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John L. Bell

Professor of Philosophy

University of Western Ontario, Canada

Why were you initially drawn to the foundations of mathematics and/or the philosophy of mathematics?

My route to the foundations and philosophy of mathematics was somewhat circuitous. In youth I was attracted to physics, especially relativity theory and cosmology—I actually attended one of Fred Hoyle's lecture courses on the subject in Cambridge in the early 1960s. (Parenthetically, I may mention that it was through Hoyle's lectures that I first heard the name Gödel, not of course in connection with his discoveries in logic, of which I was then wholly ignorant, but as the deviser of cosmological models containing closed timelike lines.) While I was, I suppose, quite clever at solving problems in mathematical physics and analysis – as I still joke, I could raise and lower a tensor index with the best of 'em! – after a while I began to realize that I had no genuine understanding of what I was actually doing. In particular, I was not even sure what a tensor really *was.* At the risk of joining the fabled centipede whose effort to understand its mode of locomotion reduced it to complete immobility, I decided to turn away from physics, my first love, and concentrate on pure mathematics. While mathematics lacked, in my eyes, the romantic appeal of cosmology, it had the compensating merit that its concepts and methods could, in principle at least, be fully presented to the understanding. My flight to mathematics was fuelled by my discovery of John Kelley's classic work *General Topology.* Its unique combination of mathematical elegance and dry wit, together with its extraordinary collection of exercises, stimulating but never oppressive, made a big impact on me. In particular, I was intrigued by the series of exercises on Boolean algebras (rings, but no matter) which I attempted to work through. Kelley also furnished my first introduction to set theory. Reading Kelley led me to study Gödel's monograph, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis.* The first two-thirds of this mathematical tour-de-force, in which Gödel presents his axiom system for set theory and develops its essential properties, seemed reasonably clear. But, despite my best efforts, I was unable to fathom the final part of the work, its grand finale, so to speak, in which, accompanied by an inaudible clash of cymbals, the consistency of the GCH is established. A good few years were to pass before I felt I truly understood what was going on.

Another influence was Bourbaki's *Éléments de Mathématique.* On first coming across some volumes of this monumental work in Blackwell's bookshop I was excited to find that it was intended to be a complete, systematic account of abstract mathematics, precisely the kind of mathematics to which I had already been converted by Kelley's *General Topology.* The *oeuvre Bourbachique* included not only *Topologie Génerale*, but *Algèbr*e, *Thèorie des Ensembles, Espaces Vectoriels Topologiques, Algèbre Commutatif*—magical titles in my eyes. I bought as many volumes as I could afford, often in obsolete – and so cheaper – editions (the whole enterprise seemed to be undergoing constant revision), and commenced to work my way through the collections of challenging exercises at the end of each section. I toiled mightily, in particular, to formulate solutions to the exercises on ordered sets in Chapter 3 of the *Thèorie des Ensembles.* It was from these that I first learned about ordinals, which Bourbaki presents in the original Cantorian manner as order types of well-ordered sets.

Kelley, Bourbaki, Gödel: it was through their influence that I was led to the foundations of mathematics. My interest in philosophy, on the other hand, derived from my being a voracious and eclectic reader. As an undergraduate I recall reading Plato's *The Last Days of Socrates,* William James's "Essays on Pragmatism", G. E. Moore's philosophical essays, Hegel's *Philosophy of History*, Descartes' *Discourse on Method,* Spinoza's *Ethics* (the statements of the theorems at least, since I found the "proofs" unenlightening), Leibniz's delphic *Monadology,* some Locke, Berkeley and Hume, Schopenhauer's *Essays in Pessimism.* And of course Bertrand Russell's breezily brilliant, if irresponsible, *History of Western Philosophy.* My attempts to penetrate the profundities of Kant's *Critique of Pure Reason* were frustrated by the work's apparent indigestibility. (I was only to appreciate its depth and philosophical importance many years later.) I greatly enjoyed Hans Reichenbach's *Philosophy of Space and Time.* On Blackwell's shelves in Oxford I came across Norman Malcolm's *Wittgenstein: A Memoir.* I was deeply moved by Malcolm's portrayal of Wittgenstein, in which he emerges as an intellectual ascetic of compelling moral grandeur. Wittgenstein's tiniest defiances of convention, for example, his refusal to wear a tie at dinner in Trinity College, I found admirable. Reading Malcolm's memoir stimulated me to attempt to read Wittgenstein's philosophical works. I was intrigued by the *Tractatus Logico-Philosophicus*, a masterpiece of sybilline refinement and compression in which Wittgenstein embarks on the heroic effort of reducing philosophy to the expressible, but in the end washes up on the shores of the ineffable. The conventionalism of the later Wittgenstein's *Philosophical Investigations,* I found less appealing.

I was drawn quite early on to the foundations of mathematics, and to general philosophy, but my conscious interest in the philosophy of mathematics per se was comparatively slow to crystallize. This took place in three stages. First, as an undergraduate I had developed an interest in set theory and the philosophy of the infinite in general. Next, I was a product of the politically supercharged 1960s, a time in which many young people, myself included, began to think about the social and political implications of their own activity, which in my case was the practice of mathematics (by this time mathematical logic). I became a member of a group of like-minded *gauchiste* mathematical logicians determined to terminate the funding of logic conferences by military-imperialistic sources such as NATO. All of us, I think I may safely say, believed that mathematics, while being. like art, a beautiful sublimation of human activity, has, in the final analysis, to be understood as the product of actual human beings living in the world. Such stimuli led me (for better or worse) to see that mathematics actually has a *hidden content*, which can actually be *argued about.* This is the opposite of the unthinking Platonism/realism to which I was, I guess, initially attracted as offering the simplest account of mathematical truth, and which also possessed the additional advantage of avoiding what I then felt to be a certain cynicism inherent in Formalism. (Still, as I have come to learn, Formalism has the great merit of offering the weary ex-Platonist a refuge.) But, like the child's loss of belief in Santa Claus, I came to regard the Platonistic account of mathematical entities as a kind of fairy tale, and in any case as engendering insuperable epistemological difficulties. I may parenthetically remark that I have since come to liken Platonism to a (necessary) disease, which, like measles, must have been contracted in one's youth so as to confer an immunity in later life.

The third stage in the development of my interest in the philosophy of mathematics came through my efforts to understand *topos theory*. I was very struck by Bill Lawvere's insight that a topos is an objective presentation of the idea of *variability*, and that its internal – intuitionistic – logic may be considered as a logic of variation. Later I went so far as to attempt to use the topos concept as the basis for a "local" (as opposed to "absolute") interpretation of mathematical statements. I suggested that the unique absolute universe of sets central to the orthodox set-theoretic account of the foundations of mathematics should be replaced by a plurality of local mathematical frameworks – elementary toposes – defined in category-theoretic terms. Observing that such frameworks possess sufficiently rich internal structure to enable mathematical concepts and assertions to be interpreted within them, I maintained that they can serve as local surrogates for the usual "absolute" universe of sets. On this account mathematical concepts will in general no longer possess absolute meaning, nor mathematical assertions (e.g. the continuum hypothesis) absolute truth values, but will instead possess such meanings or truth values only locally, i.e., relative to local frameworks. The absolute truth of set-theoretical assertions would then, I held, give way to the subtler concept of invariance, that is, validity in all local frameworks. Thus, e.g., while the theorems of constructive arithmetic turn out to possess the property of invariance, the axiom of choice or the continuum hypothesis do not, because they hold true in some local frameworks but not others.

I still find this view attractive, but it is, after all, only one among many possible accounts of mathematics. If I were pressed to characterize my present attitude towards the foundations of mathematics, I would use the word *pluralistic*: no unique foundation, rather an interlocking ensemble of "foundations".

What examples from your work (or the work of others) illustrate the use of mathematics for philosophy?

There are, of course, numerous examples illustrating the use of mathematics for philosophy. "Negative" examples include the Pythagorean discovery of incommensurable magnitudes, Zeno's paradoxes, and the Gödel incompleteness theorem, each of which served to refute a certain philosophical doctrine. “Positive” examples include the Pythagorean discovery of the arithmetical basis for harmony, their invention of figurate numbers, Euclidean geometry, the infinitesimal calculus, Riemannian geometry, set theory, probability theory, relativity theory, quantum theory, and the theory of computation, all of which have been important influences in shaping philosophical views.

One can find other, more specific, examples of such influence in the 19^{th} and 20^{th} centuries. Frege’s work on the foundations of arithmetic (and Bolzano’s before him) is now held to have contained the seeds of what was later to flower into analytic philosophy. Russell’s philosophical views were profoundly influenced by his work in mathematical logic. Brouwer’s philosophy of intuitionism was first and foremost a philosophy of mathematics. Well-known is the impact Tarski’s theory of truth had on Popper’s philosophical outlook, serving, as it did in the latter’s eyes, to revive the correspondence theory of truth.

Another important source of interaction between mathematics and philosophy arises from the opposition between the continuous and the discrete. Synechism, the doctrine that the world is ultimately continuous, has been defended by the majority of philosophers in the past, including Aristotle, Descartes, and Kant. (Leibniz seems to wavered, ending up with his strange hybrid doctrine of monadism.) Atomism, the doctrine that the world is ultimately particulate, was for a long time considered a maverick position. Now however, owing primarily to work in physics and chemistry, and lately also to the emergence of computing machines, it appears to be gaining the upper hand. (The movement to reduce mathematics to set theory initiated in the 19^{th} century can already be seen as a victory for a form of atomism.) Synechism and atomism, along with the various syntheses of the two that have emerged in the history of thought, are have been developed primarily in *mathematical* terms.

I believe that a significant potential influence of mathematics on philosophy may be seen in *category theory.* Category theory arose as a general apparatus for dealing with mathematical structures and their mutual relations and transformations. From a philosophical standpoint, a category may be viewed as an explicit presentation of a *form or concept.* The objects of a category are the *instances* of the associated form and its morphisms or arrows of are the transformations between these instances which in some specified sense "preserve" this form. Functors between categories may then be considered as embodiments of morphological variation—change of form. Category theory is beginning to be seen as an appropriate language for describing not just mathematics, but the world, in structuralist terms, in terms of form. On this account there is no unique category (or topos) representing the objective world, but a number of different categories each embodying an idealization of a significant feature of the world. For instance, the topos of sets embodies the idea of discreteness, the smooth topos that of continuity and differentiability, and the effective topos that of computability. Each topos possesses properties not shared by the others: in the topos of sets the axiom of choice (and hence classical logic) holds; in the smooth topos the real line is indecomposable; and in the effective topos the space of countable sequences of natural numbers is enumerable. Each of these features can be seen as a necessary consequence of the particular form of idealization involved.

After these spectacular instances of the impact of mathematics on philosophy, it comes as something of an anticlimax to mention, as suggested, some of my own modest contributions to that area. The first of these was essentially a contribution to philosophy of science. As an ex-aspiring-physicist I had long been intrigued by quantum theory, with its mysterious superpositions of states and incompatible measurements; and as a logician my curiosity was piqued by the so-called quantum logic, whose characteristic feature is that its algebra of propositions is not a Boolean or Heyting algebra, but a certain kind of nondistributive lattice—an ortholattice. All of these facts can be, and are, formally derived from the standard Hilbert space formalism of quantum theory. I became interested in the problem of formulating some simple principles, free of the technicalities of the theory of Hilbert spaces, from which one could derive the anomalous features of quantum theory, as well as the ortholattices underlying quantum logic. I came up with two approaches. The first, essentially topological, was based on the idea of using what I called a proximity space, a set equipped with a symmetric reflexive relation "close to". The lattice of parts of such a space is an ortholattice. There is a natural way, which I called "manifestation", similar to Paul Cohen's celebrated concept of set-theoretic forcing, of relating propositions (actually attributes) to parts of the space. The propositions manifested over the whole of every proximity space are (essentially) the theses of quantum logic. Given two propositions P, Q, their superposition can be identified with $\neg\neg(P \vee Q)$, and they are incompatible if there is a proximity space with a part manifesting P but not $Q \vee \neg Q$,or vice-versa.

In my other approach to the problem, I showed how to construct the ortholattices arising in quantum logic from what I saw as the phenomenologically plausible idea of a collection of ensembles subject to passing or failing various "tests". A collection of ensembles forms a certain kind of preorderd set with an additional relation I called an orthospace: I showed that the complete ortholattices, in particular those of quantum theory, arise as canonical completions of orthospaces in much the same way as arbitrary complete lattices arise as canonical completions of partially ordered sets. I also showed that the canonical completion of an orthospace of ensembles may be identified with the lattice of properties of the ensembles, thereby showing exactly why ortholattices arise in the analysis of "tests" or experimental propositions. I went on to axiomatize the concept of "test" itself in terms of the more primitive notion of "filters" acting on ensembles. "Passing" an ensemble through a filter s produces the subensemble of entities that have "passed" the test corresponding to s.Two filters s and t can be juxtaposed to produce the compound filter st, but in general $st \neq ts$. When this latter is that case, the two tests corresponding to s and t are, like position and momentum measurements in quantum theory, not simultaneously performable, that is, *incompatible*. When (and only when) $st \neq ts$, the juxtaposition of s and t corresponds to their logical conjunction. In this setting, it is the noncommutativity or incompatibility of filters or "tests" that gives rise to "quantum logic".

A philosophical problem that had long intrigued me was: why is traditional logic *bivalent,* that is, why is it assumed that there just two truth values rather than some other number? What is it about the number 2 that gives it this special position in logic? Wittgenstein seems to take the fact for granted when (in his Notes on Logic) he says that propositions have two "poles". It is often claimed that bivalent logic is the "logic of realism", that is, logic in which propositions are construed as referring to independently existing objects, in contrast with "anti-realist" logics such as intuitionistic logic (I don't agree that intutionistic logic has to be thought of as anti-realist—but let that pass). However, this begs the question, since the thought immediately arises: what is it about the realm of independently existing objects that confers bivalence on propositions referred to it? Why shouldn't the number of objective truth values be, say, 3, like the number of spatial dimensions? Wittgenstein recognized the possibility of this question arising but simply dismissed it.

One way that occurred to me of explaining the role of the number 2 in logic is by moving from individual propositions to sets of propositions, or *theories.* Frege had suggested that the bivalence of the logic of concepts arises from their having *sharp boundaries*: one can determine with exactitude, for such a concept, when an object falls under it , or when it does not. In other words, a concept's possession of a sharp boundary means that the theory of the concept is complete with in regard to atomic propositions. It is then natural to extend this prescription to arbitrary propositions. So, metaphorically, we may say that (the concept determined by) a theory has sharp boundaries if it is *complete*, that is, if any proposition in the theory's vocabulary is provable or refutable from the theory. But it is well known that, for any complete theory T (in propositional intuitionistic or classical logic), it is possible to assign the *two* truth values 0, 1 to propositions in such a way as to respect the logical operations, and also to assign precisely the propositions in T the value 1. And conversely, if such a bivalent assignment exists, the theory is complete. That is, the number 2 is simply the numerical representative of completeness, or the possession of "sharp boundaries".

The major logical consequence of bivalence (although not equivalent to it) is the *law of excluded middle:* the assertion, for any proposition P, of the disjunction $P \vee \neg P$.This is of course the logical principle which whose affirmation distinguishes classical from intuitionistic logic. Like bivalence the law of excluded middle has been taken to be characteristic of logic in which propositions are construed as referring to independently existing objects. I found that, if one starts with intuitionistic predicate logic, and extends it to include Hilbert's ε-terms (these are essentially objects named by the use of the indefinite article: *a* such-and-such), then the law of excluded middle becomes provable. That is, the law of excluded middle is, after all, derivable from what can reasonably be construed as an ontological principle.

I also found myself attracted by the recent revival of interest in Frege's attempt to derive arithmetic from logic, in particular to the central mathematical result, now known as *Frege's Theorem*, implicit in his *Grundlagen.* Stated in set-theoretic terms, Frege's Theorem reads: for any set E, if there exists a map ν from the power set of E to E satisfying the condition

$$\forall XY[\nu(X) = \nu(Y) \Leftrightarrow \text{there is a bijection between } X \text{ and } Y],$$

then E has a subset which is the domain of a model of Peano's axioms for the natural numbers. My first piece of work on Frege's theorem was to observe that it can be proved by the same means as Zermelo used to derive the well-ordering theorem from the axiom of choice. I then became interested in the question of whether Frege's theorem can be proved constructively. I found that this was indeed the case, providing a constructive proof of a "best possible" version of Frege's theorem in which the premise is weakened so as to require only that the map ν be defined on the family of *finite* subsets of the set E. I also showed that that the postulation of such a structure (E, ν) – a *Frege structure* – is constructively equivalent to the postulation of a model of Peano's axioms.

What is the proper role of philosophy of mathematics in relation to logic, foundations of mathematics, the traditional core areas of mathematics, and science?

I think that the philosophy of mathematics should, and in fact does, play a dialectical role in relation to its sister disciplines, guiding them and, reciprocally, responding to their internal development. Let me attempt to illustrate what I mean. Cantor's philosophy of the infinite (and his associated, if lesser-known, championship of the reduction of the continuous to the discrete) played a major part in his development of set theory, which, as is well-known, came to permeate mathematics. Partly in reaction to the unrestricted use of Cantorian set theory in mathematics, Brouwer formulated his philosophy of intuitionism which in *its* turn radically influenced his mathematical practice and that of his immediate followers, a practice which was to prove seminal for constructive and computational mathematics. In *its* turn the latter is generating its own philosophy... And so it goes.

What do you consider the most neglected topics and/or contributions in late 20th century philosophy of mathematics?

In my view contemporary philosophers of mathematics (or at least those who can be described as "mainstream") have paid far too much attention to set theory, ignoring the philosophical import of other major developments in mathematics such as category theory, type theory, and constructive mathematics. The impressive– and they *are* impressive – achievements of set theory in advancing mathematical knowledge have, perhaps, (mis)led these philosophers into thinking that, as far as philosophy is concerned, mathematics just *is* set theory. (This is the same "mistake" I believe Russell made when he claimed that mathematics just *is* logic, only in his case the "mistake" had the positive – if, with hindsight, accidental – consequence of leading to type theory.) But in truth set theory represents only one side of an opposition – that between the continuous and the discrete – which is still stimulating the growth of mathematics. With the introduction of set theory, mathematics was reduced to pure discreteness (in the eyes of certain philosophers) and those aspects of continuity incompatible with discreteness (e.g. infinitesimals and indecomposable continua) were driven out. With the emergence of category theory, type theory and constructive mathematics, set theory, while still dominant, can now be seen as no more than one among a number of ways of depicting the mathematical universe. I believe that it would benefit philosophers of mathematics to become aware of this fact.

Another topic which I think has been, on the whole, neglected by contemporary philosophers of mathematics (there are, admittedly, exceptions) is the *applicability* of mathematics. Mathematics is perhaps unique in being at once art and science. As an art, it is free to develop aesthetically pleasing internal practices of its own, practices which are capable of reduction to simpler, but equally beautiful practices which can then function as rules. (The art with which it is natural to compare mathematics in this regard is music, in which the simple rules governing the diatonic scale came to serve as the "foundation" for musical composition.) But mathematics is also a *science*; it serves to describe the natural world – in the terms of idealist philosophy, a *transcendent* world – a world that exists independently of it. The correlation between the internal practice of mathematics and the properties of the natural world is remarkable and seems to demand some kind of explanation. Galileo's explanation was that mathematics was the language of "the book of nature"; but with the rise of quantum theory and other esoteric physical theories, couched in exotic mathematical terms, physicists have become less comfortable with this explanation. It seems almost a miracle, for example, that the mathematics of Hilbert space, invented for an entirely different purpose, serves perfectly to represent the mechanics of the microworld. (This and other such "coincidences" led the physicist Eugene Wigner to entitle a famous paper "The Unreasonable Effectiveness of Mathematics".) I believe that philosophers of mathematics should enlarge their program of explicating the internal workings of mathematics to embrace the connection between mathematics and the outer world.

What are the most important open problems in the philosophy of mathematics and what are the prospects for progress?

Here are some problems in the philosophy of mathematics which to me possess significance, and are unquestionably still "open".

To explicate the applicability of mathematics

I have discussed this in the previous section.

To understand how the brain/mind generates mathematical concepts

This seems to me a problem as perplexing and intriguing as that of how the brain "generates" consciousness. A most interesting attempt to grapple with this problem has been made by the linguist George Lakoff and the psychologist Rafael Nuñez. In their book *Where Mathematics Comes From,* they fashion a sophisticated naturalistic explanation of the origins of mathematics. They advance the thesis that mathematics is not the product of some mysterious synergy between the mind and some putative empyrean world. They contend that mathematics, in all its richness and elaboration, emerges through the natural interaction of the cognitive processes common to us all with our experience of living in the actual world. We are from birth equipped with certain rudimentary mathematical abilities, for instance, the ability to distinguish objects, that of grasping and comparing the size of small pluralities instantly, and that of adding and subtracting small whole numbers. It is Lakoff's and Nuñez's contention that mathematics has emerged from our modest initial cognitive endowments through the brain/mind's distillation of *conceptual metaphors* from its/our experience of the external world, e.g. that arising from bodily movement, physical force, and spatial orientation. In terms reminiscent of category theory, Lakoff and Nuñez define a metaphor as a correlation or mapping grasped by the mind between two conceptual domains, the first of which, the source domain, is relatively concrete and familiar, and the second, the target domain, is of a more abstract character. Like morphisms in a category, these correlations must preserve structure. Thus, for example, in the metaphorical correlation between, say, heaps of stones (the source domain) and numbers (the target domain), our grasp of the fact that the combination of two piles of stones each of a given size always results in another pile of stones of a certain related size is projected onto the domain of numbers as the operation of addition. According to Lakoff and Nuñez', metaphorical correspondences such as this are the *fons et origo* of mathematical thought. While their claim is, of course, unprovable in a literal sense, I like it both for what I see as its essential plausibility and for the fact that it addresses a problem that has always nagged me.

To explicate the relationship between the continuous and the discrete—in particular, to explain how, continuity emerges from a discrete world

I have already touched on this problem – the significance of which extends far beyond the philosophy of mathematics – in a number of places above.

Here let me mention what seems to me an important special case of the problem: how is the continuity of perception (that of vision, for example) engendered by a discrete system of receptors? Actual perceptual fields can be modelled by *proximity spaces.* A proximity space is a set S equipped with a *proximity relation*, that is, a symmetric reflexive binary relation $\approx$. Here we think of S as a field of perception, its points as *locations* in it, and the relation $\approx$ as representing *indiscernibility of locations,* so that $x \approx y$ means that x and y are "too close" to one another to be perceptually distinguished. Let us call a proximity space $(S, \approx)$ *continuous* if for any $x, y \in S$ there exist $z_1, ..., z_n$ such that $x \approx z_1, z_1 \approx z_2, ..., z_{n-1} \approx z_n, z_n \approx y$. Continuity in this sense means that any two points can be joined by a finite sequence of points, each of which is indistinguishable from its immediate predecessor. If d is a metric on S such that the metric space (S, d) is connected, then every proximity structure determined by d is continuous. When S is a perceptual field such as that of vision, the fact that it does not fall into separate parts means that it is connected as a metric space with the inherent metric. Accordingly every proximity structure on S determined by that metric is continuous. Note that this continuity emerges even when S i s itself an assemblage of discrete "points". In this way continuity of perception could be produced by an discrete system of receptors.

To explicate the role of computability in mathematics

How, in particular, does the computational structure of a mathematical result reflect its content? What is the relationship between the content of a mathematical theorem and the length or complexity of its proof? In the case of spectacular recent mathematical achievements such as the proofs of the Fermat theorem and the Poincaré conjecture, the comprehensibility of the proposition proved and the complexity of its proof would seem to be in inverse relationship. This is to be contrasted, with, say, category theory, in which propositions and their proofs are virtually on an equal footing as regards intelligibility.

To characterize how mathematics as a formal/symbolic practice differs from a practice such as fiction

Of course both are language-based (despite Brouwer's contrary claim that mathematics is a "languageless activity"). But more particularly, mathematics resembles fiction in its systematic introduction of concepts such as numbers, circles, sets, etc. which are then *reified*, that is, treated as if they possessed independent existence—this is as true of constructive as of classical mathematics, by the way. In fiction, characters and events are treated, in accordance with Coleridge's "willing suspension of disbelief", as if they were real. Now one important difference between *classical* mathematics and the practice of fiction is that the reified concepts of the former, but not the latter, are treated as if their properties were fully determinate. For instance, it is accepted (I would surmise) by the majority of mathematicians that it is objectively determined whether the number $10^{10^{10}} + 3$ is prime or not—even if, as is likely, *we* shall never know the answer. But in the case of fiction the case is otherwise. Scholars may debate Shakespeare's identity, but the question of whether Hamlet's breeches was, say, green, lacks determinacy, indeed borders on the absurd, since no scrutiny of Shakespeare's play could reveal their colour. Here the play is indeed the thing!

By contrast, the manner in which reified objects are treated both in constructive and in structuralist/axiomatic mathematics (category theory, for example) bears a closer resemblance to fiction. Constructive mathematicians acknowledge that the concepts and devices of mathematics are *invented* or *constructed*, even if under such (objective) constraints as to make it seem plausible later to describe them as the products of *discovery.* While in constructive mathematics *finite* objects such as individual natural numbers are treated as if their (finitistic) properties were fully determinate, (potentially) *infinite* objects such as the set of natural numbers, numerical functions, and individual real numbers are treated in a manner similar to fictional characters in that their properties are taken to be open to further determinations. The same can be said of structuralist mathematics. Just as Sherlock Holmes or Philip Marlowe have been the protagonists of numerous sequels to those works in which they made their debuts, so in structuralist mathematics there are a number of different ways of spelling out the properties of, for example, the real number system—"sequels", as it were, to its original conception. Models have been constructed in which every function on the real numbers is continuous, and also models in which every such function is computable. Like the practice of fiction, structuralist mathematics is *pluralistic.* I think the analogies—and the differences—between mathematics and fiction deserve further investigation.

Let me conclude by saying that I believe the ultimate purpose of the philosophy of mathematics is to demystify mathematics while at the same time celebrating it.