when the evaluation is made at an *ideal* point of I.

Given an ideal point U of I, i.e., a non-principal ultrafilter over I, how shall we "evaluate" functions in A^I at U? To this end, observe that the result of evaluating at a standard point i_0 of I is essentially the same as *identifying* functions in A^I provided their values at i_0 coincide, i.e., stipulating that for $f, g \in A^I$,

$$f \sim_{i_0} g \leftrightarrow f(i_0) = g(i_0)$$
$$\leftrightarrow \{i \in I : f(i) = g(i)\} \in U_{i_0}.$$

We use this last equivalence as the basis for evaluating functions in A^I at an ideal point U of I. That is, we define

$$f \sim_U g \leftrightarrow \{i \in I : f(i) = g(i)\} \in U.$$

Then the result of "evaluating" all the elements of A^I at U is the set A^* of equivalence classes of A^I modulo $\sim_U$. (Thus A^* is the ultrapower A^I/U.) If I is infinite (and U an ideal point of I), then A^* is well known *never* to be the same as A. In particular if, for example, A is the real line R, then R^* will have the same elementary properties as R but will in addition contain new "infinitesimal" elements. Thus R^* will be a nonstandard model of the real line. This, in essence, is the basis of Robinson's nonstandard analysis.

In sum, we get Robinson's infinitesimals by the dialectical process of first negating the constancy of the classical real line, and then negating the resulting variation by arresting it at an ideal point.

If we arrest the variation of all the objects of $\mathbf{S}^I$ *simultaneously* at an ideal point of I we obtain a new classical framework $\mathbf{S}^*$ (an ultrapower or enlargement of $\mathbf{S}$) which has the same elementary properties as $\mathbf{S}$. So this instance of negation of negation leads to a classical framework which, although different, is nonetheless *internally indistinguishable* from the initial classical framework. The whole purpose of *Cohen's* method of forcing in set theory is to obtain new classical frameworks which are internally *distinguishable* from the initial one. We now describe this process, bringing out its dialectical character.

Let P be a partially ordered set in $\mathbf{S}$: think of the elements of P as states of knowledge and $p \leq q$ as meaning that q is a deeper (or later) state of knowledge than p. *A set varying over P* is a map X which assigns to each $p \in P$ a set $X(p)$ and to each pair $p, q \in P$ such that $p \leq q$ a map $X_{pq} : X(p) \to X(q)$ such that $X_{pr} = X_{qr} \circ X_{pq}$ whenever $p \leq q \leq r$. Let $\mathbf{E}$ be the framework defined over $\mathbf{S}$ whose objects are all sets varying over P (and in which an arrow $f : X \to Y$ is a collection of maps $F_p : X(p) \to Y(p)$ such that $f_q \circ X_{pq} = Y_{pq} \circ f_p$ for $p \leq q$).

Within $\mathbf{E}$ we consider objects X for which $X(p) \subseteq X(q)$ and X_{pq} is the inclusion map for $p \leq q$. Such an object will be called a set *varying* *steadily* over P. If we think of $X(p)$ as the collection of elements of the variable set X *secured* at state p, then the "steadiness" condition means that no secured elements are ever lost. For $p \in P$ and sets X, Y varying steadily over P we write

$$p \Vdash X \subseteq Y$$

for

$$\forall q \geq p \forall x \in X(q) \exists r \geq q[x \in Y(r)],$$

that is, given state p, X is *eventually contained* in Y. We write

$$p \Vdash X = Y$$

for

$$p \Vdash X \subseteq Y \text{ and } p \Vdash Y \subseteq X,$$

that is, given state p, X *eventually coincides* with Y.

Two elements $p, q \in P$ are *mutually consistent* if $\exists r \in P[p \leq r$ & $q \leq r]$. A set of mutually consistent elements of P is called a *body of knowledge*. A body of knowledge K is said to be *complete* if whenever $p \in P$ is mutually consistent with every member of K, then p belongs to K.

Given a complete body of knowledge K, define the equivalence relation $\sim_K$ on the collection of sets varying steadily over P by

$$X \sim_K Y \leftrightarrow \exists p \in K[p \Vdash X = Y].$$

Thus, $X \sim_K Y$ means that our body of knowledge K yields the assertion that X and Y eventually coincide. The collection **S*** of equivalence classes modulo $\sim_T$ of steadily varying sets forms a *new* classical framework, in general internally distinguishable from **S** in the sense that it does not possess all the elementary properties of **S**: **S*** is in fact a (possibly nonstandard) *Cohen extension of* **S**.

To recapitulate: the framework **E** was obtained by negating constancy in allowing variation ("growth") over states of knowledge, and the Cohen extension **S*** obtained from (the steadily varying objects in) **E** by using a complete body of knowledge to determine the "eventual" identities between the variable sets, in other words, to negate their variation.

In the "Cohen extension" case passage from **S** to **S*** (negation of negation) preserves some of the principles associated with constancy of sets (e.g., axiom of choice, classical logic) but, as Cohen showed, P

may be chosen in such a way – now familiar to every set theorist – so as to ensure that other such principles (e.g., axiom of constructibility, continuum hypothesis) are *violated* in this passage. In passing from **S** to **E** (negation of constancy) the classical bivalent logic of **S** is replaced by the intuitionistic logic of **E**. And passage from **E** to **S***(negation of negation) restores classical logic and constancy but, as we have remarked, not all principles associated with constancy.

Now, we could have *refrained* from performing the return passage to constancy (i.e., the second "negation") and instead remained in the framework **E** of variable sets. The set-theoretic independence proofs can be obtained by scrutinizing the internal properties of **E** (more precisely, by employing the Scott-Solovay method of replacing **E** by its associated Boolean-valued framework). If we agree more generally to abstain from returning to constancy then some startling possibilities begin to emerge. For example in certain more radical choices of the framework **E** of variable sets (where the "sets" vary over a category of rings in a natural way), the "real line" in **E** will contain non-trivial *square zero infinitesimals*, i.e., real numbers $\epsilon \neq 0$ such that $\epsilon^2 = 0$. In such frameworks (cf. [11] or [13]) every function defined on the real line is infinitesimally linear, hence smooth, and therefore corresponds to the motion of a classical body. In these circumstances one can then proceed to develop the calculus along the lines of Fermat and Newton, with no mention of infinite processes or limits. But for this to be possible we must remain within a framework of *variable sets*, resolutely adopting a local viewpoint in which constancy and classical logic no longer prevail.

4. SUMMARY AND CONCLUSION

My proposal is that the absolute universe of sets be relinquished in favour of a plurality of local mathematical frameworks. Mathematics interpreted in any such framework is appropriately called *local mathematics*; an admissible transformation between frameworks amounts to a (definable) *change of local mathematics*. The *reference* of any mathematical concept is accordingly not fixed, but *changes* with the choice of local mathematics.

Constructive provability of a mathematical statement now means that it is *invariant* or valid in *every* local mathematics. So, on this account, the use of constructive proof procedures, far from hobbling mathematical activity as (classical) mathematicians are wont to claim, has instead the opposite effect of extending the validity of mathematical reasoning to the widest possible spectrum of contexts, including those in which "variation" is taking place. The role of the axiom of choice is to *eliminate* as much of this "variation" as possible, ensuring that any framework in which it is satisfied is sufficiently similar to the classical universe of "constant" sets to allow *classical* mathematical (i.e., set-theoretical) reasoning to become valid there.

The replacement of absolute by local mathematics results, in my view, in a considerable gain in *flexibility of application* of mathematical ideas, and so offers the possibility of providing an explanation of their "unreasonable effectiveness" (cf. [17]). For now, instead of being obliged to force an intuitively given concept onto the Procrustean bed of absolute mathematics, with the attendant distortion of meaning, we are at liberty to *choose* the local mathematics naturally fitted to express and develop the concept. To the extent that the given concept embodies aspects of (our experience of) the objective world, so also will the associated local mathematics; the "effectiveness" of the latter, i.e., its conformability with the objective world, thus loses its "unreasonableness" and instead is shown to be a product of design.

So the local interpretation of mathematics implicit in category theory accords closely with the unspoken belief of many mathematicians that their science is ultimately concerned, not with abstract sets, but with the structure of the real world.

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