The Development of Categorical Logic

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WHILE CATEGORICAL LOGIC is of very recent origin, its evolution is inseparable from that of category theory itself, so rendering it effectively impossible to do the subject full justice within the space of a comparatively short article. I have elected to begin with a historical survey in which I hope to have touched on most of the major developments (as well as a number of interesting minor ones) in categorical logic, following which I offer a more detailed presentation of a number of aspects of the subject.

1. Categorical Logic: An Historical Sketch.

Category theory was invented by Eilenberg and Mac Lane in 1945, and by the 1950s its concepts and methods had become standard in algebraic topology and algebraic geometry. But few topologists and geometers had much interest in logic. And although the use of algebraic techniques was long-established in logic, starting with Boole and continuing right up to Tarski and his school, logicians of the day were, by and large, unacquainted with category theory. It was in fact an algebraist, J. Lambek, who first discerned analogies between category theory and logic, pointing out the similarities between the axioms for categories and deductive systems such as those introduced in the 1930s by Gentzen. However, the idea of recasting the entire apparatus of logic—semantics as well as syntax—in category-theoretic terms, creating thereby a full-fledged "categorical logic", rests with the visionary category-theorist F. William Lawvere. In Lawvere's pioneering work, which came to be known as *functorial semantics* (first presented in his 1963 Columbia thesis and summarized that same year in the *Proceedings of the National Academy of Sciences of the U.S.A:* Lawvere 1963*;* see also Lawvere 1965), an algebraic theory is identified as a category of a particular kind, an interpretation of one theory in another as a functor between them, and a model of such a theory as a functor on it to the category of sets. Lawvere also began to extend the categorical description of algebraic (or equational) theories to full first-order, or *elementary* theories, building on his observation that existential and universal quantification can be seen as left and right adjoints, respectively, of substitution (Lawvere 1966a and 1967). This necessitated the introduction of the notion—one which was to prove of great importance in later developments—of an object of *truth values* so as to enable relations and partially defined operations to be described. The idea of a truth-value object was further explored in Lawvere (1969) and (1970); the latter paper contains in particular the observation that the presence of such an object in a category enables the comprehension principle to be reduced to an elementary statement about adjoint functors.

But Lawvere had a still greater goal in view. He was convinced that category theory, resting on the bedrock concepts of map and map composition, could serve as a foundation for mathematics reflecting its essence in a direct and faithful fashion unmatchable by its "official" foundation, set theory. A first step taken toward this goal was Lawvere's publication, in 1964, of an "elementary" description of the category of sets (Lawvere 1964) followed in 1966 by a similar description of the category of categories. In the former paper is to be found the first published appearance of the categorical characterization of the natural number system, later to become known as the *Peano-Lawvere* axiom. In another direction, reflecting his abiding interest in physics, Lawvere hoped to develop a foundation for continuum mechanics in category-theoretic terms. In lectures given in several places during 1967 (as later reported in Lawvere 1979) he suggested that such a foundation should be sought "on the basis of a direct axiomatization of the essence of differential topology using results and methods of algebraic geometry." And to achieve this, in turn, would require " [an] axiomatic study of categories of smooth sets, similar to the topos of Grothendieck"—in other words, an elementary axiomatic description of *Grothendieck toposes*—categories of sheaves on a site, that is, a category equipped with a Grothendieck topology.

In 1969–70 Lawvere, working in collaboration with Myles Tierney, finally wove these strands together (see Lawvere 1971 and Tierney 1972, 1973). Lawvere had previously observed that every Grothendieck topos had an object of truth values, and that Grothendieck topologies are closely connected with self-maps on that object. Accordingly Lawvere and Tierney began to investigate in earnest the consequences of taking as an axiom the existence of an object of truth values: the result was a concept of amazing fertility, that of *elementary topos*—a cartesian closed category equipped with an object of truth values[1]. In addition to providing a natural generalization of elementary—i.e., first-order—theories, the concept of elementary topos, more importantly, furnished the appropriate elementary notion of a category of sheaves; in Lawvere's powerful metaphor, a topos could be seen as a category of *variable sets* (varying over a topological space, or a general category). The usual category of sets (itself, of course, a topos) could then be thought of as being composed of "constant" sets.

Lawvere and Tierney's ideas were taken up with enthusiasm and quickly developed further by several mathematicians, notably P. Freyd, J. Bénabou and his student J. Celeyrette, G. C. Wraith, and A. Kock and his student C. J. Mikkelsen. (See Freyd 1972, Bénabou and Celeyrette 1971, Kock and Wraith 1971, and Mikkelsen 1976.)

The fact that a topos possesses an object of truth values made it apparent that logic could be "done" in a topos; from the fact that this object is not normally a Boolean but a Heyting algebra it followed that a topos's "internal" logic is, in general, *intuitionistic.* (A *Boolean* topos, e.g. the topos of sets, is one whose object of truth values is a Boolean algebra; in that event its internal logic is classical.) It was William Mitchell (Mitchell (1972)) who first put forward in print the idea that logical "reasoning" in a topos should be performed in an explicit language resembling that of set theory, the topos's *internal language.* (The internal language was also invented, independently, by André Joyal and Jean Bénabou.) This language and its logic soon became identified as a form of *intuitionistic type theory*, or *higher-order intuitionistic logic,* and underwent rapid development at many hands. Systems include those of Boileau (1975), Coste (1974), Boileau and Joyal (1981), Lambek and P. J. Scott (1986, based on earlier work of theirs), Zangwill (1977), and Fourman (1974, 1977), whose system, incorporating suggestions of Dana Scott, includes a description operator and an existence predicate. In Scott (1979) a system of higher-order intuitionistic logic is described, suitable for interpretation in toposes, which includes a description operator and an existence predicate. Fourman and Scott (1979) discuss in great detail the models of this logic given by sheaves and presheaves over a complete Heyting algebra Ω, the so-called Ω-*sets.* It should also be mentioned that Higgs (1973) independently defined the notion of Ω-set and established the equivalence, for a complete Boolean algebra *B,* of the topos of *B*-sets both with category of sets and maps in the Boolean extension $V^{(B)}$ of the universe of sets and with the category of canonical set-valued sheaves on *B.*

Just as each topos determines a type theory, so each type theory generates a topos, a fact discovered independently by several mathematicians, including Coste (1974), Fourman (1974), Lambek (1974), Boileau (1975), and Volger (1975).

A major stimulus behind Lawvere and Tierney's development of elementary topos theory was the desire to provide a categorical formulation of Cohen's proof (see Cohen 1965) of the independence of the continuum hypothesis from the axioms of set theory. This categorical version is given in Tierney (1972); later Bunge (1974) extended the method to give a categorical proof of the independence of Souslin's hypothesis. These results made it a pressing matter to determine the precise relationship between elementary topos theory and axiomatic set theory. The problem was resolved independently by Cole (1973), Mitchell (1972) and Osius (1975), each of whom showed that elementary topos theory, with the addition of certain "set-like" axioms, is logically equivalent to the weak (in fact finitely axiomatizable) version of Zermelo set theory in which the axiom scheme of separation is taken in its "predicative" form, that is, in which only formulas containing bounded quantifiers are admitted.

The relationship between (intuitionistic) Zermelo-Fraenkel set theory IZF and elementary topos theory has also been the focus of a number of investigations. In

[1] Lawvere and Tierney's definition of elementary topos originally required the presence of finite limits and colimits—these were later shown to be eliminable.

Grayson (1975), (1978), (1979), the Ω-valued universe of sets $V^{(\Omega)}$—equivalent to the topos of canonical sheaves on Ω—was introduced, shown to be a model of IZF, and used to establish various independence results. In Fourman (1980) , it was shown how the construction of the usual cumulative hierarchy of sets generated by a collection of atoms can be carried out within a locally small complete topos **E** (in particular, any Grothendieck topos). This leads to a topos **E***—in a certain sense a *well-founded* part of **E**—which is a model of IZF with atoms. By starting with suitable Grothendieck toposes **E**, the associated well-founded toposes **E*** were shown to be the topos-theoretic counterparts of the various models—permutation, Boolean-valued and symmetric—used for set-theoretic independence proofs. In Freyd (1980) a novel proof of the independence of the (countable) axiom of choice (AC) from (classical) ZF is presented, pivoting on the construction of a Boolean well-founded topos **E** (i.e. such that **E = E***) containing a sequence of nonempty objects B_1, B_2, … with empty product. Stimulated by Freyd's work, Blass and Scedrov (1989) forged a general correspondence between Boolean-valued models of set theory and Boolean Grothendieck toposes, explicating the relationship between Freyd's topos-theoretic methods for showing the independence of AC and the "classical" methods of Fraenkel and Cohen. By introducing the idea of a Grothendieck topos representing a Boolean-valued model of set theory, Blass and Scedrov showed (as did also, independently, Robert Solovay), that the Boolean-valued model represented by Freyd's simplest topos-theoretic model of ¬AC was actually a proper submodel, hitherto unnoticed by set-theorists, of the one that Cohen had originally constructed for the independence of AC.

The relationship between topos theory and IZF was later fully explicated by Joyal and Moerdijk (1991), and especially their (1995). Their objective was to provide a construction of (counterparts to) cumulative hierarchies of sets within a topos—actually, in somewhat more general categories known as Heyting pre-toposes—which is as independent as possible of the external notion of "completeness" of the topos. To this end they observed that the universe *V(A)* of sets built cumulatively from a collection *A* of atoms is a complete sup-lattice under the set-theoretic union operation, where by "completeness" is meant that all set-indexed suprema exist. In addition, *V(A)* is equipped with the unary operation sending *x* to {*x*}. Joyal and Moerdijk's key insight was that, with this algebraic structure—union and singleton—*V(A)* is freely generated by *A*: in (1995) such structures are termed *Zermelo-Fraenkel algebras.* They show how to define *V(A)* when *A* is an object in a topos **E**. For this, it was necessary to specify what the term "set-indexed" should be taken to mean in the definition of completeness, for if all the objects of **E** were to be regarded as sets, *V(A)* could not exist as an object of **E.** So **E** was assumed to be endowed with sufficient additional structure to specify exactly which objects are "set-sized" or "small". Given this, it was shown that various free Zermelo-Fraenkel algebras exist in **E,** not only *V(A)* itself, but also, for example, that corresponding to the class of von Neumann ordinals. Joyal and Moerdijk's work, which unifies all previous constructions of cumulative hierarchies in toposes, may be considered a synthesis of intuitionistic set theory and topos theory.

The connection between elementary topos theory and the set-theoretic independence proofs, in which Cohen's notion of *forcing* plays a pivotal role, led Joyal to observe in the early 1970s that the various notions of forcing (which in addition included those of Robinson and Kripke), could all be conceived as instances of a general truth-conditional scheme for sentences in a (sheaf) topos. This concept, which first explicitly appeared in print in Osius (1975), came to be known as *Kripke-Joyal semantics, sheaf semantics* or *topos semantics*. Topos semantics showed, in particular, that the rules for Cohen forcing coincide with the truth conditions for sentences in the topos of presheaves over the partially ordered set of forcing conditions, thus explaining the somewhat puzzling fact that, while Cohen's independence proofs employ only classical models of set theory, his (original) notion of forcing obeys *intuitionistic* rules.

R. Diaconescu (1975) established the important fact, conjectured by Lawvere, that, in a topos, the axiom of choice implies that the topos is Boolean. This means that, in IZF, the axiom of choice implies the law of excluded middle. This latter formulation of Diaconescu's result was refined by Goodman and Myhill (1978) to show that, in IZF, the law of excluded middle follows from the axiom of choice for unordered pairs. (It was later realized that the law or excluded middle in fact follows from the mere existence of a choice function on the power set of a set with two elements.) In Fourman and Scedrov (1982), it was shown, using a topos of presheaves, that in intuitionistic set theory even what they termed the "world's simplest axiom of choice"—the assertion that every family, all of whose elements are doubletons and which itself has at most a single member—may fail. (By contrast, in Grayson 1975 it is shown that Zorn's lemma, while classically equivalent to the axiom of choice, is *consistent* with IZF; see also Bell 1997.) In Bell (1993), a version of Diaconescu's theorem was used to show that the law of excluded middle is derivable within the intuitionistic version of Hilbert's ε-calculus.

Of major significance in the development of topos theory was the emergence of the concept of *classifying topos* for a first-order theory, that is, a topos obtained by freely adjoining a model of the theory to the topos of constant sets. The roots of this idea lie in the work of the Grothendieck school and in Lawvere's functorial semantics, but it was Joyal and Reyes (see Reyes 1974) who, in 1972, identified a general type of first-order theory, later called a *coherent* or *geometric* theory, which could be shown always to possess a classifying topos. This work was later extended by Reyes and Makkai to infinitary geometric theories: they showed that any topos is the classifying topos of such a theory (see Makkai and Reyes 1977). In 1973 Lawvere (see Lawvere 1975a) pointed out that, in virtue of Joyal and Reyes's work, a previous theorem of Deligne (see Artin *et al.* 1964, VI 9.0) on coherent toposes—the classifying toposes of (finitary) geometric theories—was equivalent to the Gödel-Henkin completeness theorem for geometric theories. He also observed that, analogously, the theorem of Barr (1974) on the existence of "enough" Boolean toposes was equivalent to a "Boolean-valued" completeness theorem for infinitary geometric theories.

The practice of topos theory quickly spawned an associated philosophy—jocularly known as "toposophy"— whose chief tenet is the idea that, like a model of set theory, any topos may be taken as a taken as an autonomous universe of discourse or "world" in which mathematical concepts can be interpreted and constructions performed. Accordingly topos theorists sought to "relativize"—i.e., suitably interpret and prove—within an arbitrary topos results which had originally been proved for the topos of sets. One of the most important of these results was the basic theorem of J. Giraud, originally presented in Grothendieck's Paris seminars of the early 1960s, which characterized categories of sheaves over a site in terms of exactness and size conditions. Giraud himself (1972) had proved a relative version of his theorem for Grothendieck toposes, but it was W. Mitchell who first formulated the correct form of the theorem for elementary toposes. However, Mitchell was able to prove only a special case of the theorem; it was Diaconescu (1975) who succeeded in proving the "relative Giraud theorem" in full generality. (Later Zangwill 1980 and Chapman and Rowbottom 1991 gave purely logical proofs of the theorem within the internal language.) The principal tool introduced by Diaconescu—his theorem characterizing geometric morphisms (the natural maps between toposes) to categories of internal presheaves in a topos—turned out also to be the key to proving the relative versions of the Joyal-Reyes results on classifying toposes. The major step in this direction was taken in by G. C. Wraith, who, in 1973, constructed the classifying topos for the theory of equality—the object classifier—over an arbitrary topos with a natural number object. The general existence theorem for classifying toposes for "internal" geometric theories soon followed: this was proved independently by Joyal, Bénabou (1975) and Tierney (1976). Classifying toposes for internal algebraic theories were constructed in Johnstone and Wraith (1978), and Blass (1989) showed that the existence of an object classifier over a given topos implies that the topos has a natural number object.

Much early work in topos theory was devoted to investigating the properties of the natural and real numbers within a topos. Freyd (1972), for example, showed that any object X of a topos isomorphic to $X + 1$ has a natural number object as a subobject. Bénabou (1973) associated a finite cardinal number with each natural number in a topos and showed that finite cardinals are decidable objects which behave as expected with respect to addition, multiplication and exponentiation, and, moreover, that the category of finite cardinals in a topos is itself a topos satisfying the axiom of choice, hence Boolean. In Coste-Roy *et al.* (1980) the concept of partial recursive function in a topos is defined and a version of Kleene's normal form theorem formulated and proved. Work on the natural numbers and finiteness in a topos also includes Acuna-Ortega and Linton (1979), Bell (1999a,b), Grayson (1978), Johnstone and Linton (1978), Kock *et al.* (1975), Stout (1987), and van de Wauw-de Kinder (1975). While the properties of systems of natural numbers in toposes proved not to differ greatly from those of their classical counterpart (apart from the fact, apparently first pointed out by Sols (1975) that, in a topos, the natural numbers cannot be well-ordered without the topos being Boolean), a marked divergence emerged in respect of the real numbers. It was found, for instance, that in a topos the constructions of the real numbers by means of Dedekind sections and Cauchy sequences in the rationals may yield non-isomorphic results: the first published account of the situation was provided by Johnstone in his 1977 book. In Johnstone (1979a) it is shown that, in a topos, the order-completeness of the Dedekind reals is equivalent to the truth of the logical law $\neg p \vee \neg\neg p$. Further study of the real numbers, and, more generally, of analysis and topology in topos-theoretic contexts was undertaken by Burden (1980), Burden and Mulvey (1979), Fourman (1976), Fourman and Hyland (1979), Grayson (1978), (1981), (1982), Hyland (1979), Johnstone (1977b), Mulvey (1974), (1980), Rousseau (1977), (1979), Stout (1975), (1976), (1978).

The growth of topos theory, and more particularly the study of sheaf toposes, stimulated the development of what came to be known, punningly, as "pointless topology". This arose from the observation, originating with Ehresmann (1957) and Bénabou (1958) that the essential characteristics of a topological space are carried, not by its set of points, but by the complete Heyting algebra of its open sets. Thus complete Heyting algebras came to be regarded as "generalized topological spaces" in their own right. As "frames" these were studied by C. H. Dowker and D. Papert Strauss throughout the 1960s and 70s (see, e.g., their 1966 and 1972. Isbell (1972) observed that not the category of frames itself, but rather its *opposite*—whose objects he termed *locales*—was in fact the appropriate generalization of the category of topological spaces. Locales accordingly became known as "pointless" spaces and the study of the properties of the category of locales "pointless topology". It was Joyal who first observed that the notion of locale provides the correct concept of topological space within a topos (a view later exploited to great effect in Joyal and Tierney 1984) and, more generally, in any context where the axiom of choice is not available. This latter observation was strikingly confirmed by Johnstone (1981) who showed that Tychonoff's theorem that the product of compact spaces is compact, known to be equivalent to the axiom of choice, can, suitably formulated in terms of locales, be proved without it. Johnstone became one of the champions of pointless topology, expounding the subject most persuasively in his book (1982), and elsewhere (e.g. in 1983a).

A "logical" approach to pointless topology—*formal spaces*—was introduced by Fourman and Grayson (1982). Here the (constructive) theory of locales was developed in a logical framework using the concept of *intuitionistic propositional theory*. Each such theory was shown to engender (the dual of) a locale—its formal space—whose properties reflect those of the theory: in particular, semantic completeness of the theory (that is, possession of sufficient models for a completeness theorem to hold for it) was shown to correspond to the condition that the formal space be a genuine space (that is, possess enough points). Under the name *formal topology,* this approach has been considerably refined and developed by G. Sambin and his students and associates within the more demanding constructive framework of Martin-Löf type theory (see, e.g. Sambin 1987, 1988, 1993, 1995, Valentini 1996 and Coquand *et al.* 2000).

Two major further developments in topos theory must also be mentioned: the construction of topos models of *synthetic differential geometry* (or *smooth infinitesimal analysis*) and of *recursion theory.* The first of these originated in Lawvere's 1967 lectures, in which he outlined a category-theoretic account of smoothness for maps on manifolds and proposed a revival of the use of nilpotent infinitesimals in the calculus and differential geometry. The 1967 lectures include the first construction of a (Grothendieck) topos containing an "infinitesimal" object Δ for which the tangent bundle of any space S may be identified with the function space S^{Δ}—so that, in particular, the tangent bundle $R \times R$ of the real line R is isomorphic to R^{Δ}. In Kock (1977) this latter isomorphism is given its explicit form (which led quickly to its becoming known as the *Kock-Lawvere axiom*) and systematically exploited in a development of the differential calculus. Dubuc (1979) contains the first construction of so-called *well-adapted models* for synthetic differential geometry, that is, of toposes realizing the Kock-Lawvere axiom and into which, in addition, the usual category of manifolds can be "nicely" embedded: in view of the fact that in well-adapted models every map between objects is "smooth", such models became known as *smooth toposes.* Kock (1981) is the first systematic presentation both of the axiomatic theory and of the models of synthetic differential geometry. Moerdijk and Reyes (1991) contains a full account of the construction of well-adapted models. Further work of importance in synthetic differential geometry was carried out by Bunge (1983a), Bunge and Dubuc (1986), (1987), Bunge and Gago (1988), and Bunge and Heggie (1984), Kock (1981b), Kock and Reyes (1979), (1981), McLarty (1983), Moerdijk (1987), Moerdijk and Reyes (1984), (1987), Penon (1981).

In the second of these developments, Hyland (1982), building on suggestions by Dana Scott, introduced a topos **Eff**—the *effective* or *realizability* topos—in which truth of first-order arithmetic sentences coincides with Kleene realizability and Church's thesis holds in the strong sense that every function from the natural numbers to themselves is recursive. Indeed, in **Eff** all functions between objects constructed from the natural numbers, for example the rationals and reals, are recursive, which makes it a natural setting for higher order recursion theory: Rosolini (1986) provides an axiomatization of a significant part of this theory. Hyland *et al.* (1980) develop a general apparatus for constructing **Eff**; an alternative method, due to Freyd, is presented in McLarty (1992). Realizability interpretations of IZF have been studied in McCarty [1984a, b].

A further strand within the evolution of categorical logic derives from the *algebraic analysis of deductive systems*, which, as has been pointed out, was adumbrated by Lambek in the 1950s. In the 1960s Lambek and M. E. Szabo promulgated the view that a deductive system is just a graph with additional structure, or equivalently, a category with missing equations (see, e.g. Lambek 1968, 1969, Szabo 1978, and Lambek and Scott 1986). In his work Lambek also systematically exploited the inverse view that, by assimilating its objects to statements or formulas, and its arrows to proofs or deductions, a category can be presented "equationally" as a certain kind of deductive system with an equivalence relation imposed on proofs.

As touched on above in connection with toposes, categories can also be viewed as *type theories.* From this perspective, the objects of a category are regarded as *types* (or sorts) and the arrows as *mappings* between the corresponding types. In the 1970s Lambek established that, viewed in this way, cartesian closed categories correspond to the typed λ-calculus (see Lambek 1972, 1974, 1980 and Lambek and Scott 1986). Seely (1984) proved that locally Cartesian closed categories correspond to Martin-Löf, or predicative, type theories (see Martin-Löf 1984, Beeson 1985, and the papers collected in Sambin and Smith 1998). Lambek and Dana Scott independently observed that *C-monoids, i.e.,* categories with products and exponentials and a single, nonterminal object correspond to the *untyped* λ-calculus. Scott showed that such monoids can be constructed from certain topological spaces, *continuous lattices* (see Scott 1972, 1973)—a startling discovery that touched off the development of *domain theory*—see, e.g. Taylor (1999). From the work of Hyland (1988) it may also be inferred that C-monoids exist in **Eff.** The analogy between type theories and categories has since led to what Jacobs (1999) terms a "type-theoretic boom", with much input from, and applications to, computer science: see, e.g. Carboni *et al.* (1988), Crole (1993), Ehrhardt (1988), Hyland and Pitts (1989) Obtulowicz and Wiweger (1979), Obtulowicz (1989), Seely (1987), Maietti (1998).

A number of books have appeared on categorical logic, topos theory, and type theory. The first of these was Johnstone (1977), a landmark work summing up virtually everything that had been done up in topos theory up to that time. Next to appear was Goldblatt (1979), which provided an elementary introduction to first-order logic in toposes. Lambek and Scott (1986) cover a wide array of topics in the higher order logic of Cartesian closed categories and toposes, while Barr and Wells (1986) elected to present topos theory entirely from the standpoint of categorical algebra. In Bell (1988) topos theory is developed almost exclusively through the internal language. Freyd and Scedrov (1990) is a compressed but highly original account of toposes and what they term *allegories*, categories generalizing the category of sets and relations. McLarty (1992) is a lucid introduction to topos theory, including descriptions of the effective and smooth toposes. Mac Lane and Moerdijk (1992) provide a detailed account of topos theory, with the concept of sheaf occupying central stage. The two volumes of Johnstone (2002) provide a monumental survey of topos theory. Vickers (1988) presents topology from the point of view of categorical logic. Taylor (1998) is a wide-ranging account of mathematical foundations in which categorical logic plays a major role. Barendrecht (1981) has become the standard work on the λ -calculus; categorical models of the λ-calculus are developed in Crole (1993). Jacobs (1999) is an exhaustive presentation of the categorical logic of type theories. The theory of continuous lattices is developed in Gierz *et al.*(1980). Martin-Löf type theory is presented in Martin-Löf (1984) and Nördstrom *et al.* (1990). Works on synthetic differential geometry include Kock (1981a), Moerdijk and Reyes (1991)—both mentioned above—Lavendhomme (1996) and an elementary book by Bell (1998).

2. Categories and Deductive Systems

The relationship between category theory and logic is most simply illustrated in the case of the (intuitionistic) propositional calculus. Accordingly let S be a set of sentences in a propositional language $\mathscr{L}$: S gives rise to a category $\mathscr{C}_S$ specified as follows. The objects of $\mathscr{C}_S$ are the sentences of $\mathscr{L}$, while the arrows of $\mathscr{C}_S$ are of two types. First, for sentences p, q of $\mathscr{L}$ we count as an arrow $\pi: p \to q$ any pair $<P, p>$ where P is a proof of q from $S \cup \{p\}$ in the intuitionistic propositional calculus. And second, for each sentence p the pair $<\varnothing, P>$ is to count as an arrow $p \to p$. Given two arrows $\pi = <P, p>: p \to q$ and $\sigma: <Q, q>: q \to r$ the composite $\sigma \circ \pi: p \to r$ is defined to be the pair $<P\bigstar Q, p>$ where $P\bigstar Q$ is the proof obtained by concatenating P and Q. The identity arrow on p is taken to be $<\varnothing, p>$.

The category $\mathscr{C}_S$ is called the *syntactic category* determined by S. Syntactic categories have a number of special properties. To begin with, they are Cartesian closed, with terminal object given by the identically true proposition $\top$, the product of two sentences p, q given by the conjunction $p \wedge q$ (with canonical proofs of conjuncts from conjunctions as projection arrows), and the implication sentence $p \Rightarrow q$ playing the role of the exponential object q^p. (This last follows from the fact that, for any r, there is a natural bijection between proofs of q from $S \cup \{r \wedge p\}$ and proofs of $p \Rightarrow q$ from $S \cup \{r\}$.) Moreover, $\mathscr{C}_S$ has an initial object in the form of the identically false proposition $\perp$ and coproducts given by disjunctions in the same way as products correspond to conjunctions. We may sum all this up in by saying that syntactic categories are *bicartesian closed.*

So a certain kind of deductive system—intuitionistic propositional calculi—can be regarded as a category of a special sort. Conversely, it is possible to furnish the concept of deductive system with a definition of sufficient generality to enable every category to be regarded as a special sort of deductive system. This is done by introducing the concept of a *graph.*

A (*directed*) *graph* consists of two classes: a class $\mathscr{A}$ whose members are called *arrows,* and a class $\mathscr{O}$ whose members are called *objects* (or *vertices*), together with two mappings from $\mathscr{A}$ to $\mathscr{O}$ called *source* and *target.* For $f \in \mathscr{A}$ we write $f: A \to B$ to indicate that *source*(f) = A and *target*(f) = B.

Now a *deductive system* is a graph in which with each object A there is associated an arrow $1_A: A \to A$, called the *identity arrow* on A, and with each pair of arrows $f: A \to B$ and $g: B \to C$ there is associated an arrow $g \circ f: A \to C$ called the *composite* of f and g. It is natural for a logician to think of the objects of a deductive system as *statements* and of the arrows as *deductions* or *proofs.* In this spirit the arrow compostion operation

$$\frac{f: A \to B \qquad g: B \to C}{g \circ f: A \to C}$$

may be thought of as a *rule of inference.*

A *category* may then be defined as a deductive system in which the following equations hold among arrows:

$$(*) \qquad f \circ 1_A = f = 1_B \circ f \qquad\qquad (h \circ g) \circ f = h \circ (g \circ f)$$

for all $f: A \to B$, $g: B \to C$, $h: C \to D$. Thus each category is a certain kind of deductive system. Conversely, by identifying proofs in such a way as to make equations (*) hold, any deductive system engenders a unique category.

3. Functorial Semantics

As observed in §1, Lawvere took the first step in providing the theory of models with a categorical formulation by introducing *algebraic theories,* the categorical counterparts of equational theories. Here the key insight was to view the logical operation of substitution in equational theories as composition of arrows in a certain sort of category. Lawvere showed how models of such theories can be naturally identified as functors of a certain kind, so launching the development of what has come to be known as *functorial semantics.*

An *algebraic theory* T is a category whose objects are the natural numbers and which for each m is equipped with an m-tuple of arrows, called *projections,*

$$\pi_i\colon m \to 1 \quad i = 1, \ldots, m$$

making m into the m-fold power of 1: $m = 1^m$. (Here 1 is *not* a terminal object in T.)

In an algebraic theory the arrows $m \to 1$ play the role of m-ary operations. Consider, for example, the algebraic theory *Rng* of *commutative rings*. Here arrows $m \to n$ are n-tuples of polynomials in the variables $x_1, \ldots, x_m$, with substitution of polynomials as composition. The projection arrow $\pi_i\colon m \to 1$ is just x_i considered as a polynomial in the variables $x_1, \ldots, x_m$. Each polynomial in m variables, as an arrow $m \to 1$, may be regarded as an m-ary operation in *Rng.*

In a similar way every equational theory—groups, lattices, Boolean algebras—may be assigned an associated algebraic theory.

Now suppose given a category $\mathscr{E}$ with finite products. A *model* of an algebraic theory T in $\mathscr{E}$, or a *T-algebra* in $\mathscr{E}$, is defined to be a finite product preserving functor $A\colon T \to \mathscr{E}$. The full subcategory of the functor category $\mathscr{E}^T$ whose objects are all T-algebras is called the *category of T-models* or *T-algebras* in $\mathscr{E}$, and is denoted by Alg(T, $\mathscr{E}$).

When T is the algebraic theory associated with an equational theory E, it is not hard to see that the category of T-models in $\mathbb{S}et$, the category of sets, is equivalent to the category of algebras axiomatized by E, that is, the category of commutative rings.

Lawvere later extended functorial semantics to *first-order* logic. Here the essential insight was that existential and universal quantification can be seen as left and right adjoints, respectively, of substitution.

To see how this comes about, consider two sets A and B and a map $f\colon A \to B$. The power sets PA and PB of A and B are partially ordered sets under inclusion, and so can be considered as categories. We have the map $f^{-1}\colon PB \to PA$ given by:

$$f^{-1}(Y) = \{x\colon f(x) \in Y\},$$

which, being inclusion-preserving, may be regarded as a functor between the categories PB and PA. Now define the maps $\forall_f, \exists_f\colon PA \to PB$ by

$$\forall_f(X) = \{y\colon \forall x(f(x) = y \Rightarrow x \in X\} \qquad \exists_f(X) = \{y\colon \exists x(x \in X \wedge f(x) = y\}.$$

These maps $\exists_f$ and $\forall_f$, which correspond to the existential and universal quantifiers, are easily checked to be respectively left and right adjoint to f^{-1}. Finally, think of the members of PA and PB as corresponding to *attributes* of the members of A and B (under which the attribute corresponding to a subset is just that of belonging to it), so that inclusion corresponds to entailment. Then, for any attribute Y on B, the definition of $f^{-1}(Y)$ amounts to saying that, for any $x \in A$, x has the attribute $f^{-1}(Y)$ just when $f(x)$ has the attribute Y. That is to say, the attribute $f^{-1}(Y)$ is obtained from Y by "substitution" along f. This is the sense in which quantification is adjoint to substitution.

Lawvere's concept of *elementary existential doctrine* (which we shall abbreviate to "elexdoc") presents this analysis of the existential quantifier in a categorical setting. Accordingly an elexdoc is given by the following data: a category T with finite products—here the objects of T are to be thought of as *types* and the arrows of T as *terms*—and for each object A of T a category $\mathcal{A}(A)$ called the *category of attributes* of A. For each arrow f: $A \to B$ we are also given a functor $\mathcal{A}(f)$: $\mathcal{A}(B) \to \mathcal{A}(A)$, to be thought of as substitution along f, which is stipulated to possess a left adjoint $\exists_f$—existential quantification along f.

The category $\mathbb{S}et$ provides an example of an elexdoc: here for each set A, the category of attributes $\mathcal{A}(A)$ is just PA and for f: $A \to B$, $\mathcal{A}(f)$ is f^{-1}. This elexdoc is *Boolean* in the sense that each category of attributes is a Boolean algebra and each substitution along maps a Boolean homomorphism.

Functorial semantics for elexdocs is most simply illustrated in the Boolean case. Thus a (set-valued) *model* of a Boolean elexdoc $(T, \mathcal{A})$ is defined to be a product preserving functor M: $T \to \mathbb{S}et$ together with, for each object A of T, a Boolean homomorphism $\mathcal{A}(A) \to P(MA)$ satisfying certain natural compatibility conditions.

This concept of model can be related to the usual notion of model for a first-order theory **T** in the following way. First one introduces the so-called "Lindenbaum" doctrine of **T**: this is the elexdoc $(T, \mathcal{A})$ where T is the algebraic theory whose arrows are just projections among the various powers of 1 and in which $\mathcal{A}(n)$ is the Boolean algebra of equivalence classes modulo provable equivalence from **T** of formulas having free variables among $x_1, \dots, x_n$. For $f : m \to n$, the action of $\mathcal{A}(f)$ corresponds to syntactic substitution, and in fact $\exists_f$ can be defined in terms of the syntactic $\exists$. It is not difficult to see that each model of **T** in the usual sense gives rise to a model of the corresponding elexdoc $(T, \mathcal{A})$.

4. Cartesian Closed Categories and the Typed λ-Calculus

As we have seen, when a category is considered as a deductive system, its objects are assimilated to statements or formulas, and its arrows to proofs or deductions. But, as remarked in §1, categories can also be viewed in another light, namely, as *type theories.* Here the objects of a category are to be regarded as *types* (or sorts) and the arrows as *mappings* between the corresponding types. In this spirit Lambek has shown that the *typed λ-calculus* corresponds to a *cartesian closed category.* Here we give a brief sketch of how this correspondence is established.

A *typed λ-calculus*[2] is a theory formulated within a certain kind of formal language which we shall call a *λ-language.* A language Λ of this kind is equipped with the following *basic symbols:*

- a *type symbol* **1**
- *ground types* **A, B, C, ...** (possibly none of these)
- *function symbols* **f, g, h,** … (possibly none of these)
- *variables* $x_{\mathbf{A}}$, $y_{\mathbf{A}}$, $z_{\mathbf{A}}$, … of each type **A,** where a *type* is as defined below
- unique entity ★ of type **1.**

The *types* of Λ are defined recursively as follows:

- **1** and each ground type are types
- $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is a type whenever $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are, where, if $n = 1$, $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is $\mathbf{A}_1$, while if $n = 0$, $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is **1** (*product types*)
- $\mathbf{B}^{\mathbf{A}}$ is a type whenever **A, B** are.

Each function symbol **f** is assigned a *signature* of the form $\mathbf{A} \to \mathbf{B}$, where **A, B** are types. We normally write $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ to indicate this.

Terms of Λ and their associated *types* are defined recursively as follows. We write τ: **A** to indicate that the term τ has type **A.**

Term **: type**	Proviso
★: **1**	
$x_{\mathbf{A}}$: **A** (we shall usually omit the subscript	
$\mathbf{f}(\tau)$: **B**	$\mathbf{f}: \mathbf{A} \to \mathbf{B}$ $\quad \tau: \mathbf{A}$
$\langle \tau_1, \ldots, \tau_n \rangle: \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, where $\langle \tau_1, \ldots, \tau_n \rangle$ is τ_1 if $n = 1$, and ★ if $n = 0$	$\tau_1: \mathbf{A}_1, \ldots, \tau_n: \mathbf{A}_n$
$(\tau)_i: \mathbf{A}_i$ where $(\tau)_i$ is τ if $n = 1$	$\tau: \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, $1 \le i \le n$
$\tau`\sigma$: **B**	$\tau : \mathbf{B}^{\mathbf{A}}$ $\quad \sigma : \mathbf{A}$
$\lambda_x \tau : \mathbf{B}^{\mathbf{A}}$	$x : \mathbf{A}$ $\quad \tau : \mathbf{B}$

We write $\tau(x/\sigma)$ or $\tau(\sigma)$ for the result of substituting σ at each free occurrence of x in τ, where an occurrence of x is *free* if it does not appear within a term of the form $\lambda_x\rho$. A term σ is *substitutable* for a variable x in a term τ if variable free in σ becomes bound when so substituted.

[2] For simplicity we consider only the typed lambda calculus without iterators.

An *equation* in Λ is an expression of the form $\sigma = \tau$, where σ and τ are terms of the same type. A *theory* in Λ (or simply a *λ-theory*) is a set T of equations satisfying the following conditions:

- $\sigma = \sigma \in T \quad \sigma = \tau \in T \Rightarrow \tau = \sigma \in T$
  $\sigma = \tau \in T \ \&\ \tau = \upsilon \in T \Rightarrow \sigma = \upsilon \in T$
- $\sigma = \upsilon \in T \Rightarrow \tau`\sigma = \tau`\upsilon \in T \qquad \sigma = \tau \in T \Rightarrow \lambda_x\sigma = \lambda_x\tau \in T$
- T contains all equations of the following forms:
  $\tau = \bigstar$ for $\tau : \mathbf{1}$
  $(<\tau_1, \ldots, \tau_n>)_i = \tau_i \qquad \tau = <(\tau)_1, \ldots, (\tau)_n>$
  $(\lambda_x\tau)`\sigma = \tau(x/\sigma)$ for all σ substitutable for x in τ
  $\lambda_x(\tau`x) = \tau$ for all $\tau : \mathbf{B}^{\mathbf{A}}$, provided x: $\mathbf{A}$ is not free in τ.

Now λ-theories comprise the objects of a category λ-$\mathfrak{Th}$ whose arrows are the so-called *translations* between such theories. Let us define a *translation* $\mathbf{K}$ of a λ-language $\mathscr{L}$ into a λ-language $\mathscr{L}'$ to be a map which assigns to each type $\mathbf{A}$ of $\mathscr{L}$ a type $\mathbf{KA}$ of $\mathscr{L}'$ and to each function symbol $\boldsymbol{f}$: $\mathbf{A} \to \mathbf{B}$ a function symbol of $\mathscr{L}'$ of signature $\mathbf{KA} \to \mathbf{KB}$ in such a way that

$$\mathbf{K1} = \mathbf{1}, \ \mathbf{K}(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n) = \mathbf{KA}_1 \times \ldots \times \mathbf{KA}_n, \ \ \mathbf{K}(\mathbf{B}^{\mathbf{A}}) = \mathbf{KB}^{\mathbf{KA}}.$$

Any translation $\mathbf{K}$: $\mathscr{L} \to \mathscr{L}'$ may be extended to the terms of $\mathscr{L}$ in the evident recursive way—i.e., by defining $\mathbf{K}x_{\mathbf{A}} = x_{\mathbf{KA}}$, $\mathbf{K}\bigstar = \bigstar$, $\mathbf{K}(\boldsymbol{f}(\tau)) = \mathbf{K}\boldsymbol{f}(\mathbf{K}\tau)$, $\mathbf{K}(\lambda_x\tau) = \mathbf{K}(\lambda_{\mathbf{K}x}\mathbf{K}\tau)$ etc.—so that if $\tau : \mathbf{A}$, then $\mathbf{K}\tau : \mathbf{KA}$. We shall sometimes write $\alpha_{\mathbf{K}}$ for $\mathbf{K}\alpha$ If T, T' are λ-theories in $\mathscr{L}$, $\mathscr{L}'$ respectively, a translation $\mathbf{K}$: $\mathscr{L} \to \mathscr{L}'$ is a *translation of T into T'*, and is written $\mathbf{K}$: $T \to T'$ if, for any equation $\sigma = \tau$ in T, the equation $\mathbf{K}\sigma = \mathbf{K}\tau$ is in T'.

As we shall see, λ-theories correspond to Cartesian closed categories in such a way as to make the category λ-$\mathfrak{Th}$ equivalent to the category $\mathcal{C}_{axt}$ of Cartesian closed categories. To establish this, we first introduce the notion of an *interpretation* I of a λ-language $\mathscr{L}$ in a cartesian closed category $\mathcal{C}$. This is defined to be an assignment

- to each type $\mathbf{A}$, of a $\mathcal{C}$-object $\mathbf{A}_I$ such that:
  $(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n)_I = (\mathbf{A}_1)_I \times \ldots \times (\mathbf{A}_n)_I$,
  $(\mathbf{B}^{\mathbf{A}})_I = \mathbf{B}_I{}^{\mathbf{A}I}$
  $\mathbf{1}_I = 1$, the terminal object of $\mathcal{C}$,
- to each function symbol $\boldsymbol{f}$: $\mathbf{A} \to \mathbf{B}$, a $\mathcal{C}$-arrow $\boldsymbol{f}_I$: $\mathbf{A}_I \to \mathbf{B}_I$.

We shall sometimes write $A_{\mathcal{C}}$ or just A for $\mathbf{A}_I$.

We extend I to terms of $\mathscr{L}$ as follows. If $\tau : \mathbf{B}$, write $\boldsymbol{x}$ for $(x_1, \ldots, x_n)$, any sequence of variables containing all variables of τ (and call such sequences *adequate* for τ. Define the $\mathcal{C}$-arrow

$$[\![\tau]\!]_{\boldsymbol{x}}: A_1 \times \ldots \times A_n \to B$$

recursively as follows:

$$[\![\bigstar]\!]_{\boldsymbol{x}} = A_1 \times \ldots \times A_n \longrightarrow B$$
$$[\![x_i]\!]_{\boldsymbol{x}} = \pi_i : A_1 \times \ldots \times A_n \longrightarrow 1$$
$$[\![\boldsymbol{f}(\tau)]\!]_{\boldsymbol{x}} = \boldsymbol{f}_I \circ [\![\tau]\!]_{\boldsymbol{x}}$$
$$[\![< \tau_1, \ldots, \tau_n >]\!]_{\boldsymbol{x}} = \ < [\![\tau_1]\!]_{\boldsymbol{x}}, \ldots, [\![\tau_n]\!]_{\boldsymbol{x}} >$$

$$[\![(\tau)_i]\!]_{\boldsymbol{x}} = \pi_i \circ [\![\tau]\!]_{\boldsymbol{x}}$$
$$[\![\tau\text{'}\sigma]\!]_{\boldsymbol{x}} = ev \circ \langle [\![\tau]\!]_{\boldsymbol{x}}, [\![\sigma]\!]_{\boldsymbol{x}} \rangle$$
$$[\![\lambda_y \tau]\!]_{\boldsymbol{x}} = [\![\tau]\!]_{\boldsymbol{xy}}$$

Here ^ denotes exponential transpose and ev the appropriate evaluation arrow in the definition of the exponential.

Next, one shows that any λ-theory T determines a Cartesian closed category $\mathcal{C}(T)$. The objects of $\mathcal{C}(T)$ are taken to be the types of the language of T. The arrows of $\mathcal{C}(T)$ are pairs of the form (x, τ), where τ is a term with no free variables other than x, two such pairs (x, τ) and (y, σ) being identified whenever the equation $\tau = \sigma$ is a member of T. The identity arrow on a type **A** is the pair $(x_{\mathbf{A}}, x_{\mathbf{A}})$. The composite of (x, τ): $\mathbf{A} \to \mathbf{B}$ and (y, σ): $\mathbf{B} \to \mathbf{C}$ is given by the pair $(x, \sigma(y/\tau))$. It is now readily checked that $\mathcal{C}(T)$ is a Cartesian closed category with terminal object the type **1** and in which products and exponentials are given by the analogous operations on types. Moreover,

$$1_{\mathbf{A}} = (x_{\mathbf{A}}, \bigstar)$$
$$\pi_i = (z, (z)_i) \ \ (\text{with } z\text{: } \mathbf{A}_1 \times \ldots \times \mathbf{A}_n)$$
$$<(z, \sigma), (z, \tau)> = (z, <\sigma, \tau>)$$
$$(z, \tau)\hat{} = (x, \lambda_y\tau(<x, y>) \text{ with } z\text{: } \mathbf{A} \times \mathbf{B}, \ x\text{: } \mathbf{A}, \ y\text{: } \mathbf{B}$$
$$\mathrm{ev}_{\mathbf{C},\mathbf{A}} = (y, (y)_2\text{'}(y)_1) \ \ \text{with } y\text{: } \mathbf{C}^{\mathbf{A}} \times \mathbf{A}.$$

There is a canonical interpretation of $\mathcal{L}$ in $\mathcal{C}(T)$—which we denote by the same expression $\mathcal{C}(T)$—given by

$$\mathbf{A}_{\mathcal{C}(T)} = \mathbf{A} \qquad \boldsymbol{f}_{\mathcal{C}(T)} = (x, \boldsymbol{f}(x)).$$

A straightforward induction shows that, for any term τ with free variables $\boldsymbol{x} = (x_1, \ldots, x_n)$ of types $\mathbf{A}_1, \ldots, \mathbf{A}_n$, we have

$$[\![\tau]\!]_{\boldsymbol{x}} = (z, \tau)$$

where z is a variable of type $\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$.

Inversely, each Cartesian closed category $\mathcal{E}$ determines a λ-theory Th($\mathcal{E}$) in the following way. First, one defines the *internal language* $\mathcal{L}_{\mathcal{E}}$ of $\mathcal{E}$: the ground type symbols of $\mathcal{L}_{\mathcal{E}}$ are taken to match the objects of $\mathcal{E}$ are taken to match the objects of $\mathcal{E}$ other than 1, that is, for each $\mathcal{E}$-object A other than 1 we assume given a ground type **A.** Next, we with each type symbol **A** of $\mathcal{L}_{\mathcal{E}}$ we associate the $\mathcal{E}$-object $\mathbf{A}_{\mathcal{E}}$ defined by

$$\mathbf{A}_{\mathcal{E}} = A \text{ for ground types } \mathbf{A}, \ \ (\mathbf{A} \times \mathbf{B})_{\mathcal{E}} = \mathbf{A}_{\mathcal{E}} \times \mathbf{B}_{\mathcal{E}}, \ \ (\mathbf{B}^{\mathbf{A}})_{\mathcal{E}} = \mathbf{B}_{\mathcal{E}}{}^{\mathcal{A}_{\mathcal{E}}}.$$

As function symbols in $\mathcal{L}_{\mathcal{E}}$ we take triples of the form $(f, \mathbf{A}, \mathbf{B}) = \boldsymbol{f}$ with f: $\mathbf{A}_{\mathcal{E}} \to \mathbf{B}_{\mathcal{E}}$ in $\mathcal{E}$. The signature of $\boldsymbol{f}$ is $\mathbf{A} \to \mathbf{B}$.

The *natural interpretation*—denoted by $\mathcal{E}$—of $\mathcal{L}_{\mathcal{E}}$ in $\mathcal{E}$ is determined by the assignments

$$\mathbf{A}_{\mathcal{E}} = A \ \text{ for ground types } \mathbf{A}, \ (f, \mathbf{A}, \mathbf{B}) = f.$$

The λ-*theory* Th($\mathcal{E}$) *of* $\mathcal{E}$ is now defined in the following way. Given a pair of terms σ, τ of the same type in $\mathcal{L}_{\mathcal{E}}$ write $\boldsymbol{x}$ for the sequence of variables occurring free in either σ or τ.

The theory Th($\mathscr{E}$) consists of all equations $\sigma = \tau$ for which the arrows $([\![\sigma]\!]_{\boldsymbol{x}})_{\mathscr{E}}$ and $([\![\tau]\!]_{\boldsymbol{x}})_{\mathscr{E}}$ coincide.

It can then be shown that the canonical functor F: $\mathscr{E} \to \mathscr{C}(\mathrm{Th}(\mathscr{E}))$ defined by

$$FA = \mathbf{A} \ \text{ for each } \mathscr{E}\text{-object } A$$
$$Ff = (x, \boldsymbol{f}(x)) \text{ for each } \mathscr{E}\text{-arrow } f\colon A \to B$$

is an *isomorphism of categories.*

Similarly, for each λ-theory T there is a canonical translation G: $T \to \mathrm{Th}(\mathscr{C}(T))$ given by

$$G\mathbf{A} = \mathbf{A} \ \text{ for each type } \mathbf{A}$$
$$G\mathbf{f} = ((x, \boldsymbol{f}(x)), \mathbf{A}, \mathbf{B}) \text{ for } \boldsymbol{f}\colon \mathbf{A} \to \mathbf{B}.$$

This translation is clearly an isomorphism in the category λ-$\mathscr{T}\!h$. Now let $\mathscr{C}art$ the category whose objects are cartesian closed categories and whose arrows are functors between these preserving terminal objects, products, and exponentials. The mappings $\mathscr{C}$ and Th act as functors λ-$\mathscr{T}\!h \to \mathscr{C}art$ and $\mathscr{C}art \to \lambda$-$\mathscr{T}\!h$ respectively; in view of the fact that, for any objects $\mathscr{E}$ of $\mathscr{C}art$, T of λ-$\mathscr{T}\!h$, $\mathscr{C}(\mathrm{Th}(\mathscr{E})) \cong \mathscr{E}$ and $\mathrm{Th}(\mathscr{C}(T)) \cong T$, these functors are equivalences. Accordingly λ-$\mathscr{T}\!h$ and $\mathscr{C}art$ are equivalent categories. This is one exact sense in which a formal theory is completely representable in categorical terms.

5. Local Set Theories /Intuitionistic Type Theories

In the previous section we have described the categorical counterparts to equational and first-order logic. It is natural now to ask: what sort of category corresponds to *higher-order* logic? As remarked in §1, the answer to this question—an *elementary topos*— was provided in 1969 by Lawvere and Tierney. It was soon realized that the system of higher-order logic associated with a topos is most conveniently formulated as a generalization of classical set theory within intuitionistic logic: *intuitionistic type theory.* The system to be described here, *local set theory,* is a modification, due to Zangwill (1977), of that of Joyal and Boileau, later published as their (1981).

The category of sets is a prime example of a topos, and the fact that it is a topos is a consequence of the axioms of classical set theory. Similarly, in a local set theory the construction of a corresponding "category of sets" can also be carried out and shown to be a topos. In fact *any* topos is obtainable (up to equivalence of categories) as the category of sets within some local set theory. Toposes are also, in a natural sense, the *models* or *interpretations* of local set theories. Introducing the concept of *validity* of an assertion of a local set theory under an interpretation, such interpretations are *sound* in the sense that any theorem of a local set theory is valid under every interpretation validating its axioms and *complete* in the sense that, conversely, any assertion of a local set theory valid under every interpretation validating its axioms is itself a theorem. The basic axioms and rules of local set theories are formulated in such a way as to yield as theorems precisely those of higher-order intuitionistic logic. These basic theorems accordingly coincide with those statements that are valid under *every* interpretation.

In a local set theory the set concept, as a primitive, is replaced by that of *type.* A type in this sense may be thought of as a *natural kind* or *species* from which sets are extracted as subspecies. The resulting set theory is *local* in the sense that, for example, the inclusion relation will obtain only among sets which have the same type, i.e. are subspecies of the same species.

A local set theory, then, is a type-theoretic system built on the same primitive symbols =, ∈, {:} as classical set theory, in which the set-theoretic operations of forming products and powers of types can be performed, and which in addition contains a "truth value" type acting as the range of values of "propositional functions" on types. A local set theory is determined by specifying a collection of *axioms* formulated within a *local language* defined as follows.

5.1 Local languages and local set theories

A *local language* $\mathscr{L}$ has the following *basic symbols:*

- **1** (*unity type*) **Ω** (*truth value type*)
- **S, T, U,...** (*ground types*: possibly none of these)
- **f, g, h,...** (*function symbols*: possibly none of these)
- $x_{\mathbf{A}}$, $y_{\mathbf{A}}$, $z_{\mathbf{A}}$, **...** (*variables of each type* **A**, where a *type* is as defined below)
- (*unique entity of type* **1**)

The *types* of $\mathscr{L}$ are defined recursively as follows:

1, Ω are types

any ground type is a type

$\mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ is a type whenever $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are, where, if $n = 1$, $\mathbf{A}_1 \times .. \times \mathbf{A}_n$ is $\mathbf{A}_1$, while if $n = 0$, $\mathbf{A}_1 \times .. \times \mathbf{A}_n$ is **1** (*product types*)

- **PA** is a type whenever **A** is (*power types*)

Each function symbol **f** is assigned a *signature* of the form $\mathbf{A} \to \mathbf{B}$, where **A, B** are types; this is indicated by writing $\mathbf{f}: \mathbf{A} \to \mathbf{B}$.

Terms of $\mathscr{L}$ and their associated *types* are defined recursively as follows. We write τ: **A** to indicate that the term τ has type **A.**

Term: **type**	Proviso
$\bigstar$: **1**	
$x_{\mathbf{A}}$: **A**	
$\mathbf{f}(\tau)$: **B**	$\mathbf{f}: \mathbf{A} \to \mathbf{B}$ $\quad\tau: \mathbf{A}$
$<\tau_1, \ldots, \tau_n>: \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, where $<\tau_1, \ldots, \tau_n>$ is τ_1 if $n = 1$, and $\bigstar$ if $n = 0$.	$\tau_1: \mathbf{A}_1, \ldots, \tau_n: \mathbf{A}_n$
$(\tau)_i: \mathbf{A}_i$ where $(\tau)_i$ is τ if $n = 1$	$\tau: \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, $1 \leq i \leq n$
$\{x_{\mathbf{A}}: \alpha\}$: **PA**	$\alpha : \Omega$
$\sigma = \tau: \Omega$	σ, τ of same type
$\sigma \in \tau$	σ: **A,** τ: **PA** for some type **A**

Terms of type Ω are called *formulas, propositions,* or *truth values.* Notational conventions we shall adopt include:

$\omega, \omega', \omega''$	variables of type Ω
α, β, γ	formulas
$x, y,, z \ldots$	$x_{\mathbf{A}}, y_{\mathbf{A}}, z_{\mathbf{A}} \ldots$
$\tau(x/\sigma)$ or $\tau(\sigma)$	result of substituting σ at each free occurrence of x in τ: an occurrence of x is *free* if it does not appear within $\{x: \alpha\}$
$\alpha \leftrightarrow \beta$	$\alpha = \beta$
$\Gamma : \alpha$	sequent notation:Γ a finite set of formulas
$: \alpha$	$\varnothing: \alpha$

A term is *closed* if it contains no free variables; a closed term of type Ω is called a *sentence.*

The *basic axioms* for $\mathscr{L}$ are as follows:

Unity $\quad : x_1 = \bigstar$
Equality $\quad x = y, \alpha(z/x) : \alpha(z/y) \quad (x, y$ free for z in $\alpha)$
Products $\quad : (<x_1, \ldots, x_n>)_i = x_i$
$\quad : x = <(x)_1, \ldots, (x)_n>$
Comprehension $\quad : x \in \{x: \alpha\} \leftrightarrow \alpha$

The *rules of inference* for $\mathscr{L}$ are:

Thinning

$$\frac{\Gamma : \alpha}{\beta, \Gamma: \alpha}$$

Restricted Cut

$$\frac{\Gamma : \alpha \quad \alpha, \Gamma : \beta}{\Gamma : \beta} \qquad \text{(any free variable of } \alpha \text{ free in } \Gamma \text{ or } \beta)$$

Substitution

$$\frac{\Gamma : \alpha}{\Gamma(x/\tau) : \alpha(x/\tau)} \qquad (\tau \text{ free for } x \text{ in } \Gamma \text{ and } \alpha)$$

Extensionality

$$\frac{\Gamma : x \in \sigma \leftrightarrow x \in \tau}{\Gamma : \sigma = \tau} \qquad (x \text{ not free in } \Gamma, \sigma, \tau)$$

Equivalence

$$\frac{\alpha, \Gamma : \beta \quad \beta, \Gamma : \alpha}{\Gamma : \alpha \leftrightarrow \beta}$$

These axioms and rules of inference yield a system of *natural deduction* in $\mathscr{L}$. If S *is* any collection of sequents in $\mathscr{L}$, we say that the sequent $\Gamma : \alpha$ is *deducible from S*, and write $\Gamma \vdash_S \alpha$ provided there is a deduction of $\Gamma : \alpha$ using the basic axioms, the sequents in S, and the rules of inference. We shall also write $\Gamma \vdash_S \alpha$ for $\Gamma \vdash_\varnothing \alpha$ and $\vdash_S \alpha$ for $\varnothing \vdash_S \alpha$.

A *local set theory* in $\mathscr{L}$ is a collection S of sequents closed under deducibility from S. Any collection of sequents S *generates* the local set theory S^* comprising all the sequents deducible from S. The local set theory in $\mathscr{L}$ generated by $\varnothing$ is called *pure* local set theory in $\mathscr{L}$.

5.2. Logic in a local set theory.

The *logical operations* in $\mathscr{L}$ are defined as follows:

Logical Operation	**Definition**
$\top$ (true)	$\star = \star$
$\alpha \wedge \beta$	$\langle\alpha, \beta\rangle = \langle\top, \top\rangle$
$\alpha \rightarrow \beta$	$(\alpha \wedge \beta) \leftrightarrow \alpha$
$\forall x\,.\alpha$	$\{x: \alpha\} = \{x: \top\}$
$\bot$ (false)	$\forall\omega.\ \omega$
$\neg\alpha$	$\alpha \rightarrow \bot$
$\alpha \vee \beta$	$\forall\omega[(\alpha \rightarrow \omega \wedge \beta \rightarrow \omega) \rightarrow \omega]$
$\exists x\, \alpha$	$\forall\omega[\forall x(\alpha \rightarrow \omega) \rightarrow \omega]$

We also write $x \neq y$ for $\neg(x = y)$, $x \notin y$ for $\neg(x \in y)$, and $\exists! x\, \alpha$ for $\exists x[\alpha \wedge \forall y\, \alpha(x/y) \rightarrow x = y]$.

It can now be shown that the logical operations on formulas just defined satisfy the axioms and rules of free intuitionistic logic. (For this reason local set theories are also known as *intuitionistic type theories.*) We present just a few of the relevant derivations.

We write

$$\frac{\Gamma_1 : \alpha_1, \ldots, \Gamma_n : \alpha_n}{\Delta : \beta}$$

for deducibility of $\Delta : \beta$ from $\Gamma_1 : \alpha_1, \ldots, \Gamma_n : \alpha_n$.

$\vdash x = x.$

Derivation: $: (x)_1 = x.$

$\alpha \vdash \alpha$

Derivation:

$$\frac{\omega, \omega = \omega : \omega \qquad \dfrac{}{: \omega = \omega}}{\dfrac{\omega : \omega}{\alpha : \alpha}}$$

$$\frac{\Gamma : \alpha \qquad \Gamma : \boldsymbol{\beta}}{\Gamma : \alpha \wedge \beta}$$

Derivation:

$$\frac{\Gamma : \alpha \qquad \dfrac{\dfrac{}{\beta : \beta = \top} \qquad \dfrac{\dfrac{}{\alpha : \alpha = \top} \qquad \dfrac{}{\alpha = \top, \beta = \top : \alpha \wedge \beta}}{\alpha, \beta = \top : \alpha \wedge \beta}}{\Gamma, \beta : \alpha \wedge \beta}}{\Gamma : \alpha \wedge \beta}$$

$$\frac{\Gamma : \alpha \leftrightarrow \beta}{\Gamma : \{x : \alpha\} = \{x : \beta\}}$$

Deduction:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma : \alpha \leftrightarrow \beta}{\quad}}{\alpha, \Gamma : \beta} \qquad \cfrac{\cfrac{\Gamma : \alpha \leftrightarrow \beta}{\quad}}{\beta, \Gamma : \alpha}}{x \in \{x : \alpha\}, \Gamma : x \in \{x : \beta\} \qquad x \in \{x : \beta\}, \Gamma : x \in \{x : \alpha\}}}{\Gamma : x \in \{x : \alpha\} \leftrightarrow x \in \{x : \beta\}}}{\Gamma : \{x : \alpha\} = \{x : \beta\}}$$

In any local set theory the following *Modified Cut Rule* holds:

(i) $$\frac{\Gamma : \alpha \qquad \alpha, \Gamma : \beta}{\exists x_1(x_1 = x_1), \ldots, \exists x_n(x_n = x_n), \Gamma : \beta}$$

where $x_1, \ldots, x_n$ are the free variables of α not occurring freely in Γ or β.

(ii) $$\frac{\Gamma : \alpha \qquad \alpha, \Gamma : \beta}{\Gamma : \beta}$$ provided that, whenever **A** is the type of a free variable of α with no free occurrences in Γ or β, there is a closed[3] term of type **A.**

5.3. Set theory in a local language

We can now introduce the concept of *set* in a local language. A *set-like* term is a term of power type; a *closed* set-like term is called an ($\mathscr{L}$-) *set.* We shall use upper case italic letters *X, Y, Z, ...* for sets, as well as standard abbreviations such as $\forall x \in X.\alpha$ for $\forall x(x \in X \to \alpha)$. The set theoretic *operations* and *relations* are defined as follows. Note that in the definitions of $\subseteq$, $\cap$, and $\cup$, X and Y *must be of the same type*:

Operation	**Definition**
$\{x \in X: \alpha\}$	$\{x: x \in X \wedge \alpha\}$
$X \subseteq Y$	$\forall x \in X.\ x \in Y$
$X \cap Y$	$\{x: x \in X \wedge x \in Y\}$
$X \cup Y$	$\{x: x \in X \vee x \in Y\}$
$x \notin X$	$\neg(x \in X)$
$U_{\mathbf{A}}$ or A	$\{x_{\mathbf{A}}: \top\}$
$\varnothing_{\mathbf{A}}$ or $\varnothing$	$\{x_{\mathbf{A}}: \perp\}$
$E - X$	$\{x: x \in E \wedge x \notin X\}$
PX	$\{u: u \subseteq X\}$
$\bigcap U$ $(U: \mathbf{PPA})$	$\{x: \forall u \in U.\ x \in u\}$
$\bigcup U$ $(U: \mathbf{PPA})$	$\{x: \exists u \in U.\ x \in u\}$
$\bigcap_{i \in I} X_i$	$\{x: \forall i \in I.\ x \in X_i\}$
$\bigcup_{i \in I} X_i$	$\{x: \exists i \in I.\ x \in X_i\}$
$\{\tau_1, \ldots, \tau_n\}$	$\{x : x = \tau_1 \vee \ldots \vee x = \tau_n\}$
$\{\tau : \alpha\}$	$\{z : \exists x_1 \ldots \exists x_n (z = \tau \wedge \alpha)\}$
$X \times Y$	$\{<x,y>: x \in X \wedge y \in Y\}$
$X + Y$	$\{<\{x\},\varnothing>: x \in X\} \cup \{<\varnothing,\{y\}>: y \in Y\}$
$Fun(X,Y)$	$\{u: u \subseteq X \times Y \wedge \forall x \in X\, \exists! y \in Y.\ <x,y> \in u\}$

The standard facts concerning the set-theoretic operations and relations now follow as straightforward consequences of their definitions.

Given a term τ such that

$$<x_1, \ldots, x_n> \in X \vdash_S \tau \in Y$$

we write $(<x_1, \ldots, x_n> \mapsto \tau)$ or simply $\boldsymbol{x} \mapsto \tau$ for

$$\{<<x_1, \ldots, x_n>, \tau>: <x_1, \ldots, x_n> \in X\}.$$

Clearly we have

[3] A term is *closed* if it contains no free variables.

$$\vdash_S (<x_1, \ldots, x_n> \mapsto \tau)) \in Fun(X, Y),$$

and so we may think of $(<x_1, \ldots, x_n> \mapsto \tau)$ as the function from X to Y determined by τ.

We now show that each local set theory determines a topos. Let S be a local set theory in a local language $\mathscr{L}$. Define the relation $\approx_S$ on the collection of all $\mathscr{L}$-sets by

$$X \approx_S Y \quad \text{iff} \quad \vdash_S X = Y.$$

This is an equivalence relation. An *S-set* is an equivalence class $[X]_S$ —which we normally identify with X—of $\mathscr{L}$-sets under the relation $\approx_S$. An *S-map* $f\colon X \to Y$ is a triple (f, X, Y)—normally identified with f—of S-sets such that $\vdash_S f \in Y^X$. X and Y are, respectively, the *domain* $\mathrm{dom}(f)$ and the *codomain* $\mathrm{cod}(f)$ of f. It is now readily shown that the collection of all S-sets and maps forms a category $\mathscr{C}(S)$, the *category of S-sets,* in which the composite of two maps $f\colon X \to Y$ and $g\colon Y \to Z$ is given by

$$g \circ f = \{<x, z>: \exists y(<x, y> \in f \wedge <y, z> \in g)\}.$$

In fact, $\mathscr{C}(S)$ is a *topos,* the *topos of sets determined by S.* It has terminal object U_1**,** the product of two objects (S-sets) X, Y is the S-set $X \times Y$, with projections given by

$$\pi_1 = (<x, y> \mapsto x)\colon X \times Y \to X, \qquad \pi_2 = (<x, y> \mapsto x)\colon X \times Y \to Y,$$

its truth-value object is (Ω, t), where $t\colon 1 \to \Omega$ is the S-map $\{<\bigstar, \top>\}$, and the power object of an object X is (PX, e_X), where $e_X\colon X \times PX \to \Omega$ is the S-map $<x, z> \mapsto x \in z$. All this is proved in much the same way as for classical set theory.

5.4. Interpreting a local language in a topos. The soundness and completeness theorems

Let $\mathscr{L}$ be a local language and $\mathscr{E}$ a topos. An *interpretation I* of $\mathscr{L}$ in $\mathscr{E}$ is an assignment:

- to each type $\mathbf{A}$, of an $\mathscr{E}$-object $\mathbf{A}_I$ such that:
  $(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n)_I = (\mathbf{A}_1)_I \times \ldots \times (\mathbf{A}_n)_I,$
  $(\mathbf{PA})_I = P\mathbf{A}_I,$
  $\mathbf{1}_I = 1$, the terminal object of $\mathscr{E}$,
  $\mathbf{\Omega}_I = \Omega$, the truth-value object of $\mathscr{E}$.

- to each function symbol $\mathbf{f}\colon \mathbf{A} \to \mathbf{B}$, an $\mathscr{E}$-arrow $\boldsymbol{f}_I\colon \mathbf{A}_I \to \mathbf{B}_I$.

We shall sometimes write $A_{\mathscr{E}}$ or just A for $\mathbf{A}_I$.

We extend I to terms of $\mathscr{L}$ as follows. If $\tau : \mathbf{B}$, write $\boldsymbol{x}$ for $(x_1, \ldots, x_n)$, any sequence of variables containing all variables of τ (and call such sequences *adequate* for τ). Define the $\mathscr{E}$-arrow

$$\llbracket \tau \rrbracket_{\boldsymbol{x}} : A_1 \times \ldots \times A_n \longrightarrow B$$

recursively as follows:

$$\llbracket \bigstar \rrbracket_{\boldsymbol{x}} = A_1 \times \ldots \times A_n \longrightarrow B$$

$$\llbracket x_i \rrbracket_{\boldsymbol{x}} = \pi_i : A_1 \times \ldots \times A_n \longrightarrow 1$$

$$\llbracket \mathbf{f}(\tau) \rrbracket_{\boldsymbol{x}} = \boldsymbol{f}_I \circ \llbracket \tau \rrbracket_{\boldsymbol{x}}$$

$$\llbracket < \tau_1, \ldots, \tau_n > \rrbracket_{\boldsymbol{x}} = \ < \llbracket \tau_1 \rrbracket_{\boldsymbol{x}}, \ldots, \llbracket \tau_n \rrbracket_{\boldsymbol{x}} >$$

$$\llbracket (\tau)_i \rrbracket_{\boldsymbol{x}} = \pi_i \circ \llbracket \tau \rrbracket_{\boldsymbol{x}}$$

$$\llbracket \{y : \alpha\} \rrbracket_{\boldsymbol{x}} = (\llbracket \alpha(y/u) \rrbracket_{u\boldsymbol{x}} \circ can)^{\wedge}$$

where in this last clause u differs from $x_1, \ldots, x_n$, is free for y in α, y is of type **C**, (so that B is of type **PC**), *can* is the canonical isomorphism $C \times (A_1 \times \ \ldots \times A_n) \cong C \times A_1 \times \ldots \times A_n$, and f is as defined for power objects (see appendix). To understand why, consider the diagrams

$$\begin{tikzcd}
C \times A_1 \times \ldots \times A_n \arrow[r, "[\![\alpha(y/u)]\!]_{u\boldsymbol{x}}"] & \Omega \\
C \times (A_1 \times \ldots \times A_n) \arrow[u, "can"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
A_1 \times \ldots \times A_n \xrightarrow{f} PC$$

In set theory, $f(a_1, \ldots, a_n) = \{y \in C: \alpha(y, a_1, \ldots, a_n)\}$, so we take $[\![\{y: \alpha\}]\!]_{\boldsymbol{x}}$ to be f.

Finally,

$[\![\sigma = \tau]\!]_{\boldsymbol{x}} = eq_C \circ [\![<\sigma, \tau>]\!]_{\boldsymbol{x}}$ (with σ, τ **: C**)

$[\![\sigma \in \tau]\!]_{\boldsymbol{x}} = e_C \circ [\![<\sigma, \tau>]\!]_{\boldsymbol{x}}$ (with σ **: C**, τ **: PC** and where e_C is as defined for power objects.)

If τ **: B** is closed, then $\boldsymbol{x}$ may be taken to be the empty sequence $\varnothing$. In this case we write $[\![\tau]\!]$ for $[\![\tau]\!]_\varnothing$; this is an arrow $1 \to B$.

We note that

$$[\![\top]\!]_{\boldsymbol{x}} = [\![\bigstar = \bigstar]\!]_{\boldsymbol{x}} = eq \circ < [\![\bigstar]\!]_{\boldsymbol{x}}, [\![\bigstar]\!]_{\boldsymbol{x}} > \ = T\,.$$

For any finite set $\Gamma = \{\alpha_1, \ldots, \alpha_m\}$ of formulas write

$[\![\Gamma]\!]_{I,\boldsymbol{x}}$ for $[\![\alpha_1]\!]_{I,\boldsymbol{x}} \wedge \ldots \wedge [\![\alpha_m]\!]_{I,\boldsymbol{x}}$ if $m \geq 1$ or T if $m = 0$.

Given a formula α, let $\boldsymbol{x} = (x_1, \ldots, x_n)$ list all the free variables in $\Gamma \cup \{\alpha\}$; write

$$\Gamma \vDash_I \alpha \text{ for } [\![\Gamma]\!]_{I,\boldsymbol{x}} \leq [\![\alpha]\!]_{I,\boldsymbol{x}}.$$

$\Gamma \vDash_I \alpha$ is read : "$\Gamma : \alpha$ is *valid* under the interpretation I in $\mathscr{E}$." If S is a local set theory, we say that I is a *model* of S if every meer of S is valid under I. Notice that

$$\vDash_I \beta \quad \text{iff} \quad [\![\beta]\!]_{\boldsymbol{x}} = T.$$

So if I is an interpretation in a *degenerate* topos, i.e., a topos possessing just one object up to isomorphism, then $\vDash_I \alpha$ for all α.

We write:

$\Gamma \vDash \alpha$ for $\Gamma \vDash_I \alpha$ for every I

$\Gamma \vDash_S \alpha$ for $\Gamma \vDash_I \alpha$ for every model I of S.

It can be shown (laboriously) that the basic axioms and rules of inference of any local set theory are valid under every interpretation. This yields the

Soundness Theorem.

$\Gamma \vdash \alpha \;\Rightarrow\; \Gamma \vDash \alpha \qquad \Gamma \vdash_S \alpha \;\Rightarrow\; \Gamma \vDash_S \alpha.$

A local set theory S is said to be *consistent* if it is not the case that $\vdash_S \bot$. The soundness theorem yields the

Corollary. *Any pure local set theory is consistent.*

Proof. Set up an interpretation I of $\mathscr{L}$ in the topos $\mathcal{F}inset$ of finite sets as follows: $\mathbf{1}_I = 1$, $\mathbf{\Omega}_I = \{0, 1\} = 2$, for any ground type $\mathbf{A}$, $\mathbf{A}_I$ is any nonempty finite set. Extend I to arbitrary types in the obvious way. Finally $\mathbf{f}_I \colon \mathbf{A}_I \to \mathbf{B}_I$ is to be any map from $\mathbf{A}_I$ to $\mathbf{B}_I$.

If $\vdash \bot$, then $\vdash \alpha$, so $\vDash_I \alpha$ for any formula α. In particular $\vDash_I u = \upsilon$, where u, υ are variables of type $\mathbf{P1}$. Hence $[\![u]\!]_{I,\, u\upsilon} = [\![\upsilon]\!]_{I,\, u\upsilon}$, that is, the two projections $P1 \times P1 \to P1$ would have to be identical, a contradiction.

Given a local set theory S in a language $\mathscr{L}$, define the *canonical interpretation* $C(S)$ of $\mathscr{L}$ in $\mathscr{C}(S)$ by:

$$\mathbf{A}_{C(S)} = U_{\mathbf{A}} \qquad \mathbf{f}_{C(S)} = (x \mapsto \mathbf{f}(x)) \colon U_{\mathbf{A}} \to U_{\mathbf{B}} \qquad \text{for } \mathbf{f} \colon \mathbf{A} \to \mathbf{B}$$

A straightforward induction establishes

$$[\![\tau]\!]_{C(S),\, \boldsymbol{x}} = (\boldsymbol{x} \mapsto \tau)$$

This yields

$$(*) \qquad \Gamma \vDash_{C(S)} \alpha \iff \Gamma \vdash_S \alpha.$$

For:

$$\begin{aligned} \vDash_{C(S)} \alpha &\iff [\![\alpha]\!]_{C(S)\, \boldsymbol{x}} = T \\ &\iff (\boldsymbol{x} \mapsto \alpha) = (\boldsymbol{x} \mapsto \top) \\ &\iff \vdash_S \alpha = \top \\ &\iff \vdash_S \alpha. \end{aligned}$$

Since $\Gamma \vdash_S \alpha \iff \vdash_S \gamma \to \alpha$, where γ is the conjunction of all the formulas in Γ, the special case yields the general one.

Equivalence (*) may be read as asserting that $\mathscr{C}(S)$ is a *canonical model* of S. This fact yields the

Completeness Theorem.

$$\Gamma \vDash \alpha \;\Rightarrow\; \Gamma \vdash \alpha \qquad\qquad \Gamma \vDash_S \alpha \;\Rightarrow\; \Gamma \vdash_S \alpha$$

Proof. We know that $C(S)$ is a model of S. Therefore, using (*),

$$\Gamma \vDash_S \alpha \;\Rightarrow\; \Gamma \vdash_{C(S)} \alpha \;\Rightarrow\; \Gamma \vdash_S \alpha\,.$$

5.5. Every topos is linguistic: the equivalence theorem

A topos of the form $\mathscr{C}(S)$ is called a *linguistic* topos. It can be shown that every topos is, in a certain sense, equivalent to a linguistic one. This fact was apparently established independently by a number of mathematicians: it appears for example, in Volger (1975), Fourman (1977), Zangwill (1977), and Boileau and Joyal (1981).

Given a topos $\mathscr{E}$, we exhibit a theory $Th(\mathscr{E})$ and an equivalence $\mathscr{E} \sqcup C(Th(\mathscr{E}))$.

The local language $\mathscr{L}_{\mathscr{E}}$ associated with $\mathscr{E}$—also called the *internal language* of $\mathscr{E}$—is defined as follows. The ground type symbols of $\mathscr{L}_{\mathscr{E}}$ are taken to match the objects of $\mathscr{E}$ other than its terminal and truth-value objects, that is, for each $\mathscr{E}$-object A (other than

1, Ω) we assume given a ground type **A** in $\mathscr{L}_{\mathscr{E}}$. Next, we define for each type symbol **A** an $\mathscr{E}$-object $\mathbf{A}_{\mathscr{E}}$ by

$\mathbf{A}_{\mathscr{E}} = \mathbf{A}$ for ground types **A**,
$(\mathbf{A} \times \mathbf{B})_{\mathscr{E}} = \mathbf{A}_{\mathscr{E}} \times \mathbf{B}_{\mathscr{E}}$ [4]
$(\mathbf{PA})_{\mathscr{E}} = P(\mathbf{A})_{\mathscr{E}}$.

The function symbols of $\mathscr{L}$ are triples $(f, \mathbf{A}, \mathbf{B}) = \boldsymbol{f}$ with $f\colon \mathbf{A}_{\mathscr{E}} \to \mathbf{B}_{\mathscr{E}}$ in $\mathscr{E}$. The signature of $\boldsymbol{f}$ is $\mathbf{A} \to \mathbf{B}$.[5]

The *natural interpretation*—denoted by $\mathscr{E}$—of $\mathscr{L}_{\mathscr{E}}$ in $\mathscr{E}$ is determined by the assignments

$$\mathbf{A}_{\mathscr{E}} = \mathbf{A} \text{ for each ground type } \mathbf{A} \qquad (f, \mathbf{A}, \mathbf{B})_{\mathscr{E}} = \boldsymbol{f}.$$

The local set theory $Th(\mathscr{E})$, the *theory of* $\mathscr{E}$, is the theory in $\mathscr{L}_{\mathscr{E}}$ generated by the collection of all sequents $\Gamma : \alpha$ such that $\Gamma \vDash_{\mathscr{E}} \alpha$ under the natural interpretation of $\mathscr{L}_{\mathscr{E}}$ in $\mathscr{E}$. Then we have

$$\Gamma \vdash_{Th(\mathscr{E})} \alpha \iff \Gamma \vDash_{\mathscr{E}} \alpha.$$

For if $\Gamma \vdash_{Th(\mathscr{E})} \alpha$ then by Soundness $\Gamma \vDash_{Th(\mathscr{E})} \alpha$, i.e., $\Gamma : \alpha$ is valid in every model of $Th(\mathscr{E})$. But by definition $\mathscr{E}$ is a model of $Th(\mathscr{E})$.

It can be shown that the canonical functor $F\colon \mathscr{E} \to \mathrm{C}(\mathrm{Th}(\mathscr{E}))$ defined by

$FA = U_{\mathbf{A}}$ for each $\mathscr{E}$-object A
$Ff = (x \mapsto \boldsymbol{f}(x))\colon U_{\mathbf{A}} \to U_{\mathbf{B}}$ for each $\mathscr{E}$-arrow $f\colon A \to B$

is an equivalence of categories. This is the **Equivalence Theorem.**

Finally, we state two more facts about $Th(\mathscr{E})$.
A local set theory S in a language $\mathscr{L}$ is said to be

- *well-termed* if whenever $\vdash_S \exists! x\alpha$, there is a term τ of $\mathscr{L}$ whose free variables are those of α with x deleted such that $\vdash_S \alpha(x/\tau)$;
- *well-typed* if for any S-set X there is a type symbol **A** of $\mathscr{L}$ such that $U_{\mathbf{A}} \cong X$ in $\mathscr{C}(S)$.

Then, *for any topos* $\mathscr{E}$, $Th(\mathscr{E})$ *is well-termed and well-typed.*

5.6. Translations of local set theories.

A *translation* $\mathbf{K}\colon \mathscr{L} \to \mathscr{L}'$ of a local language $\mathscr{L}$ into a local language $\mathscr{L}'$ is a map which assigns to each type **A** of $\mathscr{L}$ a type **KA** of $\mathscr{L}'$ and to each function symbol $\boldsymbol{f}\colon \mathbf{A} \to \mathbf{B}$ of $\mathscr{L}$ a function symbol $\mathbf{Kf}\colon \mathbf{KA} \to \mathbf{KB}$ of $\mathscr{L}'$ in such a way that

---

[4] Note that, if we write C for $A \times B$, then while **C** is a ground type, $\mathbf{A} \times \mathbf{B}$ is a product type. Nevertheless $\mathbf{C}_{\mathrm{E}} = (\mathbf{A} \times \mathbf{B})_{\mathrm{E}}$.

[5] Note the following: if $f\colon A \times B \to D$, in e, then, writing C for $A \times B$ as in the footnote above, $(f, \mathbf{C}, \mathbf{D})$ and $(f, \mathbf{A} \times \mathbf{B}, \mathbf{D})$ are both function symbols of $\mathscr{L}_{\mathscr{E}}$ associated with f. But the former has signature $\mathbf{C} \to \mathbf{D}$, while the latter has the different signature $\mathbf{A} \times \mathbf{B} \to \mathbf{D}$.

$$\mathbf{K1} = \mathbf{1},\ \mathbf{K}\Omega = \Omega,\ \mathbf{K}(\mathbf{A}_1 \times \ldots \times \mathbf{A}_n) = \mathbf{KA}_1 \times \ldots \times \mathbf{KA}_n,\ \mathbf{K}(\mathbf{PA}) = \mathbf{PKA}.$$

Any translation $\mathbf{K}: \mathscr{L} \to \mathscr{L}'$ may be extended to the terms of $\mathscr{L}$ in the evident recursive way—i.e., by defining $\mathbf{K}\bigstar = \bigstar$, $\mathbf{K}(\boldsymbol{f}(\tau)) = \mathbf{K}\boldsymbol{f}(\mathbf{K}\tau)$, $\mathbf{K}(\sigma \in \tau) = \mathbf{K}\sigma \in \mathbf{K}\tau$ etc.—so that if $\tau : \mathbf{A}$, then $\mathbf{K}\tau : \mathbf{KA}$. We shall sometimes write $\alpha_{\mathbf{K}}$ for $\mathbf{K}\alpha$.

If S, S' are local set theories in $\mathscr{L}$, $\mathscr{L}'$ respectively, a translation $\mathbf{K}: \mathscr{L} \to \mathscr{L}'$ is a *translation of S into S'*, and is written $\mathbf{K}: S \to S'$ if, for any sequent $\Gamma : \alpha$ of $\mathscr{L}$,

(*) $\qquad \Gamma \vdash_S \alpha \ \Rightarrow\ \mathbf{K}\Gamma \vdash_{S'} \mathbf{K}\alpha,$

where if $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$, $\mathbf{K}\Gamma = \{\mathrm{K}\alpha_1, \ldots, \mathrm{K}\alpha_n\}$. If the reverse implication to (*) also holds, $\mathbf{K}$ is called a *conservative translation* of S into S'. If S' is an extension of S and the identity translation of S into S' is conservative, S' is called a *conservative extension of S*.

There is a natural correspondence between models of S in a topos $\mathscr{E}$ and translations of S into $Th(\mathscr{E})$: in particular the *identity translation* $Th(\mathscr{E}) \to Th(\mathscr{E})$ corresponds to the *natural interpretation* of $Th(\mathscr{E})$ in $\mathscr{E}$.

Now let $\mathscr{E}$, $\mathscr{E}'$ be toposes with specified terminal objects, products, projection arrows, truth-value objects, power objects and evaluation arrows: a functor $F: \mathscr{E} \to \mathscr{E}'$ which preserves all these is called a *logical functor*. It is easily seen that the canonical functor $\mathscr{E} \to \mathscr{C}(Th(\mathscr{E}))$ is logical.

If $\mathbf{K}: S \to S'$ is a translation, then for terms σ, τ of $\mathscr{L} \vdash_S \sigma = \tau$ implies $\vdash_{S'} \mathbf{K}\sigma = \mathbf{K}\tau$, so that $\mathbf{K}$ induces a map $\mathrm{C}_{\mathbf{K}}$ from the class of S-sets to the class of S'-sets via

$$\mathrm{C}_{\mathbf{K}}([\sigma]_S) = [\mathbf{K}\sigma]_{S'}.$$

$\mathrm{C}_{\mathbf{K}}$ is actually a *logical functor* $\mathscr{C}(S) \to \mathscr{C}(S')$. Writing $\mathscr{Loc}$ for the category of local set theories and translations, and $\mathscr{Top}$ for the category of toposes and logical functors, C is a functor $\mathscr{Loc} \to \mathscr{Top}$. And inversely any logical functor $F: \mathscr{E} \to \mathscr{E}'$ induces a translation $Th(F): Th(\mathscr{E}) \to Th(\mathscr{E}')$ in the natural way, so yielding a functor $Th: \mathscr{Loc} \to \mathscr{Top}$. C and Th are "almost" inverse, making $\mathscr{Loc}$ and $\mathscr{Top}$ "almost" equivalent.

Given a local set theory S in a language $\mathscr{L}$, define a translation $\mathbf{K}: \mathscr{L} \to \mathscr{L}_{\mathscr{C}(S)}$ by

$$\mathbf{KA} = \mathbf{U}_{\boldsymbol{A}}, \qquad \mathbf{K}\boldsymbol{f} = (f, \mathbf{A}, \mathbf{B}) \ \text{ if } \boldsymbol{f}: \mathbf{A} \to \mathbf{B}.$$

An easy induction on the formation of terms shows that, for any term τ of $\mathscr{L}$, $[\![\tau]\!]_{\mathscr{C}(S),\, \boldsymbol{x}} = [\![\mathrm{K}\tau]\!]_{\mathscr{C}(S),\, \mathbf{K}\boldsymbol{x}}$. It follows from this that $\mathbf{K}$ is a *conservative* translation of S into $Th(\mathscr{C}(S))$. For

$$\Gamma \vdash_S \alpha \ \Leftrightarrow\ \Gamma \vDash_{\mathscr{C}(S)} \alpha \ \Leftrightarrow\ \mathbf{K}\Gamma \vDash_{\mathscr{C}(S)} \mathbf{K}\alpha \ \Leftrightarrow\ \mathbf{K}\Gamma \vdash_{\mathrm{Th}(\mathscr{C}(S))} \mathbf{K}\alpha.$$

Accordingly *any local set theory can be conservatively embedded in one which is well-termed and well-typed.*

A particularly important instance of translation is the *adjunction of indeterminates* to a local set theory. Let us define a *constant* of type $\mathbf{A}$ in a local language $\mathscr{L}$ to be a term of the form $\boldsymbol{f}(\bigstar)$, where $\boldsymbol{f}: \mathbf{1} \to \mathbf{A}$. Write $\mathscr{L}(\boldsymbol{c})$ for the language obtained from $\mathscr{L}$ by adding a new function symbol $\boldsymbol{c}: \mathbf{1} \to \mathbf{A}$ and write c for $\boldsymbol{c}(\bigstar)$. Given a local set theory S in $\mathscr{L}$, and a formula α of $\mathscr{L}$ with exactly one free variable x of type $\mathbf{A}$, write $S(\alpha)$ for the theory in $\mathscr{L}(\mathbf{c})$ generated by S together with all sequents of the form $:\beta(x/c)$, where $\alpha \vdash_S \beta$.

Since clearly $\vdash_{S(\alpha)} \alpha(x/c)$, it follows that $\vdash_{S(\alpha)} \exists x\alpha$. Also, an analysis of derivations establishes that proofs in $S(\alpha)$ are related to proofs in S by the condition:

(*) $$\Gamma(x/c) \vdash_{S(\alpha)} \gamma(x/c) \iff \alpha, \Gamma \vdash_S \gamma.$$

for any sequent $\Gamma : \gamma$ of $\mathscr{L}$.

It is not hard to show that:

$$\vdash_S \exists x\, \alpha \iff S(\alpha) \text{ is a conservative extension of } S$$

$$\vdash_S \neg\exists x\, \alpha \iff S(\alpha) \text{ is inconsistent.}$$

In $S(\alpha)$, c behaves as an *indeterminate* of sort α in the sense that it can be arbitrarily assigned any value satisfying α. To be precise, the following can be proved:

Let S^* be a local set theory in a local language $\mathscr{L}^*$ and let $\mathbf{K}$: $S \to S^*$. *Then for any constant c^* of $\mathscr{L}^*$ of type* **KA** *such that $\vdash_{S^*} \alpha_{\mathbf{K}}(c^*)$, there is a unique translation* $\mathbf{K^*}$: $S(\alpha) \to S$ *extending* **K** *such that* $\mathbf{K^*}(c) = c^*$.

If I is an S-set and α the formula $x \in I$, we write S_I for $S(\alpha)$ and call it the theory obtained from S by *adjoining an indeterminate element of I.* It follows from (*) above that, for any formula γ of $\mathscr{L}$ in which i is free for x,

$$\vdash_{S(I)} \gamma(x/c) \iff \vdash_S \forall i{\in}I\ \gamma(x/i).$$

If α is the formula $x = x$ with $x : \mathbf{A}$, then $S(\alpha)$ is written $S(\mathbf{A})$ and called the theory obtained from S by *adjoining an indeterminate of type* **A.** In particular, let S_0 be the pure local set theory in the local language $\mathscr{L}_0$ with no ground types or function symbols. Evidently S_0 is an *initial object* in the category $\mathscr{Loc}$: there is a unique translation of L_0 into any given local set theory S. (Similarly, the topos $\mathscr{C}(S_0)$ is an initial object in the category top.) Now consider the theory $S_0(\mathbf{A})$, where **A** is a type symbol of $\mathscr{L}_0$: **A** may be considered a type symbol of *any* local language $\mathscr{L}$. If d is a constant of type **A** in $\mathscr{L}$, and S a local set theory in $\mathscr{L}$, there is then a unique translation $\mathbf{K}$: $S_0(\mathbf{A}) \to S$ mapping c to d. So $S_0(\mathbf{A})$ may be considered *the universal theory of an indeterminate of type* **A.**

5.7. Uses of the Equivalence Theorem.

As observed, e.g. in Fourman (1977), it is a consequence of the Equivalence Theorem that, in order to establish that all toposes have a property P preserved under equivalence of categories, one need only show that any linguistic topos (i.e., of the form $\mathscr{C}(S)$) has P, and this is usually a straightforward matter. For example, it is easy to see that $\varnothing_1$ is an initial object in $\mathscr{C}(S)$, so that *any topos has an initial object. We write* 0 *for* $\varnothing_1$.

It is not hard to see that any linguistic topos has coproducts of each pair of its objects: if X and Y are S-sets, the S-set $X + Y$ as defined above is a coproduct and the arrows σ_1 and σ_2 are the S-maps $x \mapsto \langle\{x\}, \varnothing\rangle$ and $y \mapsto \langle\varnothing, \{y\}\rangle$ respectively. We conclude that *any topos has binary coproducts.*

In the same sort of way the Equivalence Theorem can be used to show that any topos $\mathscr{E}$ possesses the following properties:

- $\mathscr{E}$ is *balanced,* that is, any arrow which is simultaneously epic and monic is an isomorphism.
- $\mathscr{E}$ has *epic-monic factorization*, i.e., each arrow f is the composite of an epic and a monic: this monic is called the *image* of f.
- In $\mathscr{E}$, pullbacks of epic arrows are epic.

Let S be a local set theory. We define the *entailment relation* on Ω to be the S-set

$$\lhd\ = \{\langle \omega, \omega' \rangle : \omega \to \omega'\}$$

Given an S-set X, we define the *inclusion relation* on PX to be the S-set

$$\sqsubseteq \; = \; \sqsubseteq_X \; = \{<u, v> \in PX \times PX: u \subseteq v\}.$$

It follows from facts concerning $\rightarrow$, $\wedge$, $\vee$ already established that

$\vdash_S <\Omega, \lhd>$ *is a Heyting algebra with top element* $\top$ *and bottom element* $\perp$.

Similarly,

$\vdash_S <PX, \sqsubseteq>$ *is a Heyting algebra with top element X and bottom element* $\varnothing$.

Let $\Omega(S)$ be the collection of *sentences* (closed formulas) of $\mathscr{L}$, where we identify two sentences α, β whenever $\vdash_S \alpha \leftrightarrow \beta$. Define the relation $\leq$ on $\Omega(S)$ by

$$\alpha \leq \beta \iff \vdash_S \alpha \rightarrow \beta.$$

Then $<\Omega(S), \leq>$ is a Heyting algebra, called the (external) *algebra of truth values of S.*

Its top element is T_Ω and its bottom element is the characteristic arrow of $\varnothing \rightarrowtail 1$.

If X is an S-*set,* write $Pow(X)$ for the collection of all S-sets U such that $\vdash_S U \subseteq V$ and define the relation $\sqsubseteq$ on $Pow(X)$ by $U \sqsubseteq V \Leftrightarrow \vdash_S U \subseteq V$.

Then $(Pow(X), \sqsubseteq)$ is a Heyting algebra, called the (external) *algebra of subsets of X.*

Given a topos $\mathscr{E}$, we can apply all this to the theory $Th(\mathscr{E})$; invoking the fact that $\vdash_{Th(\mathscr{E})} \alpha \Leftrightarrow \; \vDash_{\mathscr{E}} \alpha$ then gives

$\vDash <\Omega, \lhd>$ *and* $<PA, \subseteq_A>$ *are Heyting algebras,*

where A is any $\mathscr{E}$-object. These facts are sometimes expressed by saying that Ω and PA are *internal* Heyting algebras in $\mathscr{E}$.

What are the "internal" logical operations on Ω in $\mathscr{E}$? That is, which arrows represent $\wedge$, $\vee$ $\neg$, $\rightarrow$? Working in a linguistic topos and then transferring the result to an arbitrary topos via the Equivalence Theorem shows that, in $\mathscr{E}$,

$\Omega \times \Omega \xrightarrow{\wedge} \Omega$ is the characteristic arrow of the monic $1 \xrightarrow{<\top,\top>} \Omega \times \Omega$

$\Omega \times \Omega \xrightarrow{\vee} \Omega$ is the characteristic arrow of the image of

$$\Omega + \Omega \xrightarrow{<T_\Omega, 1_\Omega> + <1_\Omega, T_\Omega>} \Omega \times \Omega$$

$\Omega \xrightarrow{\neg} \Omega$ is the characteristic arrow of $1 \xrightarrow{\perp} \Omega$

$\Omega \times \Omega \xrightarrow{\rightarrow} \Omega$ is the characteristic arrow of the equalizer of the arrows $\pi_1, \wedge$.

(Here the *equalizer* of a pair of arrows with a common domain is the largest subobject of the domain on which they both agree.)

It is now possible to show that these "logical arrows" are the natural interpretations of the logical operations in any topos $\mathscr{E}$, in the sense that, for any interpretation of a language $\mathscr{L}$ in $\mathscr{E}$,

$$\llbracket \alpha \wedge \beta \rrbracket_{\boldsymbol{x}} = \wedge \circ \llbracket < \alpha, \beta > \rrbracket_{\boldsymbol{x}}$$
$$\llbracket \alpha \vee \beta \rrbracket_{\boldsymbol{x}} = \vee \circ \llbracket < \alpha, \beta > \rrbracket_{\boldsymbol{x}}$$
$$\llbracket \neg \alpha \rrbracket_{\boldsymbol{x}} = \neg \circ \llbracket \alpha \rrbracket_{\boldsymbol{x}}$$
$$\llbracket \alpha \rightarrow \beta \rrbracket_{\boldsymbol{x}} = \rightarrow \circ \llbracket < \alpha, \beta > \rrbracket_{\boldsymbol{x}}$$

We now turn to the "external" formulation of these ideas. First, for any topos $\mathscr{E}$ and any $\mathscr{E}$-object A, $(\mathbf{Sub}(A), \subseteq)$ is a Heyting algebra. For when $\mathscr{E}$ is of the form $\mathscr{C}(S)$, and A an S-set X, we have a natural isomorphism $(Pow(X), \sqsubseteq) \cong (\mathbf{Sub}(X), \subseteq)$ given by

$$U \mapsto [(x \mapsto x) : U \rightarrowtail X]$$

for $U \in Pow(X)$. Since we already know that $(Pow(X), \sqsubseteq)$ is a Heyting algebra, so is $(\mathbf{Sub}(X), \subseteq)$. Thus the result holds in any linguistic topos, and hence in any topos.

Since $\mathbf{Sub}(A) \cong \mathscr{E}(1, PA)$, it follows that $\mathscr{E}(1, PA)$ (with the induced ordering) is a Heyting algebra. And since $(\mathscr{E}(A,\Omega), \leq) \cong (\mathbf{Sub}(A), \subseteq)$, it follows that the former is a Heyting algebra as well. Taking $A = 1$, we see that the ordered set $\mathscr{E}(1,\Omega)$ of $\mathscr{E}$-elements of Ω is also a Heyting algebra.

Recall that a partially ordered set is *complete* if every subset has a supremum (join) and an infimum (meet). Then for any local set theory S, and any S-set X,

$$\vdash_S <\Omega, \lhd> \textit{ and } <PX, \subseteq> \textit{ are complete.}$$

For it is not difficult to show that

$u \subseteq \Omega \vdash_S (\top \in u)$ is the $\lhd$-join of u,

$u \subseteq \Omega \vdash_S (\forall \omega \in u.\omega)$ is the $\lhd$-meet of u,

$v \subseteq X \vdash_S \bigcup v$ is the $\subseteq$-join of v,

$v \subseteq X \vdash_S \bigcap v$ is the $\subseteq$-meet of v.

As a consequence, for any topos $\mathscr{E}$,

$$\vDash_{\mathscr{E}} <\Omega, \lhd> \textit{ and } <PA, \subseteq> \textit{ are complete.}$$

That is, Ω and PA are *internally complete* in $\mathscr{E}$.

5.8. The natural numbers in local set theories.

The development of the properties of the natural numbers in toposes was first carried out by Freyd (1972). Much of this development can be simplified by presentation within a local set theory. Thus let S be a local set theory in a language $\mathscr{L}$. A *natural number system in S* is a triple $(\mathbf{N}, \boldsymbol{s}, \underline{0})$, consisting of a type symbol $\mathbf{N}$, a function symbol $\boldsymbol{s}: \mathbf{N} \to \mathbf{N}$ and a closed term $\underline{0}:\mathbf{N}$, satisfying the following *Peano axioms.*

(P1) $\vdash_S \boldsymbol{s}n \neq \underline{0}$

(P2) $\boldsymbol{s}m = \boldsymbol{s}n \vdash_S m = n$

(P3) $\underline{0} \in u, \forall n(n \in u \to \boldsymbol{s}n \in u \vdash_S \forall n.\ n \in u$

Here m, n are variables of type $\mathbf{N}$, u is a variable of type $\mathbf{PN}$, and we have written $\boldsymbol{s}n$ for $\boldsymbol{s}(n)$. (P3) is the *axiom of induction.*

A local set theory with a natural number system will be called *naturalized.*

In any naturalized local set theory S, $\underline{0}$ is called the *zeroth numeral.* For each natural number $n \geq 1$, the *nth numeral* $\underline{n}$ in S is defined recursively by putting $\underline{n} = \boldsymbol{s}(\underline{n-1})$. Numerals are closed terms of type $\mathbf{N}$ which may be regarded as *formal representatives* in S of the natural numbers.

It is readily shown that (P3) is equivalent to the following *induction scheme:*

For any formula α *with exactly one free variable of type* $\mathbf{N}$*, if* $\vdash_S \alpha(\underline{0})$ *and* $\alpha(n) \vdash_S \alpha(\boldsymbol{s}n)$, then $\vdash_S \forall n \alpha(n)$.

It can also be shown that functions may be defined on N by the usual process of simple recursion. In fact we have, for any naturalized local set theory S, the following *simple recursion principle SRP:*

For any S-set X: **PA,** *any closed term a* **:** **A**, *and any S-map g: X → X, there is a unique S-map f: N → X such that*

$$\vdash_S f(\underline{0}) = a \wedge \forall n[f(\boldsymbol{s}n) = g(f(n))].$$

It follows from this that a natural number system on a local set theory is determined uniquely up to isomorphism in the evident sense.

Conversely, it can be shown that *SRP yields the Peano axioms,* so that they are equivalent ways of characterizing a natural number system.

Given a naturalized local set theory S, if we denote the map $n \mapsto \boldsymbol{s}(n)$: $N \to N$ by s and the map $\bigstar \mapsto \underline{0}$: $1 \to \mathbf{N}$ by o, it is easy to see, using P! and P2, that the map $N+1 \xrightarrow{s+o} N$ is an isomorphism in $\mathscr{C}(S)$. Conversely, it can be shown that the presence of an S-set X with an isomorphism f: $X + 1 \cong X$ yields a natural number system: this was first established, in a topos-theoretic setting, in Freyd (1972). For if we define

$$U = \bigcap\{u \subseteq X\colon f\bigstar \in u \wedge \forall x{\in}u.f(x) \in u\},$$

it is straightforward to show that the triple $(U, f, f\bigstar)$ is a natural number system.

An important feature of a natural number system ($\mathbf{N}$, $\boldsymbol{s}$, $\underline{0}$) is that *the equality relation on N is decidable,* that is,

$$\vdash_S m = n \vee m \neq n.$$

As shown in Bell (1999a) and (1999b), *Frege's* construction of the natural numbers can also be carried out in a local set theory and the result shown to be equivalent to the satisfaction of the Peano axioms. Thus suppose given a local set theory *S.* We shall work entirely within *S,* so that all the assertions we make will be understood as being demonstrable in *S.* In particular, by "set", "family", etc. we shall mean "*S*-set", "*S*-family", etc.

A family $\mathcal{F}$ of subsets of a set E is *inductive* if $\varnothing \in \mathcal{F}$ and $\mathcal{F}$ is *closed under unions with disjoint singletons,* that is, if

$$\forall X{\in}\mathcal{F}\, \forall x \in E - X\, (X \cup \{x\} \in \mathcal{F}).$$

A *Frege structure* is a pair (E,ν) with ν a map to E whose domain is an inductive family of subsets of E such that, for all $X, Y \in \mathrm{dom}(\nu)$,

$$\nu(X) = \nu(Y) \Leftrightarrow X \approx Y,$$

where we have written $X \approx Y$ for *there is a bijection between X and Y.*

It can be shown that, for any Frege structure (E, ν), there is a subset N of E which is the domain of a natural number system. In fact, for $X \in \mathrm{dom}(\nu)$ write X^+ for $X \cup \{\nu(X)\}$ and call a subfamily $\mathscr{E}$ of $\mathrm{dom}(\nu)$ *weakly inductive* if $\varnothing \in \mathscr{E}$ and $X^+ \in \mathscr{E}$ whenever $X \in \mathscr{E}$ and $\nu(X) \notin X$. Let $\mathbb{N}$ be the intersection of the collection of all weakly inductive families, and define $\underline{0} = \nu(\varnothing)$, $N = \{\nu(X)\colon X \in \mathbb{N}\}$, and s: $N \to N$ by $s(\nu(X)) = \nu(X^+)$ for $X \in \mathbb{N}$. Then $(N, s, \underline{0})$ is a natural number system.

Conversely, each natural number system $(N, s, \underline{0})$ yields a Frege structure. For one can define the map g: $N \to PN$ recursively by

$$g(\underline{0}) = \varnothing \qquad g(sn) = g(n) \cup \{n\},$$

and the map ν by

$$\nu = \{(X, n) \in PN \times N\colon X \approx g(n)\}.$$

The domain of ν is the family of finite subsets of N and ν assigns to each such subset the number of its elements. (N, ν) is a Frege structure.

We next describe the interpretation of the concept of natural number system in a topos $\mathscr{E}$. Let (N, s, o) be a triple consisting of an $\mathscr{E}$-object N and $\mathscr{E}$-arrows s: $N \to N$, o: $1 \to N$. Let $\boldsymbol{s}, \boldsymbol{o}$ be the function symbols in $\mathscr{L}_{\mathscr{E}}$ corresponding to s, o respectively and let

$\underline{0}$ be the closed term $\boldsymbol{o}(\bigstar)$. The clearly $(\mathbf{N}, \boldsymbol{s}, \underline{0})$ satisfies the simple recursion principle in $Th(\mathscr{E})$ iff the following condition, known as the *Peano-Lawvere axiom,* first presented in Lawvere (1964) holds:

For any diagram $1 \xrightarrow{a} X \xrightarrow{g} X$ in $\mathscr{E}$, *there exists a unique* $N \xrightarrow{f} X$ *for which the diagram*

$$\begin{tikzcd} 1 \arrow[r, "o"] \arrow[dr, "a"'] & N \arrow[r, "s"] \arrow[d, "f"] & N \arrow[d, "f"] \\ & X \arrow[r, "g"] & X \end{tikzcd}$$

commutes.

A triple (N, s, o) satisfying this condition is called a *natural number system,* and N a *natural number object*, in $\mathscr{E}$. From previous observations it follows that a topos has a natural number object if and only if it contains an *infinite* object, that is, an object A isomorphic to $A + 1$.

Now let $\mathscr{L}_{\mathbf{N}}$ be the language with just one ground type symbol $\mathbf{N}$, one function symbol $\boldsymbol{s}\colon \mathbf{N} \to \mathbf{N}$ and one function symbol $\boldsymbol{0}\colon \mathbf{1} \to \mathbf{N}$. Write $\underline{0}$ for $\boldsymbol{0}(\bigstar)$. Let P be the local set theory in $\mathscr{L}_{\mathbf{N}}$ generated by the sequents

$$: \boldsymbol{s}n \neq \underline{0}$$
$$\boldsymbol{s}m = \boldsymbol{s}n : m = n$$
$$\underline{0} \in u, \forall n(n \in u \to \boldsymbol{s}n \in u) : \forall n.\ n \in u$$

where m, n are variables of type $\mathbf{N}$ and u is a variable of type $\mathbf{PN}$. The triple $(\mathbf{N}, \boldsymbol{s}, \underline{0})$ is then a natural number system in P, so that P is a naturalized local set theory: it is called the *free* naturalized local set theory.

P is particularly important because it is *an initial object* in the category of naturalized local set theories. Given two such theories S, S', a *natural translation* of S into S' is a translation $K\colon S \to S'$ which preserves $\mathbf{N}$, $\boldsymbol{s}$ and $\underline{0}$. Write $\mathscr{N}atloc$ for the category of naturalized local set theories and natural translations. It should be clear that P is an initial object in $\mathscr{N}atloc$. The associated topos $\mathscr{C}(P)$ is called the *free topos.*

Lambek and Scott (1986) establish a have emphasized, P has some features which recommend it from a constructive standpoint: for instance it has the *disjunction property*, namely, for sentences α, β, $\vdash_P \alpha \vee \beta \Leftrightarrow \vdash_P \alpha$ or $\vdash_P \beta$; and it is *witnessed* in that for any type symbol $\mathbf{A}$ and any formula α with at most the variable $x : \mathbf{A}$ free, if $\vdash_S \exists x\, \alpha$, then $\vdash_S \alpha(x/\tau)$ for some closed term $\tau : \mathbf{A}$. These facts have led them to suggest that P is the *ideal theory* and its model the free topos the *ideal universe*, for the constructively minded mathematician.

Lambek and Scott also show that P has a number of further features, of which the most remarkable is the *parametrized disjunction property,* namely, for any type $\mathbf{A}$ of the form $\mathbf{\Omega}$ or $\mathbf{PC}$, and x a variable of type $\mathbf{A}$, if $\vdash_P \forall x\, [\alpha(x) \vee \beta(x)]$, then either $\vdash_P \forall x\, \alpha(x)$ or $\vdash_P \forall x\, \beta(x)]$. This means that, in P, $\mathbf{\Omega}$ and all power types are *indecomposable*, that is, cannot be partitioned into two nonempty parts.

If to the axioms of P we add the *law of excluded middle*

$$: \forall \omega.\ \omega \vee \neg\omega,$$

we get the theory P^c—the *free classical naturalized local set theory*—which is the classical counterpart of P. The associated topos $\mathscr{C}(P^c)$ is called the *free Boolean topos.* It would seem natural to regard this topos as the ideal universe for the classically minded mathematician; however, the incompleteness of first-order set theory implies that P^c is not complete, so that there are more than two "truth values" in $\mathscr{C}(P^c)$, an evident drawback from the classical standpoint.

5.9. Beth-Kripke-Joyal Semantics for Local Set Theories

Joyal's topos semantics, mentioned in §1, admits a natural formulation in local set theories. Let S be a local set theory, let α be a formula of the language of S, let $u, v, \ldots$ be variables of types $\mathbf{A}, \mathbf{B}, \ldots$, let X be an S-set and let $f\colon X \to A$, $g\colon X \to B, \ldots$ be S-maps. We think of $f, g, \ldots,$ as *generalized elements* of A, B *defined* at stage X. Then we write

$$X \Vdash_S \alpha[f, g, \ldots]$$

and say that X *forces* $\alpha[f, g, \ldots]$ or the generalized elements $f, g,\ldots$ *satisfy* α *at stage* X if

$$\vdash_S \forall x{\in}X\, \alpha(u/fx, v/gx, \ldots)$$

where x does not occur in α.

Recalling that S_X denotes the theory obtained from S by adding the indeterminate X-element c, we have

$$X \Vdash_S \alpha[f, g,\ldots] \quad \Leftrightarrow \vdash_S\ \alpha(u/fc, v/gc,\ldots).$$

This reduces the concept of satisfaction by generalized elements to the more familiar concept of satisfaction by elements.

It is readily seen that

$$\vdash_S \alpha \ \Leftrightarrow X \Vdash_S \alpha[f, g,\ldots] \text{ for all } f\colon X \to A,\ g\colon X \to B, \ldots,$$

that is, *provability is equivalent to being forced at all stages.* Moreover,

$$X \Vdash_S \alpha[f] \ \Rightarrow\ Y \Vdash_S \alpha[g \circ f] \ \text{ for all } f\colon\ Y \to X,$$

that is, *satisfaction at a given stage implies satisfaction at all "previous" stages.*

The rules governing the forcing relation, stated below, are collectively known as *Beth-Kripke-Joyal semantics* (for local set theories).

These rules are as follows: given $f\colon X \to A$,

- $X \Vdash_S \top$ always
- $X \Vdash_S \bot \ \Leftrightarrow\ X = \varnothing$
- $X \Vdash_S (\alpha \wedge \beta)[f] \ \Leftrightarrow X \Vdash_S \alpha[f]\ \&\ X \Vdash_S \beta[f]$
- $X \Vdash_S (\alpha \vee \beta)[f] \ \Leftrightarrow$ there are S-sets U, V such that $U \cup V = X$ and $U \Vdash_S \alpha[f| U]$, $V \Vdash_S \beta[f| V]$, where $|$ denotes restriction in the usual sense.
- $X \Vdash_S (\alpha \to \beta)[f] \ \Leftrightarrow$ for all $g\colon Y \to X$, $Y \Vdash_S \alpha[f \circ g]$ implies $Y \Vdash_S \beta[f \circ g]$
- $X \Vdash_S \neg\alpha[f] \ \Leftrightarrow$ for all $g\colon Y \to X$, $Y \Vdash_S \alpha[f \circ g]$ implies $Y = \varnothing$
- $X \Vdash_S \forall v\alpha[f] \Leftrightarrow$ for all $g\colon Y \to X$, $h\colon Y \to B$, we have $Y \Vdash_S \alpha[f \circ g, h]$
- $X \Vdash_S \exists v\alpha[f] \Leftrightarrow$ there is an epic arrow $g\colon Y \to X$ and an arrow $h\colon Y \to B$ for which $Y \Vdash_S \alpha[f \circ g, h]$.

The equivalence theorem enables these rules to be interpreted directly in toposes, the contexts in which Joyal had originally presented his semantics. Thus, given objects $X, A, B, \ldots$ and arrows $f\colon X \to A$, $g\colon X \to B$, of a topos $\mathscr{E}$, and a formula α of $\mathscr{L}_{\mathscr{E}}$, one writes $X \Vdash_{\mathscr{E}} \alpha[f, g,\ldots]$ for $X \Vdash_{Th(\mathscr{E})} \alpha[f, g,\ldots]$ and says, as before, that X *forces* $\alpha[f, g, \ldots]$ or the generalized elements $f, g,\ldots$ *satisfy* α *at stage* X. The rules for $\Vdash_{\mathscr{E}}$ are similar to those for $\Vdash_S$. For example, the rule for forcing a disjunction becomes

$X \Vdash_{\mathscr{E}} (\alpha \vee \beta)[f] \ \Leftrightarrow$ there are monic arrows $m\colon U \to X$, $n\colon V \to X$ such that $m + n\colon U + V \to X$ is epic and $U \Vdash_{\mathscr{E}} \alpha[f \circ m]$, $V \Vdash_{\mathscr{E}} \beta[f \circ n]$.

5.10. Syntactic properties of local set theories.

There are many natural syntactical conditions which can be imposed on a local set theory: these were first presented in Bell (1988). The corresponding conditions in toposes were first subject to systematic investigation in Johnstone (1977).

Let S be a local set theory in a language $\mathscr{L}$. We make the following

Definitions.

- S is *classical* if $\vdash_S \omega \vee \neg\omega$.
- S is *sententially classical* if $\vdash_S \alpha \vee \neg\alpha$ for any sentence α.
- S is *complete* if $\vdash_S \alpha$ or $\vdash_S \neg\alpha$ for any sentence α.
- For each S-set $X : \mathbf{PA}$ let $\Delta\,(X)$ be the set of closed terms τ such that $\vdash_S \tau \in X$. X is *standard* if for any formula α with at most the variable $x : \mathbf{A}$ free the following is valid:
$$\frac{\vdash_S \alpha(x/\tau) \quad \text{for all } \tau \text{ in } \Delta(X)}{\vdash_S \forall x{\in}X\,\alpha}$$
S is *standard* if every S-set is so.
- If $X : \mathbf{PA}$ is an S-set, an X-*singleton* is an S-set U such that
$$\vdash_S U \subseteq X \wedge \forall x \in U\, \forall y \in U\, (x = y).$$
Write $\Gamma(X)$ for the set of all X-singletons. X is *near-standard* if for any formula α with at most the variable $x : \mathbf{A}$ free the following is valid:
$$\frac{\vdash_S \forall x{\in}U\,\alpha \quad \text{for all } U \text{ in } \Gamma(X)}{\vdash_S \forall x \in X\,\alpha}$$
S is *near-standard* if every S-set is so.
- S is *witnessed* if for any type symbol $\mathbf{A}$ of $\mathscr{L}$ and any formula α with at most the variable $x : \mathbf{A}$ free the following is valid:
$$\frac{\vdash_S \exists x\,\alpha}{\vdash_S \alpha(x/\tau) \quad \text{for some closed term } \tau : \mathbf{A}.}$$
- S is *choice* if, for any S-sets X, Y and any formula α with at most the variables x, y free the following version of the axiom of choice holds:
$$\frac{\vdash_S \forall x \in X\, \exists y \in Y\, \alpha(x, y)}{\vdash_S \forall x \in X\, \alpha(x, fx) \quad \text{for some } f\colon X \to Y}$$
- S is *internally choice* if under the conditions of the previous definition
$\forall x \in X\, \exists y \in Y\, \alpha(x, y) \vdash_S \exists f \in Fun(X,Y)\; \forall x \in X\, \exists y \in Y\, [\alpha\,(x, y) \wedge \langle x, y\rangle \in f]$.
- An S-set X is *discrete* if
$$\vdash_S \forall x \in X\, \forall y \in X.\ x = y \vee x \neq y.$$
- A *complement* for an S-set $X : \mathbf{PA}$ is an S-set $Y : \mathbf{PA}$ such that
$$\vdash_S X \cup Y = A \;\wedge\; X \cap Y = \varnothing.$$

An S-set that has a complement is said to be *complemented.*

Now for some facts concerning these notions.

1. *Any of the following conditions is equivalent to the classicality of S:*

(i) $\vdash_S \Omega = \{\top, \bot\}$

(ii) $\vdash_S \neg\neg\omega \to \omega$

(iii) *any S-set is complemented,*

(iv) *any S-set is discrete,*

(v) Ω *is discrete,*

(vi) $\vdash_S 2 = \{0, 1\}$ *is well-ordered under the usual ordering,*

(vii) *if S is naturalized,* $\vdash_S N$ *is well-ordered under the usual ordering.*

Proof. (iii) If S is classical, clearly $\{x\colon x \notin X\}$ is a complement for X. Conversely, if $\{\top\}$ has a complement U, then

$$\vdash_S \omega \in U \to \neg(\omega = \top) \to \neg\omega \to \omega = \bot.$$

Hence $\vdash_S U = \{\bot\}$, whence $\vdash_S \Omega = \{\top\} \cup U = \{\top, \bot\}$.

(v) If Ω is discrete, then $\vdash_S \omega = \top \vee \neg(\omega = \top)$, so $\vdash_S \omega \vee \neg\omega$.

(vi) If S is classical, then 2 is trivially well-ordered under the usual well-ordering. Conversely, if 2 is well-ordered, take any formula α, and define $X = \{x \in 2\colon x = 1 \vee \alpha\}$. Then X has a least element, a, say. Clearly $\vdash_S a = 0 \leftrightarrow \alpha$, so, since $\vdash_S a = 0 \vee a = 1$, we get $\vdash_S a = 1 \leftrightarrow \neg\alpha$, and hence $\vdash_S \alpha \vee \neg\alpha$.

2. *S standard* $\Rightarrow$ *S complete and classical.*

Proof. Let α be a sentence and suppose that not $\vdash_S \neg\alpha$. Let $x : \mathbf{1}$ and $X = \{x : \alpha\}$. Then not $\vdash_S \forall x \in X\, x \neq x$, so, if S is standard, there must be a closed term τ such that $\vdash_S \tau \in X$. It follows that $\vdash_S \alpha$, so S is complete. Since $\vdash_S \alpha$ or $\vdash_S \neg\alpha$, we certainly have $\vdash_S \alpha \vee \neg\alpha$, for any sentence α, whence $\vdash_S \forall\omega.\ \omega \vee \neg\omega$ by standardness. So S is classical.

3. *S standard and consistent* $\Rightarrow$ *S witnessed.*

Proof. If $\vdash_S \exists x\alpha$, then not $\vdash_S \neg\exists x\alpha$, whence not $\vdash_S \forall x\neg\alpha$, so that, by standardness, not $\vdash_S \neg\alpha(\tau)$ for some closed term τ. But S is complete by 2, so $\vdash_S \alpha(\tau)$ as required.

4. *For consistent S, S standard* $\Leftrightarrow$ *S classical, witnessed, and complete.*

Proof. $\Rightarrow$ has already been established. For $\Leftarrow$, suppose not $\vdash_S \forall x \in X\, \alpha(x)$. Then $\vdash_S \neg\forall x \in X\, \alpha(x)$ by completeness, whence $\vdash_S \exists x \in X\, \neg\alpha(x)$ by classicality, so that $\vdash_S \neg\alpha(x/\tau)$ for some $\tau \in \Delta(X)$ by witnessing. So not $\vdash_S \alpha(x/\tau)$ by consistency.

5. *For well-termed S, S standard* $\Leftrightarrow$ *S near-standard and complete.*

Proof. Suppose S *is* standard. Then S is complete by 2. If $\vdash_S \forall x \in U\, \alpha$ for all X-singletons U, then $\vdash_S \forall x \in \{\tau\}\, \alpha$, whence $\vdash_S \alpha(x/\tau)$ for all $\tau \in \Delta(X)$. So $\vdash_S \forall x\alpha$ by standardness.

Conversely, suppose that S is near-standard, complete, and that $\vdash_S \alpha(x/\tau)$ for all $\tau \in \Delta(X)$. If U is an X-singleton, then by completeness of S either whence $\vdash_S \exists x.\ x \in U$ or $\vdash_S \neg\exists x.\ x \in U$. In the former case, the well-termedness of S yields a closed term τ such that $\vdash_S U = \{\tau\}$ so that, since $\vdash_S \alpha(x/\tau)$, we have $\vdash_S \forall x \in U\, \alpha$. If, on the other hand, $\vdash_S \neg\exists x.\ x \in U$, then *a fortiori* $\vdash_S \forall x \in U\, \alpha$. Accordingly $\vdash_S \forall x \in U\, \alpha$ for *any* X-singleton U, and near-standardness yields $\vdash_S \forall x \in X\, \alpha$ as required.

6. *For well-termed S, S choice ⇔ S internally choice and witnessed.*

Proof. Suppose S is choice. If $\vdash_S \exists x\alpha$, let u **:** **1** and define $\beta(u, x) \equiv \alpha(x)$. Then $\vdash_S \forall u\in 1\exists x\in X\beta(u,x)$. Now choice yields an S-map f: $1 \rightarrow X$ such that $\vdash_S \forall u\in 1\beta(u,f(u))$ i.e., $\vdash_S \beta(\bigstar, f\bigstar)$ or $\vdash_S \alpha(f\bigstar)$. By well-termedness, $f\bigstar$ may be taken to be a closed term τ, and we then have $\vdash_S \alpha(\tau)$. So S is witnessed.

To derive internal choice from choice, we argue as follows: let

$$X^* = \{x \in X\text{: } \exists y\in Y\alpha(x, y)\}.$$

Then $\vdash_S \forall x\in X^*\exists y\in Y\alpha(x,y)$. Accordingly choice yields a map f: $X^* \rightarrow Y$ such that $\vdash_S \forall x\in X^*\alpha(x, fx)$, i.e. $\vdash_S \forall x\in X^*\exists y\in Y[<x, y> \in f \wedge \alpha(x, y)]$. Now

$$\forall x\in X\exists y\in Y\, \alpha(x, y)] \ \vdash_S X = X^* \ \vdash_S f \in Fun(X, Y)$$

so

$$\forall x\in X\exists y\in Y\, \alpha(x, y)] \vdash_S \forall x\in X\exists y\in Y[<x, y> \in f \wedge \alpha(x, y)].$$

Hence

$$\forall x \in X\, \exists y \in Y\, \alpha(x, y) \vdash_S \exists f \in Fun(X,Y)\ \forall x \in X\, \exists y \in Y\, [\alpha\, (x, y) \wedge <x, y>\in f],$$

as required. The converse is easy.

7. *If S is well-termed and well-typed, then S is choice* ⇔ S_X (= $S(X)$) *is witnessed for every S-set X.*

Proof. Suppose S is choice and $\vdash_{S(X)} \exists y\alpha(y)$. We may assume that X is of the form $U_{\mathbf{A}}$, in which case α is of the form $\beta(x/c, y)$ with x **: A.** From $\vdash_{S(X)} \exists y\beta(x/c, y)$ we infer $\vdash_S \forall x\exists y\beta(x/c, y)$. So using AC and the well-termedness of S we obtain a term $\tau(x)$ such that $\vdash_S \forall x\beta(x, \tau(x))$. Hence $\vdash_{S(X)} \beta(c, \tau(c))$, i.e., $\vdash_{S(X)} \alpha(\tau(c))$. Therefore S_X is witnessed.

Conversely, suppose S_X is witnessed for every S-set X, and that $\vdash_S \forall x\in X\exists y\in Y\, \alpha(x, y)]$. Then $\vdash_{S(X)} \exists y\in Y\, \alpha(c, y)]$, so there is a closed $\mathscr{L}_X$-term τ such that $\vdash_{S(X)} \tau \in Y \wedge \alpha(c, \tau)$. But τ is $\sigma(x/c)$ for some $\mathscr{L}$-term $\sigma(x)$. Thus $\vdash_{S(X)} \sigma(c)\in Y \wedge \alpha(c,\sigma(c))$, whence $\vdash_S \forall x\in X\ [\sigma(x) \in Y \wedge \alpha(x, \sigma)]$. Defining $f = (x \mapsto\sigma)$: $X \rightarrow Y$ then gives $\vdash_S \forall x \in X\, \alpha(x, fx)]$ as required.

8. Diaconescu's Theorem. *S choice ⇒ S classical.*

This is the version, for local set theories, of Diaconescu's (1975) result that any topos in which the axiom of choice holds is Boolean.

Proof. Step 1. *S choice ⇒* S_I *choice for any S-set I.*

Proof of step 1. Suppose that S is choice, and

$$\vdash_{S(I)}\ \forall x \in X(c)\ \exists y \in Y(c)\ \alpha\ (x, y, c).$$

Then

$$\vdash_S\ \forall x \in X(i)\ \exists y \in Y(i)\ \alpha\ (x, y, i).$$

Define

$$X^* = \{<x, i>\text{: } x \in X(i) \wedge i \in I\}, \quad Y^* = \bigcup_{i\in I} Y(i).$$

$$\beta(u, i) \equiv \exists x \in X(i)\exists i \in I[u = <x, i> \wedge \alpha(x, y, i) \wedge y \in Y(i)].$$

Then

$$\vdash_S\ \forall u \in X^*\ \exists y \in Y^*\ \beta(u, y).$$

So choice yields $f^*: X^* \to Y^*$ such that

$$\vdash_S \ \forall u \in X^* \ \beta(u, f^*u).$$

i.e.,

$$\vdash_S \ \forall i \in I \ \forall x \in X(i) \ \alpha(x, f^*(\langle x, i\rangle, i) \wedge f^*(\langle x, i\rangle) \in Y(i)],$$

whence

$$\vdash_S \ \forall x \in X(c) \ \alpha(x, f^*(\langle x\ c\rangle, c) \wedge f^*(\langle x, c\rangle) \in Y(c)],$$

Now define $f = (x \mapsto f^*(\langle x, c\rangle))$. Then $f: X(c) \to Y(c)$ in S_I and

$$\vdash_{S(I)} \ \forall x \in X(c) \ \alpha(x, fx, c).$$

This completes the proof of step 1.

Step 2. *S choice ⇒ S sententially classical.*

Proof of step 2. Define $2 = \{0, 1\}$ and let $X = \{u \subseteq 2: \exists y.y \in u\}$. Then

$$\vdash_S \ \forall u \in X \ \exists y \in 2 \ . \ y \in u.$$

So by choice there is $f: X \to 2$ such that

$$\vdash_S \ \forall u \in X \ . \ fu \in u.$$

Now let α be any sentence; define

$$U = \{x \in 2: x = 0 \vee \alpha\}, \ V = \{x \in 2: x = 1 \vee \alpha\},$$

Then $\vdash_S U \in X \wedge V \in X$, so, writing $a = fU$, $b = fV$, we have

$$\vdash_S \ [a = 0 \vee \alpha] \wedge [b = 1 \vee \alpha],$$

whence

$$\vdash_S \ [a = 0 \wedge b = 1] \vee \alpha,$$

so that

(*) $\qquad \vdash_S \ a \neq b \vee \alpha.$

But $\alpha \vdash_S U = V \vdash_S a = b$, so that $a \neq b \vdash_S \neg\alpha$. It follows from this and (*) that

$$\vdash_S \alpha \vee \neg\alpha,$$

as claimed. This establishes step 2.

Moral of step 2: if P2 has a choice function, then logic is classical.

Step 3 (obvious). *S classical ⇔ S_Ω sententially classical.*

Finally, to prove Diaconescu's theorem, we observe that

S choice ⇒ S_Ω choice ⇒ S_Ω sententially classical ⇒ S classical.

In Johnstone (1979a) and (1979b) an investigation was undertaken of the status in toposes of the nonconstructive De Morgan law $\neg(\omega \wedge \omega') \to \neg\omega \vee \neg\omega'$. Among the more interesting conditions shown to be equivalent to it are *every maximal ideal in a commutative ring with identity is prime*; and *the Dedekind real numbers are conditionally complete in their natural ordering.* These results were among the first establishing an equivalence, in an intuitionistic context, of a purely logical with a purely mathematical assertion.

5.11. Tarski's and Gödel's theorems in local set theories.

The type structure of local languages makes them particularly appropriate for the stating (and proving) of *undefinability* and *incompleteness theorems*. Thus let $\mathscr{L}$ be a local language containing a type symbol **C,** which we will call the *type of codes of formulas:* letters u, v will be used as variables of type **C** and the letter $\mathbf{u}$ will denote a closed term of type **C.** We shall also suppose that $\mathscr{L}$ contains terms $\sigma(u, v)$: **C**, $\delta(u)$: **C,** $\tau(u)$: $\mathbf{\Omega}$, and for each formula $\alpha(u)$, containing at most the free variable u, a closed term $\ulcorner\alpha\urcorner$: **C** called the *code* of α.

Let S be a local set theory in $\mathscr{L}$. We say that

- σ is a *substitution operator* in S if
  $\vdash_S \sigma(\ulcorner\alpha\urcorner, \mathbf{u}) = \ulcorner\alpha(\mathbf{u})\urcorner$ for any formula $\alpha(u)$ and any closed term $\mathbf{u}$**: C.**
- δ is a *diagonal operator* in S if
  $$\vdash_S \delta(\ulcorner\alpha\urcorner) = \ulcorner\alpha(\ulcorner\alpha\urcorner)\urcorner \quad \text{for any formula } \alpha(u).$$
- τ is a *truth definition* for S if
  $$\vdash_S \tau(\ulcorner\alpha\urcorner) \leftrightarrow \alpha \qquad \text{for any sentence } \alpha.$$
- τ is a *demonstration predicate* for S if
  $$\vdash_S \alpha \Leftrightarrow \ \vdash_S \tau(\ulcorner\alpha\urcorner) \qquad \text{for any sentence } \alpha.$$
- τ is a *provability predicate* for S if it satisfies, for any sentences α, β,
  (a) $\vdash_S \alpha \Rightarrow \ \vdash_S \tau(\ulcorner\alpha\urcorner)$,
  (b) $\tau(\ulcorner\alpha \rightarrow \beta\urcorner) \vdash_S \tau(\ulcorner\alpha\urcorner) \rightarrow \tau(\ulcorner\beta\urcorner)$,
  (c) $\tau(\ulcorner\alpha\urcorner) \vdash_S \tau(\ulcorner\tau(\ulcorner\alpha\urcorner)\urcorner)$.

Note that, if σ is a substitution operator, then $\sigma(u, u)$ is a diagonal operator.

We now prove, for local set theories,

Tarski's Theorem on the Undefinability of Truth. Let S be a local set theory in $\mathscr{L}$ with a diagonal operator (or a substitution operator). Then if S has a truth definition, it is inconsistent.

Proof. Let δ be a diagonal operator in S and τ a truth definition for S. Define $\beta(u)$ to be the formula $\neg\tau(\delta(u))$; write $\mathbf{u}$ for $\ulcorner\beta\urcorner$. Then $\ulcorner\beta(\mathbf{u})\urcorner$ is $\ulcorner\neg\tau(\delta(\mathbf{u}))\urcorner$, so that
$$\tau(\ulcorner\beta(\mathbf{u})\urcorner) \text{ is } \tau(\ulcorner\neg\tau(\delta(\mathbf{u}))\urcorner), \qquad (*)$$
Since δ is a substitution operator, we have
$$\vdash_S \ulcorner\beta(\mathbf{u})\urcorner = \delta(\ulcorner\beta\urcorner) = \delta(\mathbf{u}),$$
whence
$$\vdash_S \tau(\ulcorner\beta(\mathbf{u})\urcorner) \leftrightarrow \tau(\delta(\mathbf{u})),$$
i.e. by (*),
$$\vdash_S \tau(\ulcorner\neg\tau(\delta(\mathbf{u}))\urcorner) \leftrightarrow \tau(\delta(\mathbf{u})). \qquad (**)$$
But since τ is a truth definition we have
$$\vdash_S \tau(\ulcorner\neg\tau(\delta(\mathbf{u}))\urcorner) \leftrightarrow \neg\tau(\delta(\mathbf{u})).$$
This and (**) give
$$\vdash_S \neg\tau(\delta(\mathbf{u})) \leftrightarrow \tau(\delta(\mathbf{u})),$$
so that S is inconsistent.

In a similar way one can establish, for local set theories,

Gödel's First Incompleteness Theorem. Let S be a local set theory with a diagonal operator (or a substitution operator). Then if S is consistent and has a demonstration predicate, it is incomplete.

Proof. If we define the formula $\beta(u)$ and the term $\mathbf{u}$ as in the proof of Tarski's theorem, we find that

$$\vdash_S \neg\tau(\delta(\mathbf{u})) \Leftrightarrow \vdash_S \tau(\delta(\mathbf{u})),$$

so that, if S is consistent, $\tau(\delta(\mathbf{u}))$ is neither provable nor refutable in S, and accordingly S is incomplete.

Next, we can prove, for local set theories, the

Fixed Point Lemma. Suppose that S has a diagonal operator. Then any formula $\alpha(u)$ has a "fixed point", i.e., there is a sentence β such that

$$\vdash_S \beta \leftrightarrow \alpha(\ulcorner\beta\urcorner).$$

Proof. Let δ be a diagonal operator in S; given $\alpha(u)$ write $\mathbf{u} = \ulcorner\alpha(\delta(u))\urcorner$ and define β to be the sentence $\alpha(\delta(\mathbf{u}))$, i.e. $\alpha(\delta(\ulcorner\alpha(\delta(u))\urcorner))$. The fact that δ is a diagonal operator gives

$$\vdash_S \delta(\ulcorner\alpha(\delta(u))\urcorner) = \alpha(\delta(\ulcorner\alpha(\delta(u))\urcorner)) = \ulcorner\beta\urcorner .$$

It follows that

$$\vdash_S \beta \leftrightarrow \alpha(\delta(\ulcorner\alpha(\delta(u))\urcorner)) \leftrightarrow \alpha(\ulcorner\beta\urcorner),$$

as required.

This leads to a version, for local set theories, of a theorem of Löb (1955).

Löb's Theorem. Suppose that S has both a diagonal operator and a provability predicate τ. Then for any sentence α,

$$\vdash_S \tau(\ulcorner\alpha\urcorner) \to \alpha \quad \Rightarrow \quad \vdash_S \alpha.$$

Proof. Suppose that

(1) $$\vdash_S \tau(\ulcorner\alpha\urcorner) \to \alpha.$$

Applying the Fixed Point Lemma to the formula $\tau(u) \to \alpha$ yields a sentence β for which

(2) $$\vdash_S \beta \leftrightarrow (\tau(\ulcorner\beta\urcorner) \to \alpha),$$

whence

$$\vdash_S \beta \to (\tau(\ulcorner\beta\urcorner) \to \alpha).$$

Since τ is a proof predicate, it follows that

$$\vdash_S \tau(\ulcorner\beta \to (\tau(\ulcorner\beta\urcorner) \to \alpha)\urcorner).$$

For the same reason, we have also

$$\tau(\ulcorner\beta \to (\tau(\ulcorner\beta\urcorner) \to \alpha)\urcorner) \vdash_S \tau(\ulcorner\beta\urcorner) \to \tau(\ulcorner\tau(\ulcorner\beta\urcorner) \to \alpha)\urcorner).$$

Accordingly

$$\vdash_S \tau(\ulcorner\beta\urcorner) \to \tau(\ulcorner\tau(\ulcorner\beta\urcorner) \to \alpha)\urcorner),$$

i.e.,

$$\tau(\ulcorner\beta\urcorner) \vdash_S \tau(\ulcorner\tau(\ulcorner\beta\urcorner) \to \alpha)\urcorner).$$

Again,

$$\tau(\ulcorner\tau(\ulcorner\beta\urcorner) \to \alpha)\urcorner)\,) \vdash_S \tau(\ulcorner\tau(\ulcorner\beta\urcorner)\urcorner) \to \tau(\ulcorner\alpha\urcorner)),$$

so

$$\tau(\ulcorner\beta\urcorner) \vdash_S \tau(\ulcorner\tau\ulcorner\beta\urcorner)\urcorner) \to \tau(\ulcorner\alpha\urcorner).$$

But the fact that τ is a proof predicate gives

$$\tau(\ulcorner\beta\urcorner) \vdash_S \tau(\ulcorner\tau(\ulcorner\beta\urcorner)\urcorner);$$

this and the previous assertion give

$$\tau(\ulcorner\beta\urcorner) \vdash_S \tau(\ulcorner\alpha\urcorner).$$

Using (1), we get

(3) $$\tau(\ulcorner\beta\urcorner) \vdash_S \alpha,$$

whence

$$\vdash_S \tau(\ulcorner\beta\urcorner) \to \alpha.$$

This together with (2) gives

$$\vdash_S \beta,$$

so that, since τ is a proof predicate,

$$\vdash_S \tau(\ulcorner\beta\urcorner.$$

And this together with (3) gives $\vdash_S \alpha$ as claimed.

This yields finally a version within local set theories of

Gödel's Second Incompleteness Theorem. Suppose that S has both a diagonal operator and a provability predicate τ. Then if S is consistent, the sentence $\neg\tau(\ulcorner\bot\urcorner)$ expressing the consistency of S is not provable in S.

Proof. If $\vdash_S \neg\tau(\ulcorner\bot\urcorner)$, then $\vdash_S \tau(\ulcorner\bot\urcorner) \to \bot$. So it follows from Löb's theorem that $\vdash_S \bot$, i.e., S is inconsistent.

5.12. Characterization of $\mathscr{Set}$

A local set theory S is *full* if for each (intuitive) set I there is a type symbol $\hat{\mathbf{I}}$ of the language $\mathscr{L}$ of S together with a collection $\{\hat{i} : i \in I\}$ of closed terms each of type $\hat{\mathbf{I}}$ satisfying the following *universal condition:*

- For any I - indexed family $\{\tau_i\colon i \in I\}$ of closed terms of common type $\mathbf{A}$, there is a term $\tau(x) : \mathbf{A}, x : \hat{\mathbf{I}}$ such that
$$\vdash_S \tau_i = \tau(\hat{i}) \quad \text{for all } i \in I,$$
and, for any term $\sigma(x) : \mathbf{A}, x : \hat{\mathbf{I}}$, if $\vdash_S \tau_i = \sigma(\hat{i})$ for all $i \in I$, then $\vdash_S \tau = \sigma$ (uniqueness condition).

We establish:

1. *The following is valid for any formula $\alpha(x)$ with $x : \hat{\mathbf{I}}$:*

$$\frac{\vdash_S \alpha(\hat{i}) \quad \textit{for all } i \in I}{\vdash_S \forall x\, \alpha}$$

and similarly for more free variables. In particular, $\hat{I}$ is standard.

Proof. Assume the premises. Then for any $i \in I$ we have $\vdash_S \alpha(\hat{i}) = \top$ and it follows from the uniqueness condition that $\vdash_S \alpha(x) = \top$, whence have $\vdash_S \forall x \alpha$.

2. *$\hat{I}$ is discrete.*

Proof. Given $i, j \in I$, either they are the same, or not. If the same, then $\vdash_S \hat{i} = j$. If not, define for $k \in I$ the terms τ_k by $\tau_i = 0$, $\tau_j = 1 = \tau_k$ for $k \in I - \{i, j\}$. There is then a term τ such that $\vdash_S \tau_k = \tau(k)$ for all $k \in I$. It follows that $\vdash_S \tau(\hat{i}) = 0 \neq 1 = \tau(j)$, whence $\vdash_S \hat{i} \neq j$. Hence $\vdash_S \hat{i} = j$ or $\vdash_S \hat{i} \neq j$, so that $\vdash_S \hat{i} = j \vee \hat{i} \neq j$. It now follows from 1 that $\vdash_S \forall x \in \hat{I}\ \forall y \in \hat{I}\ (x = y \vee x \neq y)$; in other words, $\hat{I}$ is discrete.

3. *If S is full, $\Omega(S)$ is complete.*

Proof. Suppose $\{\alpha_i\colon i \in I\} \subseteq \Omega(S)$. Then by the universal condition on $\hat{\mathbf{I}}$ there is a formula $\beta(x)$ with $x : \hat{\mathbf{I}}$ such that if $\vdash_S \alpha_i = \beta(\hat{i})$ for all $i \in I$. We claim that the sentence $\exists x\beta(x)$ is the join of $\{\alpha_i\colon i \in I\}$ in $\Omega(S)$. First, it is an upper bound, since

$$\alpha_i \vdash_S \beta(\hat{i}) \vdash_S \exists x\beta(x).$$

And it is least, since if γ is a sentence with $\alpha_i \leq \gamma$ for all $i \in I$, then $\alpha_i \vdash_S \gamma \wedge \alpha_i$. for all $i \in I$, so that $\alpha_i \vdash_S \alpha_i = \gamma \wedge \beta(\hat{i})$ for all $i \in I$. By uniqueness of β, $\vdash_S \beta(x) = \gamma \wedge \beta(x)$, whence $\vdash_S \beta(x) \rightarrow \gamma$, $\vdash_S \exists x\beta(x) \rightarrow \gamma$ and $\exists x\beta(x) \leq \gamma$. So $\Omega(S)$ is complete as claimed.

Note the following: (i) if S is well-termed and well-typed, then $\mathscr{C}(S)$ is full iff the *I-indexed sum of* 1 exists in $\mathscr{C}(S)$ for any (intuitive) set I. Here the I-indexed sum of an object X in a category $\mathscr{C}$ is an object $I \otimes X$ together with arrows σ_i: $X \rightarrow I \otimes X$ ($i \in I$) such that, for any arrows f_i: $X \rightarrow A$, there is a unique arrow f: $I \otimes X \rightarrow A$ such that $f_i = f \circ \sigma_i$ for all $i \in I$. In $\mathcal{S}et$, $I \otimes X$ is the disjoint union of I copies of X, so that, in particular, $I \otimes 1$ is essentially I itself.

(ii) S is standard iff 1 is a *generator* in $\mathscr{C}(S)$. Here an object A of a category $\mathscr{C}$ is a generator if, for any pair of distinct $\mathscr{C}$-arrows f, g: $X \rightarrow Y$, there is an arrow h: $A \rightarrow X$ such that $f \circ h \neq g \circ h$.

Conditions (i) and (ii) are both satisfied by $\mathcal{S}et$.

We can now prove the

Theorem. *The following conditions on a well-termed, well-typed consistent local set theory S are equivalent:*

(i) *S is full and standard,*

(ii) *S is full, complete and choice,*

(iii) $\mathscr{C}(S) \simeq \mathcal{S}et$.

Proof. (i) $\Leftrightarrow$ (ii) follows from previously proved results, and (iii) $\Rightarrow$ (i) has been discussed above. Now assume (i). Since S is well-termed, for any S-map f: $X \rightarrow Y$ we can write $f(\tau)$ for each closed term τ such that $\vdash_S \tau \in X$..

We define functors Δ: $\mathscr{C}(S) \rightarrow \mathcal{S}et$, ^: $\mathcal{S}et \rightarrow \mathscr{C}(S)$ which we show establish an equivalence.

First, $\Delta(X)$ is the set of closed terms τ such that $\vdash_S \tau \in X$, where we identify σ, τ if that $\vdash_S \sigma = \tau$. Given f: $X \rightarrow Y$, we define $\Delta(f)$ to be the map $(\tau \mapsto f(\tau))$: $\Delta(X) \rightarrow \Delta(Y)$.

Next, given I in $\mathcal{S}et$, we define $\hat{I}$ to be the S-set $U_{\hat{\mathbf{I}}}$. If f: $I \rightarrow J$, there is a term $f(x)$ **:** $\mathbf{J}$ with x **:** $\hat{\mathbf{I}}$ such that that $\vdash_S f(\hat{i}) = f(i)$ for all $i \in I$. We define $f : \hat{I} \rightarrow J$ to be the S-map $x \mapsto f(x)$.

We now claim that, for any set I and any S-set X,

$$I \cong \Delta(\hat{I}) \text{ and } X \cong \Delta(X),$$

from which the equivalence of $\mathcal{S}et$ and $\mathscr{C}(S)$ follows.

For define η_I: $I \rightarrow \Delta(\hat{I})$ by $\eta_I(i) = \hat{i}$. Then η_I is a bijection. To see, e.g., that it is onto, observe that, if $\tau \in \Delta(\hat{I})$ and not $\vdash_S \tau = \hat{i}$ for all $i \in I$, then, since S is complete, $\vdash_S \tau \neq \hat{i}$ for all $i \in I$, whence $\vdash_S \forall x(\tau \neq x)$, which would make S inconsistent. Hence $\vdash_S \tau = \hat{i}$ for some $i \in I$, i.e., $\tau = \eta_I(i)$.

Finally define ε: $\Delta(X) \rightarrow X$ as follows. Take ε to be a term such that $\vdash_S \varepsilon(\hat{\tau}) = \tau$ for all $\tau \in \Delta(X)$. Then $y \mapsto \varepsilon(y)$ is an isomorphism. Again, to check that ε is onto, observe that $\vdash_S \exists y\ \varepsilon(y) = \tau$ for all $\tau \in \Delta(X)$ so that $\vdash_S \forall x{\in}X\exists y\ (\varepsilon(y) = x)$, yielding ontoness.

5.13. Toposes of variable sets: presheaves and sheaves.

While the category of sets is the most familiar, and certainly the simplest example of a topos, as pointed out in §1, the origins of the topos concept do not lie in set theory, but rather in algebraic geometry and topology. The term "topos" was introduced by Grothendieck and his school in the early 1960s as a generic name for a category satisfying certain conditions originally associated with the category of sheaves on a topological space. As remarked in §1, Lawvere came to describe such categories as toposes of *variable sets.* Here are some simple examples.

$\mathcal{S}et^{\rightarrow}$: topos of sets varying over two possible states 0 ('then"), 1 ("now"), with $0 \le 1$. An *object* X here is a pair of sets X_0, X_1 together with a "transition" map p: $X_0 \rightarrow X_1$. An *arrow* f: $X \rightarrow Y$ is a pair of maps f_0: $X_0 \rightarrow Y_0$, f_1: $X_1 \rightarrow Y_1$ compatible with the transition maps in the sense that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "p"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "q"'] & Y_1 \end{tikzcd}$$

commutes.

The truth value object Ω in $\mathcal{S}et^{\rightarrow}$has 3 (rather than 2) elements. For if (m, X) is a subobject of Y in $\mathcal{S}et^{\rightarrow}$, then we may take $X_0 \subseteq Y_0$, f_0 and f_1 identity maps, and p to be the restriction of q to X_0. Then for any $y \in Y$ there are three possibilities: (i) $q(y) \in X_1$ and $y \notin X_0$; (ii) $y \in X_0$; (iii) $q(y) \notin X_1$. So we take Ω to be the variable set $3 \rightarrow 2$, where $2 = \{0, 1\}$ nd $3 = \{0, 1, 2\}$ with $0 \mapsto 1$, $1 \mapsto 1$, $2 \mapsto 2$.

Moree generally, we may consider sets varying over $\{0, 1, \ldots, n\}$ or ω, or any totally ordered assemblage of stages. In each case there is "one more" truth value than stages: "truth" = "time" + 1.

Still more generally, we may consider the category $\mathcal{S}et^{\boldsymbol{P}}$ of *sets varying over a partially ordered set* $\boldsymbol{P}$. As objects this category has *functors* $\boldsymbol{P} \rightarrow \mathcal{S}et$, i.e., maps F which assigns to each $p \in P$ a set $F(p)$ and to each $p \le q$ such that $p \le q$ a map F_{pq}: $F(p) \rightarrow F(q)$ such that F_{pp} is the identity map for each $p \in P$ and $F_{pr} = F_{qr} \circ F_{pq}$ whenever $p \le q \le r$. An *arrow* η: $F \rightarrow G$ in $\mathcal{S}et^{\boldsymbol{P}}$ is a *natural transformation* between F and G, that is, an assignment of a map η_p: $F(p) \rightarrow G(p)$ to each $p \in P$ in such a way that, whenever $p \le q$, we have $\eta_q \circ F_{pq} = G_{pq} \circ \eta_p$. The *terminal object* $\mathbb{1}$ in $\mathcal{S}et^{\boldsymbol{P}}$ is the functor on P with constant value $1 = \{0\}$.

To determine Ω in $\mathcal{S}et^{\boldsymbol{P}}$ we define an *upper set* over $p \in P$ to be a subset U of $O_p = \{q \in P: p \le q\}$ such that $q \in U, q \ge q \Rightarrow r \in U$. Then $\Omega(p)$ is the set of all upper sets over p and $\Omega_{pq}(U) = U \cap O_q$ for $p \le q$, $U \in \Omega(p)$. The arrow t: $\mathbb{1} \rightarrow \Omega$ has $t_p(0) = O_p$ for each $p \in P$.

Objects in $\mathcal{S}et^{\boldsymbol{P}^*}$ where P^* is the partially ordered set obtained by reversing the order on P, are called *presheaves* on P. In particular, when P is the partially ordered set $\mathcal{O}(X)$ of open sets in a topological space X, objects in $\mathcal{S}et^{\mathcal{O}(X)^*}$ called *presheaves on X*. So a presheaf on X is an assignment to each $U \in \mathcal{O}(X)$ of a set $F(U)$ and to each pair of open sets U, V such that $V \subseteq U$ of a map F_{UV} : $F(U) \rightarrow F(V)$ such that, whenever $W \subseteq U \subseteq V$, $F_{UW} = F_{VW} \circ F_{UV}$ and F_{UU} is the identity map on $F(U)$.

If $s \in F(U)$, write $s|_V$ for $F_{UV}(s)$—the *restriction* of s to V. A presheaf F is a *sheaf* if whenever $U = \bigcup_{i \in I} U_i$ and we are given a set $\{s_i: i \in I\}$ such that $s_i \in F(U_i)$ for all $i \in I$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a *unique* $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. For example, $C(U)$ = set of continuous real-valued functions on U, and $s|_V$ = restriction of s to V defines the sheaf of continuous real-valued functions on X.

The full subcategory $\mathcal{Shv}(X)$ of $\mathcal{Set}^{\mathcal{O}(X)^*}$ whose objects are the sheaves on X is a topos called the *topos of sheaves* on X. Its truth-value object is the sheaf Ω on X given by $\Omega(U) = \{V \in \mathcal{O}(X): V \subseteq U\}$, so that in particular $\Omega(X)$, its set of global truth values, coincides with $\mathcal{O}(X)$ itself.

As a topos, the category of sheaves on a topological space arises from the topos of presheaves by confining attention to presheaves satisfying a certain covering condition. In the early 1960s Grothendieck extended the notion of covering to arbitrary categories, by so doing conferring a generality on the concept which proved to be of immense fertility, a fertility extending even as far as logic.

Grothendieck's covering concept—which came to be known as a *Grothendieck topology*—is formulated within an arbitrary category $\mathcal{C}$. First, one defines a *sieve* on an object C of $\mathcal{C}$ to be a collection S of $\mathcal{C}$-arrows closed under composition on the right, that is, satisfying $f \in S \Rightarrow f \circ g \in S$ for any arrow g composable with f on the right. Note that, if S is a sieve on C and $h: D \to C$ is any arrow with codomain C, then the set

$$h^*(S) = \{g: \text{codomain}(g) = D \ \& \ h \circ g \in S\}$$

is a sieve on D. A *Grothendieck topology,* or simply a *topology,* on $\mathcal{C}$ is a map J which assigns to each object C a collection $J(C)$ of sieves on C in such a way that the following conditions (i), (ii), (iii) are satisfied. For any arrow $f: D \to C$, let us say that S *covers* f if $f^*(S) \in J(D)$.

(i) **Reflexivity.** If S is a sieve on C, and $f \in S$, then S covers f.

(ii) **Stability.** If S covers $f : D \to C$, it also covers any composite $f \circ g$.

(iii) **Transitivity.** If S covers $f : D \to C$, and R is a sieve on C which covers every map in S, then R also covers f.

These three conditions are the counterparts, in an arbitrary category, of the natural conditions satisfied by covering families in the usual sense in the category of all subsets of a set, partially ordered by (reverse) inclusion. In this setting (i) reads: any covering family covers each of its members, (ii): the restriction of a covering family to a subset U of the set covered by it is a covering family of U, (iii): given a covering family S, and for each member $U \in S$ a covering family S_U, the family comprising the members of all the S_U is itself a covering family.

Now the concept of sheaf associated with a Grothendieck topology arises, as does the classical concept, from corresponding toposes of presheaves. The presheaf toposes in question are generalizations of the toposes of presheaves of sets varying over a partially ordered set; their objects are, in fact, *sets varying* (contravariantly) *over an arbitrary category* $\mathcal{C}$. Thus one defines a *presheaf* on $\mathcal{C}$ to be a functor from $\mathcal{C}^{op}$—the "opposite" category of $\mathcal{C}$ in which all arrows are "reversed"—to the category of sets. The presheaf category $\mathcal{C}^\wedge$ has as objects all presheaves on $\mathcal{C}$ and as arrows all natural transformations between them. This is, as in the previous cases, a topos. There is a natural embedding Y—the *Yoneda embedding*—of $\mathcal{C}$ into $\mathcal{C}^\wedge$, whose action on objects is defined as follows. For each object C of $\mathcal{C}$, YC is the presheaf on $\mathcal{C}$ which assigns, to each object X of $\mathcal{C}$, the set $\mathrm{Hom}(X, C)$ of $\mathcal{C}$-arrows from X to C. The presheaf YC is the natural representative of C in $\mathcal{C}$, and the two are usually identified.

Grothendieck defined a *site* to be a pair $(\mathcal{C}, J)$ consisting of a category $\mathcal{C}$ and a topology J thereon. Since any sieve S may be regarded in a natural way as a subfunctor or subobject of the object YC, it may also be considered an object of $\mathscr{C}^\wedge$. Now suppose given a $\mathscr{C}^\wedge$-object F of and a $\mathscr{C}^\wedge$ - arrow f: $S \to F$. A $\mathscr{C}^\wedge$-arrow g: $YC \to F$ is said to be an *extension* of f to YC if its restriction to the subobject F of YC coincides with f. A presheaf F is called a (J-) *sheaf* if, for any object C, and any J-covering sieve S on C, any arrow f: $S \to F$ has a *unique* extension to YC. Thus, speaking figuratively, a J-sheaf is a presheaf F which "believes", as it were, that (the canonical representative YC of) any $\mathscr{C}$-object C is "really covered" by any of its J-covering sieves, in the sense that, in $\mathscr{C}^\wedge$, any arrow from such a covering sieve to F fully determines an arrow from YC to F.

Grothendieck and his school assigned the term "topos" (in this case, a back-formation from the word "topology" to its original Greek source "topos", "place") to any category of sheaves associated with a site. Thus, for any site ($\mathscr{C}$, J), the full subcategory $\mathscr{S}h_J(\mathscr{C})$ was referred to as the topos of sheaves on that site. After Lawvere and Tierney had distilled from this the general concept of elementary topos, Grothendieck's original constructs came to be known as *Grothendieck toposes* (cf. "Abelian" groups).

Lawvere and Tierney showed that a Grothendieck topology can be regarded as an arrow on a truth value object. To be precise, a *Lawvere-Tierney* topology in a topos $\mathscr{E}$ is an arrow j: $\Omega \to \Omega$ which is such that (i) $j \circ \top = \top$, (ii) $j \circ j = j$, (iii) $j \circ \wedge = \wedge \circ (j \times j)$, that is, preserves $\top$, is idempotent, and distributes over conjunctions. It soon came to be recognized that a Lawvere-Tierney topology in a topos is essentially the same thing as a *modal operator* in its internal language. Stated in the context of local set theories, a *modality* in a local set theory S is a formula μ with exactly one free variable of type $\mathbf{\Omega}$ satisfying the following conditions: for any formulas α, β, (i) $\alpha \vdash_S \mu(\alpha)$, (ii) $\alpha \vdash_S \beta \Rightarrow \mu(\alpha) \vdash_S \mu(\beta)$, (iii) $\mu(\mu(\alpha)) \vdash_S \mu(\alpha)$. (So μ may be thought of as a *possibility* operator.)

Thus Grothendieck topologies ultimately correspond to modalities. What about sheaves? It turns out that they correspond to *absolutely closed sets.* Given a modality μ in a local set theory S, we define the (μ-) *closure* of an S-set X to be the set $\{x: \mu(x \in X)\}$; a set is said to be *closed* if it coincides with its closure, and *separated* if each of its singletons is closed. It is *absolutely closed,* or a (μ-)*sheaf,* if it is separated, and whenever it is (isomorphic to) a subset of a separated set, it is always a closed subset. Again, it can be shown that sheaves in this sense also give rise to toposes, that is, for any modality μ in S, the full subcategory $\mathscr{S}h_\mu(S)$ of $\mathscr{C}(S)$ whose objects are all μ-sheaves is a topos. Moreover, there is a natural functor L: $\mathscr{C}(S) \to \mathscr{S}h_\mu(S)$, the *associated sheaf functor,* which assigns to each S-set X a sheaf LX which is the "best approximation" to X by a sheaf (to be precise, the functor L is left adjoint to the insertion functor $\mathscr{S}h_\mu(S) \to \mathscr{C}(S)$. These constructions were originally carried out by Lawvere in the topos context, before the development of the internal language. Different constructions of L were also given by Johnstone (1974) and, within the internal language, by Veit (1981).

An important special case of a modality which received intense study during the early development of topos theory is the so-called *double negation* modality defined by $\mu(\alpha) = \neg\neg\alpha$. For any topos $\mathscr{E}$, the full subcategory $\mathscr{E}_{\neg\neg}$ of double negation sheaves is Boolean—that is, its associated theory is classical. Later it was seen that the true reason for this is that the truth of sentences of the internal language in $\mathscr{E}$ and in $\mathscr{E}_{\neg\neg}$ are related by a "translation" to all intents and purposes identical with Gödel's translation of classical into intuitionistic logic. One of the most important early uses of double negation sheaves was in forging links between topos theory and independence proofs in set theory. Indeed, if P is a partially ordered set of conditions in the usual Cohen sense, then the topos of double negation sheaves in the category $P^\wedge$ of presheaves on P turns out to be equivalent to the Scott-Solovay Boolean universe $V^{(B)}$, where B is the Boolean completion of P. This observation led Tierney (1972) and others, as observed in §1, to analyze some of the independence proofs, notably that of the continuum hypothesis, in a topos-theoretic setting.

6. First-Order Model Theory in Local Set Theories and Toposes

As observed in §1, Joyal and Reyes continued the search initiated by Lawvere for an adequate categorical formulation of the model theory of first-order languages. One way of doing this, as they came to see, was essentially to retain both the traditional presentation of first-order theories, and the concept of model, but to interpret the latter in an arbitrary topos, rather than in the usual category of sets. This led them to identify a particular class of first-order theories—the so-called *geometric* theories—whose models in toposes have a particularly simple categorical description. They showed that, associated with each such theory Σ, there is a topos—the so-called *classifying topos* of Σ—which contains a canonical or *generic* model of Σ validating exactly those (geometric) formulas holding in every model of Σ. (A systematic account of their work was given in Makkai and Reyes [1977].) Here we present the essentials of their constructions in terms of local set theories.

We begin by introducing the first-order languages appropriate to our purposes. A *(many-sorted) first-order language* $\mathbb{L}$ has the following ingredients:

- *sorts A, B, …*
- *function symbols f, g, …*each of which is assigned a *signature* $(\vec{A}, B)$ where $\vec{A} = (A_1, \ldots, A_n)$ is a finite sequence of sorts and B is a sort.
- *predicate symbols* $\alpha, \beta, \ldots$ each of which is assigned a signature $\vec{A}$
- *logical symbols* $\wedge, \vee, \rightarrow, \forall, \exists$
- *propositional constants* $\top$ (true), $\bot$ (false)
- *equality symbol* $=$
- *variables* $x_A, y_A, \ldots$ for each sort A.

Terms of $\mathbb{L}$ and their associated *sorts* are defined recursively as follows:

- each variable x_A is a term of sort A.
- if f is a function symbol of signature$(\vec{A}, B)$ and $\vec{\tau} = (\tau_1, \ldots, \tau_n)$ is a sequence of terms of sorts $A_1, \ldots, A_n$, then $f(\vec{\tau})$ is a term of sort B.

Finally, the *formulas* of $\mathbb{L}$ are given recursively by the following clauses:

- $\top$, $\bot$ are formulas
- if σ, τ are terms of the same sort, $\sigma = \tau$ is a formula
- if α is a predicate symbol of signature $\vec{A}$ and $\vec{\tau}$ a sequence of terms of sorts $A_1, \ldots, A_n$, then $\alpha(\vec{\tau})$ is a formula
- if φ, ψ are formulas and x a variable $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\forall x\varphi$, $\exists x\varphi$ are formulas

As usual, $\neg\varphi$, $\varphi \leftrightarrow \psi$ are written for $\varphi \rightarrow \bot$, $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ respectively. It is assumed that $\mathbb{L}$ is equipped with the usual deductive apparatus of intuitionistic predicate logic, *except* that the inference rule *modus ponens* is subject to the stipulation that every variable free in the rule's premises must also be free in its conclusion. (This is to allow, as in "free" logic, for the possibility that the interpretation of a sort may be empty.) This gives rise to an intuitionistic deducibility relation $\vdash$ in the usual way; we shall also write $\vdash_C$ for the corresponding *classical* deducibility relation obtained when the law of excluded middle $\varphi \vee \neg\varphi$ is added to the axioms.

Now let S be a local set theory in a local language $\mathscr{L}$. An *interpretation I* of a first-order language $\mathbb{L}$ in S assigns:

- to each sort A of $\mathbb{L}$, a type symbol $\mathbf{A}_I$ of $\mathscr{L}$ and an S-set A_I of type $\mathbf{PA}_I$

- to each function symbol f of $\mathbb{L}$ of signature $(A^{\rightarrow}, B)$, an S-map $f_I : A_{1I} \times \ldots \times A_{nI} \rightarrow B$
- to each predicate symbol α of $\mathbb{L}$ of signature $A^{\rightarrow}$,a formula $[\![\alpha]\!]_I$ of $\mathscr{L}$ with free variables $x_1, \ldots, x_n$ of types $\mathbf{A}_{1I}, \ldots, \mathbf{A}_{nI}$ such that $\vdash_S [\![\alpha]\!]_I \rightarrow x_1 \in A_{1I} \wedge \ldots \wedge x_n \in A_{nI}$.

The interpretation I can now be extended to all terms and formulas of $\mathbb{L}$ in the obvious recursive way, with the result that, if τ is a term of sort B, then $\vdash_S \tau_I \in B_I$, and if φ is a formula of $\mathbb{L}$, then $[\![\varphi]\!]_I$ is a formula of $\mathscr{L}$.

An interpretation I is a *model in S* of a set Σ of $\mathbb{L}$-formulas if $\vdash_S [\![\varphi]\!]_I$ for all $\varphi \in \Sigma$. Now write $\Sigma \vDash \varphi$ for *every model of* Σ *is a model of* φ. A straightforward induction on formulas suffices to establish the *first-order soundness theorem:* $\Sigma \vdash \varphi \Rightarrow \Sigma \vDash \varphi$.

Using the internal language, the notion of interpretation of a first order language, and the soundness theorem can be extended to toposes in the obvious way to obtain Joyal and Reyes's original concept of interpretation.

Now each set of first-order formulas Σ can be associated with certain category $\mathcal{Syn}(\Sigma)$—its *syntactic category*—described as follows. We shall write $\boldsymbol{x}, \boldsymbol{y}, ..$ for lists of variables of $\mathbb{L}$, and $\varphi(\boldsymbol{x})$, etc., to indicate that the free variables of φ are within $\boldsymbol{x}$. The *objects* of $\mathcal{Syn}(\Sigma)$ are *formal class terms,* i.e, symbols of the form $\{\boldsymbol{x}: \varphi(\boldsymbol{x})\}$. An *arrow* in $\mathcal{Syn}(\Sigma)$ between two objects $\{\boldsymbol{x}: \varphi(\boldsymbol{x})\}$ and $\{\boldsymbol{y}: \varphi(\boldsymbol{y})\}$ is defined to be an equivalence class, modulo Σ-provable equivalence, of formulas $\vartheta(\boldsymbol{x}, \boldsymbol{y})$ which are Σ-provably functional, in the obvious sense, with domain $\{\boldsymbol{x}: \varphi(\boldsymbol{x})\}$ and codomain $\{\boldsymbol{y}: \varphi(\boldsymbol{y})\}$: the arrow defined by $\vartheta(\boldsymbol{x}, \boldsymbol{y})$ is denoted by $[\boldsymbol{x} \mapsto \boldsymbol{y} \mid \vartheta(\boldsymbol{x}, \boldsymbol{y})]$.

A Grothendieck topology K_Σ called the *finite cover topology* is now introduced on $\mathcal{Syn}(\Sigma)$. A finite family of arrows

$$[\boldsymbol{x_i} \mapsto \boldsymbol{y} \mid \vartheta_i(\boldsymbol{x_i}, \boldsymbol{y})]: \{\boldsymbol{x_i}: \varphi_i(\boldsymbol{x_i})\} \rightarrow \{\boldsymbol{y}: \psi(\boldsymbol{y})\} \quad (i = 1, \ldots, n)$$

is said to Σ-*provably epic on* $\boldsymbol{y}$: $\psi(\boldsymbol{y})\}$ if the image of the family "covers" $\{\boldsymbol{y}: \psi(\boldsymbol{y})\}$ in the sense that

$$\Sigma \vdash \forall \boldsymbol{y}[\psi(\boldsymbol{y}) \rightarrow \exists \boldsymbol{x}_1 \vartheta_1(\boldsymbol{x_1}, \boldsymbol{y}) \vee \ldots \vee \exists \boldsymbol{x}_n \vartheta_n(\boldsymbol{x}_n, \boldsymbol{y})].$$

For each object $A = \{\boldsymbol{y}: \psi(\boldsymbol{y})$ we define $K_\Sigma(A)$ to consist of all cosieves on A containing a provably epic family. The resulting site $(\mathcal{Syn}(\Sigma), K_\Sigma)$ is called the *syntactic site* determined by Σ.

A formula of $\mathbb{L}$ not containing $\rightarrow$ or $\forall$ is called a *geometric* formula. A *geometric implication* is a sentence of the form $\forall \boldsymbol{x}(\varphi \rightarrow \psi)$, where φ and ψ are geometric. A set of geometric implications is called a *geometric theory.* The *geometric category* $\mathcal{Geom}(\Gamma)$ determined by a geometric theory Γ is constructed in the same way as was the syntactic category, except that throughout the construction attention is confined to *geometric* formulas. The finite cover topology K_Γ is introduced on $\mathcal{Geom}(\Gamma)$ analogously; the resulting site $(\mathcal{Geom}(\Gamma), K_\Gamma)$ is called the *geometric site* determined by Γ.

Joyal and Reyes showed that models of a geometric theory Γ in a given topos $\mathscr{E}$ correspond in a natural way to the so-called *geometric functors* to $\mathscr{E}$ defined on the category $\mathcal{Geom}(\Gamma)$. Here a geometric functor is a functor F: $\mathcal{Geom}(\Gamma) \rightarrow \mathscr{E}$ that is (left) *exact,* that is, preserves terminal objects, products and equalizers, and which in addition *preserves coverings*. By this is meant that, for any K_Γ-covering cosieve S of an object A, the family of arrows $\{Ff: f \in S\}$ covers the image FA in the sense that, for any arrows g, h in $\mathscr{E}$ with common domain A, if $g \circ Ff = h \circ Ff$ for all $f \in S$, then $g = h$.

Accordingly, models of geometric theories in toposes may be identified with geometric functors. Joyal and Reyes showed that for each geometric theory Γ, the topos $\mathcal{Set}[\Gamma]$ of sheaves over the geometric site determined by Γ is the topos obtained by adjoining a *universal model* of Γ to $\mathcal{Set}$. To be precise, there is a model (i.e., a geometric functor) M of Γ in $\mathcal{Set}[\Gamma]$ with the universal property that *any* model of Γ in a (Grothendieck) topos $\mathcal{E}$ can be "uniquely factored" through M. For this reason $\mathcal{Set}[\Gamma]$ is called the *classifying topos* of Γ. This fact leads quickly to the *completeness theorem for geometric theories*, namely, for a geometric theory Γ and a geometric implication φ, $\Gamma \vdash \varphi$ if and only if φ is true in every model of Γ. Application of a result of Barr (1974) led to the discovery that "$\vdash$" can be replaced by "$\vdash_C$", and thence to the purely logical conclusion that a geometric implication classically deducible from a geometric theory is also intuitionistically deducible from it.

The known fact that only geometric theories possess classifying toposes in the above sense seems to have led topos-theorists to neglect the use of syntactic sites for the construction of models of *arbitrary* first-order theories. The existence of a topos containing a model of an arbitrary first-order theory Σ, generic in the sense that the sentences true in M are precisely those deducible from Σ had been pointed out in the 1970s by Peter Freyd (see Freyd and Scedrov 1990) and by Fourman and Scott (1979)—from this the full *completeness theorem* for first-order theories, namely, that $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$, quickly follows. But it was not in fact until the late 1990s—in Butz and Johnstone (1998) and Palmgren (1997)— that the observation was made that, for any first-order theory Σ, the topos of sheaves over its syntactic site actually contains such a generic model of Σ.

7. Constructive Type Theory

Finally I come to *constructive type theory* (CTT). The first truly constructive theory of types, in the sense of being both predicative and based on intuitionistic logic, to undergo systematic development, was that of Per Martin-Löf (1975, 1982, 1984). In introducing it his purpose was to provide, as he put it in (1975) "a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop." The chief advantage of Martin-Löf's framework over traditional intuitionistic systems was in allowing proofs to be constituents of propositions, so enabling propositions to express properties of proofs, and not merely individuals, as in first-order predicate logic. Indeed, it provides a complete embodiment of the "propositions-as-types (or sets)" idea originally suggested by Curry and Feys (1958) and Howard (1980). At the root of the "propositions-as-types" conception lies the idealist notion, which may be traced back to Kant, that the meaning of a proposition does not derive from an absolute standard of truth external to the human mind, but resides rather in the evidence for its assertability in the form of a mental construction or proof. In the "propositions-as-types" interpretation, and more generally, in constructive type theories, each proposition is the type, or set, of its proofs. A major consequence is that *under this interpretation these are the only sets, or types*. In other words, a set *is a set of proofs*, or more generally, constructions. It is this feature that has made constructive type theory particularly suitable for developing the theory of computer programming. (Here the somewhat hazy idea of "mental constructions" has been replaced by the precise notion of a computer program.)

Here is Martin-Löf himself on the matter in (1975)

> "Every mathematical object is of a certain kind or *type*. Better, a mathematical object is always given together with its type, that is it is not just an object: it is an object of a certain type. … A type is defined by prescribing what we have to do in order to construct an object of that type… Put differently, a type is well-defined if we understand…what it means to be an object of that type. … Note that it is required, neither that we should be able to generate somehow all the objects of a given type, nor that we should so to say know all of them individually. It is only a question of understanding what it means to be an *arbitrary* object of the type in question.
>
> A *proposition* is defined by prescribing how we are allowed to prove it, and a proposition *holds* or is *true* intuitionistically if there is a proof of it. … It will not be necessary, however, to introduce the notion of proposition as a separate notion because we can represent each proposition by a certain type, namely, the type of proofs of that proposition. Conversely, each type determines a proposition, namely, the proposition that the type in question is nonempty. This is the proposition which we prove by exhibiting an object of the type in question. On this analysis, there appears to be no fundamental difference between propositions and types. Rather, the difference is one of point of view: in the case of a proposition, we are not so much interested in what its proofs are as in whether it has a proof, that is, whether it is true or false, whereas, in the case of a type, we are of course interested in what its objects are and not only in whether it is empty or nonempty."

A key element in Martin-Löf's formulation of type theory is the distinction, which goes back to Frege, between *propositions* and *judgments.* Propositions (which, as we have seen, in Martin-Löf's systems are identified with types) are syntactical objects on which mathematical operations can be performed and which bear certain formal relationships to other syntactical objects called proofs. Propositions and proofs are, so to speak, *objective* constituents of the system. Judgments, on the other hand, typically involve the *idealist* notion of "understanding" or "grasping the meaning of". Thus, for example, while $2 + 2 = 4$ is a proposition, "$2 + 2 = 4$ is a proposition" and "$2 + 2 = 4$ is a true proposition" are judgments.

Martin-Löf also follows Frege in taking the rules of inference of logic to concern judgments rather than propositions. Thus, for example, the correct form of the rule of $\rightarrow$ -elimination is not

$$\frac{A \quad A \rightarrow B}{B}$$

but

$$\frac{A \text{ true} \quad A \rightarrow B \text{ true}}{B \text{ true}}.$$

That is, the rule does not say that the proposition B follows from the propositions A and $A \rightarrow B$, but that the *truth* of the proposition B follows from the *truth* of the proposition A conjoined with that of $A \rightarrow B$. In general, judgments may be characterized as expressions which appear at the conclusions of rules of inference.

Another important respect in which Martin-Löf follows Frege is in his insistence that judgments and formal rules must be accompanied by full explanations of their *meaning.* (This is to be contrasted with the usual model-theoretic semantics which is really nothing more than a translation of one object-language into another.) In particular, the judgment *A is a proposition* may be made only when one knows what a (canonical) proof of A is, and the judgment *A is a true proposition* only when one knows how to find such a proof. Judgments, and the notion of truth, are thus seen to be mind-dependent.

Martin-Löf's various systems abound in subtle distinctions. For example, in addition to the distinction between proposition and judgment, there is a parallel distinction between *type* (or set) and *category*[6] (or species). In order to be able to judge that A is a category one must be able to tell what kind of objects fall under it, and when they are equal. To be in a position to make the further judgment that a category is a type, or set, one must be able to specify what its "canonical" or typical, elements are. In judging something to be a set, one must possess sufficient information concerning the its elements to enable quantification over it to make sense. Thus, for example, the natural numbers form a set $\mathbb{N}$, with canonical elements given by: 0 is a canonical element of $\mathbb{N}$, and if n is a canonical element of $\mathbb{N}$, then $n + 1$ is a canonical element of $\mathbb{N}$. On the other hand the collection of subsets of $\mathbb{N}$ forms a category, but not a set.

The "propositions-as-types" conception (which for convenience we abbreviate to PAT) gives rise to a correspondence between logical operators and operations on types (or sets). Tait (1994)) offers a clear exposition of the idea. To begin with, consider two propositions/types A and B. What should be required of a proof f of the implication $A \rightarrow B$? Just that, given any proof x of A, f should yield a proof of B, that is, f should be a function from A to B. In other words, the proposition $A \rightarrow B$ is just the type of functions from A to B:

$$A \rightarrow B = B^A$$

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[6] In this usage, of course, to be distinguished from the term as employed in its mathematical sense throughout the present article.

Similarly, all that should be required of a proof c of the conjunction $A \wedge B$ is that it yield proofs x and y of A and B, respectively. From the construction-theoretic point of view $A \wedge B$ is accordingly just the type $A \times B$ of all pairs (x, y), with x of type A (we write this as x: A) and y: B.

A proof of the disjunction $A \vee B$ is either a proof of A or a proof of B together with the information as to which of A or B it is a proof. That is, if we introduce the type 2 with the two distinct elements 0 and 1, a proof of $A \vee B$ may be identified as a pair (c, n) in which either c is a proof of A and n is 0, or c is a proof of B and n is 1. This means that, from the construction-theoretic point of view, $A \vee B$ is the disjoint union $A + B$ of A and B.

The true proposition $\top$ may be identified with the one element type $1 = \{0\}$: 0 thus counts as the unique proof of $\top$. The false proposition $\bot$ is taken to be a proposition which lacks a proof altogether: accordingly $\bot$ is identified with the empty set $\varnothing$. The negation $\neg A$ of a proposition A is defined as $A \to \bot$, which therefore becomes identified with the set $A^{\varnothing}$.

As we have already said, a proposition A is deemed to be true if it (i.e, the associated type) has an element, that is, if there is a function $1 \to A$. Accordingly the *law of excluded middle* for a proposition A becomes the assertion that there is a function $1 \to A + \varnothing^A$.

If a and b are objects of type A, we introduce the *identity proposition* or *type* $a =_A b$ expressing that a and b are identical objects of type A. This proposition is true, that is, the associated type has an element, if and only if a and b are identical. In that case id(a) will denote an object of type a.

In PAT one must distinguish sharply between *propositions,* which have proofs, and *judgements,* which do not. For example $0 =_2 0$ is a proposition, while "0 is of type 2" is a judgement. Rather than being true or false, a judgement is either assertable, or nonsensical.

While 2^A does not have a very natural interpretation as a proposition, it may be considered the type of all *decidable sets of objects* of type A. For given f: 2^A and x: A, if we define elementhood by

$$x \in_A f \quad \text{iff } fx = 1,$$

then it is easy to see that $x \in_A f \vee \neg\, x \in_A f$.

In order to deal with the quantifiers we require operations defined on families of types, that is, types $\varphi(x)$ depending on objects x of some type A. (It is for this reason that Martin-Löf's system is also known as *dependent type theory*: see Jacobs (1999).) By analogy with the case $A \to B$, a proof f of the proposition $\forall x{:}A\ \varphi(x)$, that is, an object of type $\forall x{:}A\ \varphi(x)$, should associate with each x: A a proof of $\varphi(x)$. So f is just a function with domain A such that, for each x: A, fx is of type $\varphi(x)$. That is, $\forall x{:}A\ \varphi(x)$ is the *product* $\Pi x{:}A\ \varphi(x)$ of the $\varphi(x)$'s. We use the λ-notation in writing f as $\lambda x fx$.

A proof of the proposition $\exists x{:}A\ \varphi(x)$, that is, an object of type $\exists x{:}A\ \varphi(x)$, should determine an object x: A and a proof y of $\varphi(x)$, and *vice-versa.* So a proof of this proposition is just a pair (x, y) with x: A and y: $\varphi(x)$. Therefore $\exists x{:}A\ \varphi(x)$ is the *disjoint union*, or *coproduct* $\coprod x{:}A\ \varphi(x)$ of the $\varphi(x)$'s.

Still following Tait (1994), we introduce the functions σ, π, π' of types $\forall x{:}A(\varphi(x) \to \exists x{:}A\ \varphi(x))$, $\exists x{:}A\ \varphi(x) \to A$, and $\forall y{:}\ (\exists x \varphi(x)).\ \varphi(\pi(y))$ as follows. If b: A and c: $\varphi(b)$, then σbc is (b, c). If d: $\exists x{:}A\ \varphi(x)$, then d is of the form (b, c) and in that case $\pi(d) = b$ and $\pi'(d) = c$. These yield the equations

$$\pi(\sigma bc) = b \qquad \pi'(\sigma bc) = c \qquad \sigma(\pi d)(\pi' d) = d.$$

The *axiom of choice* (AC) is the proposition

$$\forall x{:}A\exists y{:}B\, \varphi(x, y)) \to \exists x{:}B^A\forall x{:}A\, \varphi(x, fx)).$$

AC is true in PAT, as the following argument shows. Let u be a proof of the antecedent $\forall x{:}A\exists y{:}B\, \varphi(x, y))$. Then, for any $x{:}\, A$, $\pi(ux)$ is of type B and $\pi'(ux)$ is a proof of $\varphi(x, \pi ux)$. So $s(u) = \lambda x.\pi(ux)$ is of type B^A and $t(u) = \lambda x.\, \pi'(ux)$ is a proof of $\forall x{:}A\, \varphi(x, s(u)x)$. Accordingly $\lambda u.\sigma s(u)t(u)$ is a proof of $\forall x{:}A\exists y{:}B\, \varphi(x, y)) \to \exists x{:}B^A\, \forall x{:}A\varphi(x, fx))$.

In ordinary set theory—indeed in local set theory—this argument establishes the *isomorphism* of the sets $\Pi x{:}A\coprod y{:}B\, \varphi(x, y))$ and $\coprod f{:}B^A\, \Pi x{:}A\, \varphi(x, fx))$, but not the validity of the axiom of choice. In (local) set theory AC is not is not represented by this isomorphism, but is rather (equivalent to) the equality in which Π is replaced by $\bigcap$ and $\coprod$ by $\bigcup$, namely

$$\bigcap_{x\in A}\bigcup_{y\in B}\varphi(x,y) = \bigcup_{f\in B^A}\bigcap_{x\in A}\varphi(x,fx).$$

While in PAT AC has no "untoward" logical consequences, in local or intuitionistic set theory, or in the internal language of a topos, this is far from being the case, for there, as we know, AC implies the law of excluded middle. Let us repeat the argument in a simplified form.

Suppose given a choice function f on the power set of the set $2 = \{0, 1\}$. Let α be any proposition, and define

$$U = \{x\in 2\colon x = 0 \vee \alpha\} \qquad V = \{x\in 2\colon x = 1 \vee \alpha\}.$$

Writing $a = fU$, $b = fV$, we have $a \in U$, $b \in V$, i.e.,

(#) $$[a = 0 \vee \alpha] \wedge [b = 1 \vee \alpha].$$

It follows that

$$[a = 0 \wedge b = 1] \vee \alpha,$$

whence

(*) $$a \neq b \vee \alpha,$$

Now clearly

$$\alpha \Rightarrow U = V = 2 \Rightarrow a = b,$$

whence

$$a \neq b \Rightarrow \neg\alpha.$$

But this and (*) together imply $\neg\alpha \vee \alpha$. (In fact, we need only assume the choice function to be defined on the set $\{U, V\}$.)

Given that AC holds in PAT, it is of interest to ask why these arguments cannot be reproduced there. Now the first argument seems to hinge on two assumptions, first, that the sets U and V are well defined and satisfy the usual "eliminability" conditions leading to the assertability of (#) above. And secondly, that the choice function f satisfies extensionality in the sense that, if U and V are extensionally identical, then $fU = fV$. It seems to be the case that when subset types are added to PAT (in Martin-Löf's system), the 'eliminability" condition

$$a \in \{x: \varphi(x)\} \to \varphi(a)$$

fails. Concerning the second argument, this seems to fail essentially because in PAT the value of a function defined on a (sub)set X depends not only on the variable member x of X but also on the *proof* that x is in fact in X. Thus suppose given types A, B and a subset $X = \{x: \beta(x)\}$ of A. Write $p \vdash \alpha$ for "p is a proof of α". Then in PAT from $\forall x{:}A[\beta(x) \to \exists y{:}B\varphi(x, y)]$ we can infer the existence of a function f: $\{(x, p): p \vdash \beta(x)\} \to B$ for which $\forall x\forall p[p \vdash \beta(x) \to \varphi(x, f(x,p))]$. Now return to Diaconescu's argument. Here A is P2, the power set of 2 (supposing that to be present), $\beta(x)$ is $\exists x.\ x \in X$ (X a variable of type P2), B is 2 and $\varphi(X, y)$ is $y \in X$. Now, given a proposition α, define the subsets U and V as before. Constructively, the only proof of $\exists x.\ x \in U$ to be had is by exhibiting a member of U, and , since α is not known to be true, the only exhibitable member of U is 0. Similarly, the only exhibitable member of V is 1. Accordingly, writing $a = f(U, 0)$ and $b = f(V, 1)$, we derive (*) above as before (assuming subset eliminability). But now while $\alpha \to U = V$, we cannot infer that $U = V \to a = b$, so blocking the derivation of $\alpha \to a = b$.

So, as Maietti and Valentini (1999) showed, if extensional power sets are suitably added to PAT (or CTT), logic becomes classical there.

In the "propositions as types" interpretation each logical operation corresponds to a categorical operation: $\wedge$ to $\times$, $\vee$ to $+$, $\to$ to exponentiation, $\perp$ to an initial element 0, $\neg$ to exponentiation by 0, $\forall$ to product and $\exists$ to coproduct. This suggests that constructive type theory should be interpretable in suitable categories, just as local set theories are interpretable in toposes. For some time it was conjectured that the appropriate categories in this respect were the so-called *locally Cartesian closed* categories: these are finitely complete categories $\mathscr{E}$ such that, for each object A of $\mathscr{E}$, the "slice" category $\mathscr{E}/A$ is Cartesian closed. (Any topos is locally Cartesian closed.) It was already known that many of the mathematical constructions within a topos could be carried out within a locally Cartesian closed category. In such a category, the notion of "variable set", for example an "A-indexed set" , is represented by an arrow $B \to A$ of $\mathscr{E}$, that is, by an object of $\mathscr{E}/A$. In the interpretation of constructive type theory that was explicitly worked out by Seely (1984), types are interpreted as objects, and terms as arrows, of a locally Cartesian closed category $\mathscr{E}$. As Seely pointed out, this interpretation engenders a systematic ambiguity among the notions of type, predicate and term, and between object and proof: indeed a term of type A is an arrow into A, which is in turn a predicate over A, and an arrow $1 \to A$ may be regarded either as an object of type A or as a proof of the proposition A. (This ambiguity is, of course, shared by the "propositions as types" interpretation.) In precise analogy with the correspondence between local set theories and toposes, Seely establishes a correspondence between theories formulated in Martin-Löf's system—constructive type theories—and locally Cartesian closed categories.

These correspondences yield in turn a correspondence between local set theories and constructive type theories. To wit, starting with a local set theory S, we may consider the topos $\mathscr{C}(S)$ determined by S: as observed above, this is a locally Cartesian closed category and so has an associated constructive type theory $T(S)$. It can be shown that S is witnessed and sententially classical if and only if the law of excluded middle for all propositions holds in $T(S)$, and Awodey (1995) essentially shows that S is choice if and only if , for every S-set I, the law of excluded middle for all propositions holds in $T(S_I)$. On the other hand, that the axiom of choice automatically holds in $T(S)$ —as it does in any constructive type theory—corresponds merely to the easily verified fact that, for any family $\{X_{ij} : i \in I, j \in J\}$ of *S*-sets, there is, *S*-provably, a natural isomorphism between the *S-sets* $\prod_{i \in I} \coprod_{j \in J} X_{ij}$ and $\coprod_{f \in J^I} \prod_{i \in I} X_{if(i)}$. This shows once more the essentially different status of the axiom of choice in local set theory and constructive type theory.

The relationship between type theories and categories is investigated in a general setting in Jacobs (1999).

Appendix
Basic Concepts of Category Theory

A *category* $\mathscr{C}$ is determined by first specifying two classes $Ob(\mathscr{C})$, $Arr(\mathscr{C})$—the collections of $\mathscr{C}$*-objects* and $\mathscr{C}$*-arrows*. These collections are subject to the following axioms:

- Each $\mathscr{C}$-arrow f is assigned a pair of $\mathscr{C}$-objects dom(f), cod(f) called the *domain* and *codomain* of f, respectively. To indicate the fact that $\mathscr{C}$-objects X and Y are respectively the domain and codomain of f we write f: $X \to Y$ or $X \xrightarrow{f} Y$. The collection of $\mathscr{C}$-arrows with domain X and codomain Y is written $\mathscr{C}(X, Y)$.

- Each $\mathscr{C}$-object X is assigned a $\mathscr{C}$-arrow 1_X: $X \to X$ called the *identity arrow* on X.

- Each pair f, g of $\mathscr{C}$-arrows such that cod(f) = dom(g) is assigned an arrow $g \circ f$: dom(f) $\to$ cod(g) called the *composite* of f and g. Thus if f: $X \to Y$ and g: $Y \to Z$ then $g \circ f$: $X \to Z$. We also write $X \xrightarrow{f} Y \xrightarrow{g} Z$ for $g \circ f$. Arrows f, g satisfying cod(f) = dom(g) are called *composable.*

- *Associativity law.* For composable arrows (f, g) and (g, h), we have $h \circ (g \circ f)$ = $h \circ (g \circ f)$.

- *Identity law.* For any arrow f: $X \to Y$, we have $f \circ 1_X = f = 1_Y \circ f$.

As a basic example of a category, we have the category $\mathscr{Set}$ of sets whose objects are all sets and whose arrows are all maps between sets (strictly, triples (f, A, B) with domain(f) = A and range(f) $\subseteq B$.) Other examples of categories are the category of groups, with objects all groups and arrows all group homomorphisms and the category of topological spaces with objects all topological spaces and arrows all continuous maps. Categories with just one object may be identified with *monoids*, that is, algebraic structures with an associative multiplication and an identity element.

A *subcategory* $\mathscr{C}$ of a category $\mathscr{D}$ is any category whose class of objects and arrows is included in the class of objects and arrows of $\mathscr{D}$, respectively, and which is closed under domain, codomain, identities, and composition. If, further, for any objects C, C' of $\mathscr{C}$, we have $\mathscr{C}(C, C') = \mathscr{D}(C, C')$, we shall say that $\mathscr{C}$ is a *full* subcategory of $\mathscr{D}$.

BASIC CATEGORY-THEORETIC DEFINITIONS

Commutative diagram (in category)

Diagram of objects and arrows such that the arrow obtained by composing the arrows of any connected path depends only on the endpoints of the path.

Initial object

Object 0 such that, for any object X, there is a unique arrow $0 \to X$ (e.g., $\varnothing$ in $\mathcal{Set}$)

Terminal object

Object 1 such that, for any object X, there is a unique arrow $X \to 1$ (e.g. any singleton in $\mathcal{Set}$)

Element of an object X

Arrow $1 \to X$

Monic arrow $X \rightarrowtail Y$

Arrow $f\colon X \to Y$ such that, for any arrows $g, h\colon Z \to X$, $f \circ g = f \circ h \Rightarrow g = h$ (in $\mathcal{Set}$, *one-one* map)

Epic arrow $X \twoheadrightarrow Y$

Arrow $f\colon X \to Y$ such that, for any arrows $g, h\colon Y \to Z$, $g \circ f = h \circ f \Rightarrow g = h$ (in $\mathcal{Set}$, *onto* map)

Isomorphism $X \cong Y$

Arrow $f\colon X \to Y$ for which there is $g\colon Y \to X$ such that $g \circ f = 1_X$, $f \circ g = 1_Y$

Product of objects X, Y

Object $X \times Y$ with arrows (projections) $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ such that any diagram

$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "g"] & \\ {} & & {} \end{tikzcd}$$

can be uniquely completed to a commutative diagram

$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[d, "{<f,g>}"'] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

Product of arrows $f_1: X_1 \to Y_1$, $f_2: X_2 \to Y_2$

Unique arrow $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ making the diagram

$$\begin{tikzcd}
& & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dd, "f_1 \times f_2"] \arrow[dr, "\pi_2"] & & \\
& X_1 \arrow[dl, "f_1"'] & & X_2 \arrow[dr, "f_2"] & \\
Y_1 & & Y_1 \times Y_2 \arrow[ll, "\pi'_1"'] \arrow[rr, "\pi'_2"] & & Y_2
\end{tikzcd}$$

commute. I..e., $f_1 \times f_2 = < f_1 \circ \pi_1, f_2 \circ \pi_2>$.

Diagonal arrow on object X

Unique arrow $\delta_X : X \to X \times X$ making the diagram

$$\begin{tikzcd}
& X \arrow[dl, "1_X"'] \arrow[d, "\delta_X"] \arrow[dr, "1_X"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

commute. I.e., $\delta_X = <1_X, 1_X>$.

Coproduct of objects X, Y

Object $X + Y$ together with a pair of arrows $X \xrightarrow{\sigma_1} X + Y \xleftarrow{\sigma_2} Y$ such that for any pair of arrows $X \xrightarrow{f} A \xleftarrow{g} B$, there is a unique arrow $X + Y \xrightarrow{h} A$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\sigma_1"] \arrow[dr, "f"'] & X + Y \arrow[d, "h"] & Y \arrow[l, "\sigma_2"'] \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

commutes.

Pullback diagram or *square*

Commutative diagram of the form

$$\begin{tikzcd}
\blacksquare \arrow[r] \arrow[d] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "g"] & \bullet
\end{tikzcd}$$

such that for any commutative diagram

$$\begin{tikzcd}
\blacklozenge \arrow[r] \arrow[d] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "g"] & \bullet
\end{tikzcd}$$

there is a unique $\blacklozenge \xrightarrow{!} \blacksquare$ such that

$$\begin{tikzcd}
\blacklozenge \arrow[dr] \arrow[drr, "!"] \arrow[ddr] & & \\
& \blacksquare \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& \bullet \arrow[r] & \bullet
\end{tikzcd}$$

commutes.

Equalizer of pair of arrows $\bullet \underset{g}{\overset{f}{\rightrightarrows}} \blacksquare$

Arrow $\blacklozenge \xrightarrow{e} \bullet$ such that $f \circ e = g \circ e$ and, for any arrow $\blacktriangle \xrightarrow{e'} \bullet$ such that $f \circ e' = g \circ e'$ there is a unique $\blacktriangle \xrightarrow{u} \blacklozenge$ such that

$$\begin{tikzcd}
\blacktriangle \arrow[r, "u"] \arrow[dr, "e'"'] & \blacklozenge \arrow[d, "e"] \\
& \bullet
\end{tikzcd}$$

commutes.

Subobject of an object X

Pair (m, Y), with m a monic arrow $Y \rightarrowtail X$

Truth value object or *subobject classifier*

Object Ω together with arrow t: $1 \to \Omega$ such that every monic m: $\bullet \rightarrowtail \blacklozenge$ (i.e., subobject of $\blacklozenge$) can be uniquely extended to a pullback diagram of the form

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "t"] \\
\blacklozenge \arrow[r, "\chi(m)"] & \Omega
\end{tikzcd}$$

Power object of an object X.

An object PX together with an arrow ("evaluation") e_X: $X \times PX \to \Omega$ such that, for any f: $X \times PX \to \Omega$, there is a unique arrow $f : Y \to PX$ such that

$$\begin{tikzcd} X \times Y \arrow[d, "1_X \times f"'] \arrow[dr, "f"] & \\ X \times PX \arrow[r, "e_X"'] & \Omega \end{tikzcd}$$

commutes. (In $\mathcal{S}et$, PX is the power set of X and e_X the characteristic function of the membership relation between X and PX.)

Exponential object of objects Y, X — An object Y^X, together with an arrow ev: $X \times Y^X \to Y$ such that, for any arrow f: $X \times Z \to Y$ there is a unique arrow $f: Z \to Y^X$—the *exponential transpose* of f— such that the diagram

$$\begin{tikzcd} X \times Z \arrow[d, "1_X \times f"'] \arrow[dr, "f"] & \\ X \times Y^X \arrow[r, "ev"'] & Y \end{tikzcd}$$

commutes. In $\mathcal{S}et$, Y^X is the set of all maps $X \to Y$ and ev is the map that sends (x, f) to $f(x)$.

Product of indexed set $\{A_i: i \in I\}$ *of objects* — Object $\prod_{i\in I} A_i$ together with arrows $\prod_{i\in I} A_i \xrightarrow{\pi_i} A_i$ $(i \in I)$ such that, for any arrows $f_i: B \to A_i$ $(i \in I)$ there is a *unique* arrow $h: B \longrightarrow \prod_{i\in I} A_i$ such that, for each $i \in I$, the diagram

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f_i"'] & \prod_{i\in I} A_i \arrow[d, "\pi_i"] \\ & A_i \end{tikzcd}$$

commutes.

Coproduct of indexed set $\{A_i: i \in I\}$ *of objects* — Object $\coprod_{i\in I} A_i$ together with arrows $A_i \xrightarrow{\sigma_i} \coprod_{i\in I} A_i$ $(i \in I)$ such that, for any arrows $f_i: A_i \to B$ $(i \in I)$ there is a *unique* arrow $h : \coprod_{i\in I} A_i \longrightarrow B$

such that, for each $i \in I$, the diagram

$$\begin{tikzcd}
A_i \arrow[r, "\sigma_i"] \arrow[dr, "f_i"'] & \coprod_{i\in I} A_i \arrow[d, "h"] \\
 & B
\end{tikzcd}$$

commutes.

A category is *cartesian closed* if it has a terminal object, as well as products and exponentials of arbitrary pairs of its objects. It is *finitely complete* if it has a terminal object, products of arbitrary pairs of its objects, and equalizers. A *topos* is a category possessing a terminal object, products, a truth-value object, and power objects. It can be shown that every topos is cartesian closed and finitely complete (so that this notion of topos is equivalent to that originally given by Lawvere and Tierney).

More on products in a category. A *product* of objects $A_1, \ldots, A_n$ in a category $\mathscr{C}$ is an object $A_1 \times \ldots \times A_n$ together with arrows π_i: $A_1 \times \ldots \times A_n \to A_i$ for $i = 1, \ldots, n$, such that, for any arrows f_i: $B \to A_i$, $i = 1, \ldots, n$, there is a unique arrow, denoted by $\langle f_1, \ldots, f_n \rangle$: $B \to A_1 \times \ldots \times A_n$ such that $\pi_i \circ \langle f_1, \ldots, f_n \rangle = f_i$, $i = 1, \ldots, n$. Note that, when $n = 0$, $A_1 \times \ldots \times A_n$ is the terminal object 1. The category is said to *have finite products* if $A_1 \times \ldots \times A_n$ exists for all $A_1, \ldots, A_n$. If $\mathscr{C}$ has binary products, it has finite products, since we may take $A_1 \times \ldots \times A_n$ to be $A_1 \times (A_2 \times (\ldots \times A_n)\ldots)$. It is easily seen that the product operation is, up to isomorphism, commutative and associative. The relevant isomorphisms are called *canonical* isomorphisms.

A *functor* F: $\mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ is a map that "preserves commutative diagrams", that is, assigns to each $\mathscr{C}$-object A a $\mathscr{D}$-object FA and to each $\mathscr{C}$-arrow f: $A \to B$ a $\mathscr{D}$-arrow Ff: $FA \to FB$ in such a way that:

$$\begin{tikzcd}
A \arrow[d, "f"'] & \longmapsto & FA \arrow[d, "Ff"] \\
B & & FB
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[loop right, "1_A"] & \longmapsto & FA \arrow[loop right, "1_{FA}"]
\end{tikzcd} \qquad F(1_A) = 1_{FA}$$

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "h"'] & \bullet \arrow[d, "g"] & \longmapsto & \bullet \arrow[r, "Ff"] \arrow[dr, "Fh"'] & \bullet \arrow[d, "Fg"] \\
 & \bullet & & & \bullet
\end{tikzcd} \qquad F(g \circ f) = Fg \circ Ff$$

commutes $\qquad\qquad$ commutes

A functor F: $\mathscr{C} \to \mathscr{D}$ is an *equivalence* if it is "an isomorphism up to isomorphism", that is, if it is

- *faithful*: $Ff = Fg \Rightarrow f = g$.
- *full*: for any h: $FA \to FB$ there is f: $A \to B$ such that $h = Ff$.
- *dense*: for any $\mathscr{D}$-object B there is a $\mathscr{C}$-object A such that $B \cong FA$.

Two categories are *equivalent,* written ⊔, if there is an equivalence between them. Equivalence is the appropriate notion of "identity of form" for categories.

Given functors F, G: $\mathscr{C} \to \mathscr{D}$, a *natural transformation* between F and G is a map η from the objects of $\mathscr{C}$ to the arrows of $\mathscr{D}$ satisfying the following conditions.

- For each object A of $\mathscr{C}$, ηA is an arrow $FA \to GA$ in $\mathscr{D}$
- For each arrow $f: A \to A'$ in $\mathscr{C}$,

$$\begin{tikzcd} FA \arrow[r, "\eta A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "\eta A'"] & GA' \end{tikzcd}$$ commutes.

Finally, two functors F: $\mathscr{C} \to \mathscr{D}$ and G: $\mathscr{D} \to \mathscr{C}$ and are said to be *adjoint* to one another if, for any objects A of $\mathscr{C}$, B of $\mathscr{D}$, there is a "natural" bijection between arrows $A \to GB$ in $\mathscr{C}$ and arrows $FA \to B$ in $\mathscr{D}$. To be precise, for each such pair A, B we must be given a bijection φ_{AB}: $\mathscr{C}(A, GB) \to \mathscr{D}(FA, B)$ satisfying the "naturality" conditions

- for each $f: A \to A'$ and h: $A' \to GB$, $\varphi_{AB}(h \circ f) = \varphi_{A'B}(h) \circ Ff$
- for each $g: B \to B'$ and h: $A \to GB'$, $\varphi_{AB}(Gg \circ h) = g \circ \varphi_{AB'}(h)$.

Under these conditions F is said to be *left adjoint* to G, and G *right adjoint* to F.

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