CHAPTER 12

THE PHILOSOPHY OF MATHEMATICS.

THE CLOSE CONNECTION BETWEEN mathematics and philosophy has long been recognized by practitioners of both disciplines. The apparent timelessness of mathematical truth, the exactness and objective nature of its concepts, its applicability to the phenomena of the empirical world—explicating such facts presents philosophy with some of its subtlest problems. In this final chapter we discuss some of the attempts made by philosophers and mathematicians to explain the nature of mathematics. We begin with a brief presentation of the views of four major classical philosophers: *Plato, Aristotle, Leibniz,* and *Kant.* We conclude with a more detailed discussion of the three "schools" of mathematical philosophy which have emerged in the twentieth century: *Logicism, Formalism,* and *Intuitionism*.

Classical Views on the Nature of Mathematics.

Plato (*c*.428–347 B.C.) included mathematical entities—numbers and the objects of pure geometry such as points, lines, and circles—among the well-defined, independently existing eternal objects he called *Forms*. It is the fact that mathematical statements refer to these definite Forms that enables such statements to be true (or false). Mathematical statements about the empirical world are true to the extent that sensible objects resemble or manifest the corresponding Forms. Plato considered mathematics not as an idealization of aspects of the empirical world, but rather as *a direct description of reality*, that is, the world of Forms as apprehended by reason.

Plato's pupil and philosophical successor *Aristotle* (384–322 B.C.), on the other hand, rejected the notion of Forms being separate from empirical objects, and maintained instead that *the Forms constitute parts of objects.* Forms are grasped by the mind through a process of *abstraction* from sensible objects, but they do not thereby attain an autonomous existence detached from these latter. Mathematics arises from this process of abstraction; its subject matter is the body of idealizations engendered by this process; and mathematical rigour arises directly from the simplicity of the properties of these idealizations. Aristotle rejected the concept of *actual* (or completed) *infinity*, admitting only *potential infinity*, to wit, that of a totality which, while finite at any given time, grows beyond any preassigned bound, e.g. the sequence of natural numbers or the process of continually dividing a line.

Leibniz divided all true propositions, including those of mathematics, into two types: *truths of fact*, and *truths of reason*, also known as *contingent* and *analytic* truths, respectively. According to Leibniz, true mathematical propositions are truths of reason and their truth is therefore just logical truth: their denial would be logically impossible. Mathematical propositions do not have a special "mathematical" content—as they did for Plato and Aristotle—and so true mathematical propositions are true in all possible worlds, that is, they are *necessarily* true.[1] Leibniz attached particular importance to the *symbolic* aspects of mathematical reasoning. His program of developing a *characteristica universalis* centered around the idea of devising a method of representing thoughts by means of arrangements of characters and signs in such a way that relations among thoughts are reflected by similar relations among their representing signs.

Immanuel Kant (1724–1804) introduced a new classification of (true) propositions: *analytic*, and nonanalytic, or *synthetic*, which he further subdivided into empirical, or *a posteriori*, and nonempirical, or *a priori*. Synthetic a priori propositions are not dependent on sense perception, but are necessarily true in the sense that, if *any* propositions about the empirical world are true, *they* must be true. According to Kant, mathematical propositions are synthetic a priori because they ultimately involve reference to *space and time*. Kant attached particular importance to the idea of *a priori construction* of mathematical objects. He distinguishes sharply between mathematical *concepts* which, like noneuclidean geometries, are merely internally consistent, and mathematical *objects* whose construction is made possible by the fact that perceptual space and time have a certain inherent structure. Thus, on this reckoning, $2 + 3 = 5$ is to be regarded ultimately as an assertion about a certain construction, carried out in time and space, involving the succession and collection of units. The *logical possibility* of an arithmetic in which $2 + 3 \neq 5$ is not denied; it is only asserted that the correctness of such an arithmetic would be incompatible with the structure of perceptual space and time. So for Kant the propositions of pure arithmetic and geometry are necessary, but *synthetic a priori*. *Synthetic,* because they are ultimately about the structure of space and time, revealed through the objects that can be constructed there. And *a priori* because the structure of space and time provides the universal preconditions rendering possible the perception of such objects. On this reckoning, *pure mathematics* is the analysis of the structure of pure space and time, free from empirical material, and *applied mathematics* is the analysis of the structure of space and time, augmented by empirical material. Like Aristotle, Kant distinguishes between *potential* and *actual infinity*. However, Kant does not regard actual infinity as being a logical impossibility, but rather, like noneuclidean geometry, as an *idea of reason*, internally consistent but neither perceptible nor constructible.

Logicism

The Greeks had developed mathematics as a rigorous demonstrative science, in which geometry occupied central stage. But they lacked an abstract conception of *number*:

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[1] On the other hand, empirical propositions containing mathematical terms such as 2 *cats* + 3 *cats* = 5 *cats* are true because they hold in the *actual* world, and, according to Leibniz, this is the case only because the actual world is the "best possible" one. Thus, despite the fact that $2 + 3 = 5$ is true in all possible worlds, 2 *cats* + 3 *cats* = 5 *cats* could be false in some world.

this in fact only began to emerge in the Middle Ages under the stimulus of Indian and Arabic mathematicians, who brought about the liberation of the number concept from the dominion of geometry. The seventeenth century witnessed two decisive innovations which mark the birth of modern mathematics. The first of these was introduced by Descartes and Fermat, who, through their invention of coordinate geometry, succeeded in correlating the then essentially separate domains of algebra and geometry, so paving the way for the emergence of modern mathematical analysis. The second great innovation was, of course, the development of the infinitesimal calculus by Leibniz and Newton.

However, a price had to be paid for these achievements. In fact, they led to a considerable diminution of the deductive rigour on which the certainty of Greek mathematics had rested. This was especially true in the calculus, where the rapid development of spectacularly successful new techniques for solving previously intractable problems excited the imagination of mathematicians to such an extent that they frequently threw logical caution to the winds and allowed themselves to be carried away by the spirit of adventure. A key element in these techniques was the concept of *infinitesimal quantity* which, although of immense fertility, was logically somewhat dubious. By the end of the eighteenth century a somewhat more circumspect attitude to the cavalier use of these techniques had begun to make its appearance, and in the nineteenth century serious steps began to be taken to restore the tarnished rigour of mathematical demonstration. The situation (in 1884) was summed up by Frege in a passage from his *Foundations of Arithmetic:*

> After deserting for a time the old Euclidean standards of rigour, mathematics is now returning to them, and even making efforts to go beyond them. In arithmetic, it has been the tradition to reason less strictly than in geometry. The discovery of higher analysis only served to confirm this tendency; for considerable, almost insuperable, difficulties stood in the way of any rigorous treatment of these subjects, while at the same time small reward seemed likely for the efforts expended in overcoming them. Later developments, however, have shown more and more clearly that in mathematics a mere moral conviction, supported by a mass of succesful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to admit to a closer scrutiny of their credentials. In all directions these same ideals can be seen at work—rigour of proof, precise delimitation of extent of validity, and as a means to this, sharp definition of concepts.

Both Frege and Dedekind were concerned to supply mathematics with rigorous definitions. They believed that the central concepts of mathematics were ultimately *logical* in nature, and, like Leibniz, that truths about these concepts should be established by purely logical means. For instance, Dedekind asserts (in the Preface to his *The Nature and Meaning of Numbers*, 1888) that

> I consider the number concept [to be] entirely independent of the notions or intuitions of space and time ... an immediate result from the laws of thought.

Thus, if we make the traditional identification of logic with the laws of thought, Dedekind is what we would now call a *logicist* in his attitude toward the nature of mathematics. Dedekind's "logicism" embraced *all* mathematical concepts: the concepts of number—natural, rational, real, complex—and geometric concepts such as continuity[2]. As a practicing mathematician Dedekind brought a certain latitude to the conception of what was to count as a "logical" notion—a law of thought—as is witnessed by his remark that

> ... we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible.[3]

Thus Dedekind was not particularly concerned with providing precise formulation of the logical principles supporting his reasoning, believing that reference to self-evident "laws of thinking" would suffice. Dedekind's logicism was accordingly of a less thoroughgoing and painstaking nature than that of his contemporary Frege, whose name, together with Bertrand Russell's, is virtually synonymous with logicism. In his logical analysis of the concept of number, Frege undertook to fashion in exacting detail the symbolic language within which his analysis was to be presented.

Frege's analysis is presented in three works:

Begriffsschrift (1879): Concept-Script, a symbolic language of pure thought modelled on the language of arithmetic.

Grundlagen (1884): The Foundation of Arithmetic, a logico-mathematical investigation into the concept of number.

Grundgesetze (1893, 1903): Fundamental Laws of Arithmetic, derived by means of concept-script.

In the *Grundgesetze* Frege refines and enlarges the symbolic language first introduced in the *Begriffsschrift* so as to undertake, in full formal detail, the analysis of the concept of number, and the derivation of the fundamental laws of arithmetic. The logical universe of *Grundgesetze* comprises two sorts of entity: *functions*, and *objects*. Any function f associates with each value ξ of its argument an object $f(\xi)$: if this object is always one of the two *truth values* **0** (false) or **1** (true), then f is called a *concept* or *propositional function,* and when $f(\xi) = \mathbf{1}$ we say that ξ *falls under* the concept f. If two functions f and g assign the same objects to all possible values of their arguments, we should naturally say that they have the same *course of values*; if f and g are concepts, we would say that they both have the same *extension*. Frege's decisive step in the *Grundgesetze* was to introduce a new kind of object expression—which we shall write as $\widehat{f}$ —to symbolize the course of values of f and to lay down as a basic principle the assertion

$$\widehat{f} = \widehat{g} \leftrightarrow \forall\xi\ [f(\xi) = g(\xi)].^4 \qquad (1)$$

[2]In fact, it was the imprecision surrounding the concept of continuity that impelled him to embark on the program of critical analysis of mathematical concepts.

[3]This idea of *correspondence* or *functionality,* taken by Dedekind as fundamental, is in fact the central concept of *category theory* (see Chapter 6).

[4]Here and in the sequel we employ the logical operators introduced at the beginning of Appendix 2. Thus "∀" stands for "for every", "∃" stands for "there exists", "¬" for "not"," ∧" for 'and", "∨" for "or", "→" for

Confining attention to concepts, this may be taken as asserting that *two concepts have the same extension exactly when the same entities fall under them.*

The notion of the extension of a concept underpins Frege's definition of number, which in the *Grundlagen* he had argued persuasively should be taken as a measure of the size of a concept's extension[5]. As already mentioned in Chapter 3, he introduced the term *equinumerous* for the relation between two concepts that obtains when the fields of entities falling under each can be put in biunique correspondence. He then defined cardinal number by stipulating that the cardinal number of a concept F is the extension of the concept *equinumerous with the concept F.* In this way a number is associated with a *second-order* concept—a concept about concepts. Thus, if we write $\nu(F)$ for the cardinal number of F so defined, and $F \approx G$ for *the concept F is equinumerous with the concept G*, then it follows from (1) that

$$\nu(F) = \nu(G) \leftrightarrow F \approx G. \qquad (2)$$

And then the natural numbers can be defined as the cardinal numbers of the following concepts:

$$N_0\colon x \neq x \qquad 0 = \nu(N_0)$$

$$N_1\colon x = 0 \qquad 1 = \nu(N_1)$$

$$N_2\colon x = 0 \vee x = 1 \qquad 2 = \nu(N_2)$$

etc.

In a technical *tour-de-force* Frege established that the natural numbers so defined satisfy the usual principles expected of them.

Unfortunately, in 1902 Frege learned of *Russell's paradox*, which can be derived from his principle (1) and shows it to be *inconsistent.* Russell's paradox, as formulated for sets or classes in the previous chapter, can be seen to be attendant upon the usual supposition that *any property determines a unique class*, to wit, the class of all objects possessing that property (its "extension"). To derive the paradox in Frege's system, classes are replaced by Frege's extensions: we define the concept R by

$$R(x) \leftrightarrow \exists F[x = \widehat{F} \wedge \neg F(x)]$$

(in words: *x falls under the concept R exactly when x is the extension of some concept under which it does not fall*). Now write r for the extension of R, i.e.,

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"implies" and "$\leftrightarrow$" for "is equivalent to".

[5]It is helpful to think of the extension of a concept as the class of all entities that fall under it, so that, for example, the extension of the concept *red* is the class of all red objects. However, it is by no means necessary to identify extensions with classes; all that needs to be known about extensions is that they are objects satisfying (1).

$$r = \widehat{R}\,.$$

Then

$$R(r) \leftrightarrow \exists F[r = \widehat{F} \wedge \neg F(r)]. \qquad (3)$$

Now suppose that $R(r)$ holds. Then, for some concept F,

$$r = \widehat{F} \wedge \neg F(r).$$

But then

$$\widehat{F} = r = \widehat{R}\,,$$

and so we deduce from (1) that

$$\forall x[F(x) \leftrightarrow R(x)].$$

Since $\neg F(r)$, it follows that $\neg R(r)$. We conclude that

$$R(r) \rightarrow \neg R(r).$$

Conversely, assume $\neg R(r)$. Then

$$r = \widehat{R} \wedge \neg R(r),$$

and so *a fortiori*

$$\exists F[r = \widehat{F} \wedge \neg F(r)].$$

It now follows from the definition of R that $\neg R(r)$. Thus we have shown that

$$\neg R(r) \rightarrow R(r).$$

We conclude that Frege's principle (1) yields the contradiction

$$R(r) \leftrightarrow \neg R(r).$$

Thus Frege's system in the *Grundgesetze* is, as it stands, inconsistent. Later investigations, however, have established that the definition of the natural numbers and the derivation of the basic laws of arithmetic can be salvaged by suitably restricting (1) so that it becomes consistent, leaving the remainder of the system intact. In fact it is only necessary to make the (consistent) assumption that the extensions of a certain special type of concept—the *numerical concepts*[6]—satisfy (1). Alternatively, one can abandon extensions altogether and instead take the cardinal number $\nu(F)$ as a primitive notion, governed by equivalence (2). In either case the whole of Frege's derivation of the basic laws of arithmetic can be recovered.

Where does all this leave Frege's (and Dedekind's) claim that arithmetic can be derived from logic? Both established beyond dispute that arithmetic can be formally or logically derived from principles which involve no explicit reference to spatiotemporal intuitions. In Frege's case the key principle involved is that certain concepts have extensions satisfying (1). But although this principle involves no reference to spatiotemporal intuition, it can hardly be claimed to be of a purely logical nature. For it is an *existential* assertion and one can presumably conceive of a world devoid of the objects ("extensions") whose existence is asserted. It thus seems fair to say that, while Frege (and Dedekind) did succeed in showing that the concept and properties of number are "logical" in the sense of being independent of spatiotemporal intuition, they did not (and it would appear could not) succeed in showing that these are "logical" in the stronger Leibnizian sense of holding in every possible world.

The logicism of *Bertrand Russell* was in certain respects even more radical than that of Frege, and closer to the views of Leibniz. In *The Principles of Mathematics* (1903) Russell asserts that mathematics and logic are *identical.* To be precise, he proclaims at the beginning of this remarkable work that

> Pure mathematics is the class of all propositions of the form "p implies q" where p and q are propositions ... and neither p nor q contains any constants except logical constants.[7]

The monumental, and formidably recondite[8] *Principia Mathematica,* written during 1910 –1913 in collaboration with *Alfred North Whitehead* (1861–1947), contains a complete system of pure mathematics, based on what were intended to be purely logical principles, and formulated within a precise symbolic language. A central concern of *Principia Mathematica* was to avoid the so-called *vicious circle paradoxes,* such as those of Russell and Grelling-Nelson—mentioned in the previous chapter—which had come to trouble mathematicians concerned with the ultimate soundness of their discipline. Another is *Berry's paradox*, in one form of which we consider the phrase *the least integer not definable in less than eleven words*. This phrase defines, in less than eleven words (ten, actually), an integer which satisfies the condition stated,

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[6] A numerical concept is one expressing equinumerosity with some given concept.

[7] Thus at the time this was asserted Russell was what could be described as an "implicational logicist".

[8] One may get an idea of just how difficult this work is by quoting the following extract from a review of it in a 1911 number of the London magazine *The Spectator:*

> It is easy to picture the dismay of the innocent person who out of curiosity looks into the later part of the book. He would come upon whole pages without a single word of English below the headline; he would see, instead, scattered in wild profusion, disconnected Greek and Roman letters of every size interspersed with brackets and dots and inverted commas, with arrows and exclamation marks standing on their heads, and with even more fantastic signs for which he would with difficulty so much as find names.

that is, of not being definable in less than eleven words. This is plainly self-contradictory.

If we examine these paradoxes closely, we find that in each case a term is defined by means of an implicit reference to a certain class or domain which contains the term in question, thereby generating a vicious circle. Thus, in Russell's paradox, the defined entity, that is, the class R of all classes not members of themselves is obtained by singling out, from the class V of all classes *simpliciter*, those that are not members of themselves. That is, R is defined in terms of V, but since R is a member of V, V cannot be obtained without being given R in advance. Similarly, in the Grelling-Nelson paradox, the definition of the adjective *heterological* involves considering the concept *adjective* under which *heterological* itself falls. And in the Berry paradox, the term *the least integer not definable in less than eleven words* involves reference to the class of all English phrases, including the phrase defining the term in question.

Russell's solution to these problems was to adopt what he called the *vicious circle principle* which he formulated succinctly as: *whatever involves* all *of a collection must not be one of a collection.* This injunction has the effect of excluding, not just self-contradictory entities of the above sort, but *all* entities whose definition is in some way circular, even those, such as the class of all classes which are members of themselves, the adjective "autological", or the least integer definable in less than eleven words, the supposition of whose existence does not appear to lead to contradiction.[9]

The vicious circle principle suggests the idea of arranging classes or concepts (propositional functions) into distinct *types* or *levels,* so that, for instance, any class may only contain classes (or individuals) of lower level as members[10], and a propositional function can have only (objects or) functions of lower level as possible arguments. Under the constraints imposed by this theory, one can no longer form the class of all possible classes as such, but only the class of all classes of a given level. The resulting class must then be of a higher type than each of its members, and so cannot be a member of itself. Thus Russell's paradox cannot arise. The Grelling-Nelson paradox is blocked because the property of heterologicality, which involves self-application, is inadmissible.

Unfortunately, however, this simple theory of types does not circumvent paradoxes such as Berry's, because in these cases the defined entity is clearly of the same level as the entities involved in its definition. To avoid paradoxes of this kind Russell was therefore compelled to introduce a further "horizontal" subdivision of the totality of entities at each level, into what he called *orders*, and in which the *mode of definition* of these entities is taken into account. The whole apparatus of types and orders is called the *ramified theory of types* and forms the backbone of the formal system of *Principia Mathematica.*

To convey a rough idea of how Russell conceived of orders, let us confine attention

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[9] The self-contradictory nature of the "paradoxical" entities we have described derives as much from the occurrence of *negation* in their definitions as it does from the circularity of those definitions.

[10] The idea of stratifying classes into types had also occurred to Russell in connection with his analysis of classes as genuine *pluralities*, as opposed to *unities.* On this reckoning, one starts with individual objects (lowest type), pluralities of these comprise the entities of next highest type, pluralities of these pluralities the entities of next highest type, etc. Thus the evident distinction between individuals and pluralities is "projected upwards" to produce a hierarchy of types.

to propositional functions taking only individuals (type 0) as arguments. Any such function which can be defined without application of quantifiers[11] to any variables other than individual variables is said to be of *first order*. For example, the propositional function *everybody loves x* is of first order. Then *second order* functions are those whose definition involves application of quantifiers to nothing more than individual and first order variables, and similarly for third, fourth,..., n^{th} order functions. Thus *x has all the first-order qualities that make a great philosopher* represents a function of second order and first type.

Distinguishing the order of functions enables paradoxes such as Berry's to be dealt with. There the word *definable* is incorrectly taken to cover not only definitions in the usual sense, that is, those in which no functions occur, but also definitions involving functions of all orders. We must instead insist on specifying the orders of all functions figuring in these definitions. Thus, in place of the now illegitimate *the least integer not definable in less than eleven words* we consider *the least integer not definable in terms of functions of order* n *in less than eighteen words*. This integer is then indeed *not* definable in terms of functions of order *n* in less than eighteen words, but *is* definable in terms of functions of order $n+1$ in less than eighteen words. There is no conflict here.

While the ramified theory of types circumvents all known paradoxes (and can in fact be proved consistent from some modest assumptions), it turns out to be too weak a system to support unaided the development of mathematics. To begin with, one cannot prove within it that there is an infinity of natural numbers, or indeed that each natural number has a distinct successor. To overcome this deficiency Russell was compelled to introduce an *axiom of infinity*, to wit, that there exists a level containing infinitely many entities. As Russell admitted, however, this can hardly be considered a principle of logic, since it is certainly possible to conceive of circumstances in which it might be false. In any case, even augmented by the axiom of infinity, the ramified theory of types proves inadequate for the development of the basic theory of the real numbers. For instance, the theorem that every bounded set of real numbers has a least upper bound, upon which the whole of mathematical analysis rests, is not derivable without further *ad hoc* strengthening of the theory, this time by the assumption of the so-called *axiom of reducibility*. This asserts that any propositional function of any order is equivalent to one of first order in the sense that the same entities fall under them. Again, this principle can hardly be claimed to be a fact of logic.

Various attempts have been made to dispense with the axiom of reducibility, notably that of *Frank Ramsey* (1903–1930). His idea was to render the whole apparatus of orders superfluous by eliminating quantifiers in definitions. Thus he proposed that a universal quantifier be regarded as indicating a conjunction, and an existential quantifier a disjunction[12], even though it may be impossible in practice to write out the resulting expressions in full. On this reckoning, then, the statement ***Citizen Kane*** *has all the qualities that make a great film* would be taken as an abbreviation for something

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[11]Here by a *quantifier* we mean a an expression of the form "for every" (∀) or "there exists" (∃).

[12]Thus, if a domain of discourse *D* comprises entities *a, b, c,...*, then *for every x in D, P*(*x*) is construed to mean *P*(*a*) *and P*(*b*) and *P*(*c*) *and...*, and *there exists x in D such that P*(*x*) to mean *P*(*a*) *or P*(*b*) *or P*(*c*) *or ...*

like ***Citizen Kane*** *is a film, brilliantly directed, superbly photographed, outstandingly performed, excellently scripted, etc.* For Ramsey, the distinction of orders of functions is just a complication imposed by the structure of our language and not, unlike the hierarchy of types, something inherent in the way things are. For these reasons he believed that the simple theory of types would provide an adequate foundation for mathematics.

What is the upshot of all this for Russell's logicism? There is no doubt that Russell and Whitehead succeeded in showing that mathematics can be derived within the ramified theory of types from the axioms of infinity and reducibility. This is indeed no mean achievement, but, as Russell admitted, the axioms of infinity and reducibility seem to be at best contingent truths. In any case it seems strange to have to base the truth of mathematical assertions on the proviso that there are infinitely many individuals in the world. Thus, like Frege's, Russell's attempted reduction of mathematics to logic contains an irreducible mathematical residue.

Formalism

In 1899 Hilbert published his epoch-making work *Grundlagen der Geometrie* ("Foundations of Geometry"). Without introducing any special symbolism, in this work Hilbert formulates an absolutely rigorous axiomatic treatment of Euclidean geometry, revealing the hidden assumptions, and bridging the logical gaps, in traditional accounts of the subject. He also establishes the *consistency* of his axiomatic system by showing that they can be interpreted (or as we say, possess a *model*) in the system of real numbers. Another important property of the axioms he demonstrated is their *categoricity*, that is, the fact that, up to isomorphism they have exactly *one* model, namely, the usual 3-dimensional space of real number triples. Although in this work Hilbert was attempting to show that geometry is entirely self-sufficient *as a deductive system*[13], he nevertheless thought, as did Kant, that geometry is ultimately *the logical analysis of our intuition of space.* This can be seen from the fact that as an epigraph for his book he quotes Kant's famous remark from the *Critique of Pure Reason:*

> Human knowledge begins with intuitions, goes from there to concepts, and ends with ideas.

The great success of the method Hilbert had developed to analyze the deductive system of Euclidean geometry—we might call it the *rigorized axiomatic method*, or the *metamathematical method*—emboldened him to attempt later to apply it to pure mathematics as a whole, thereby securing what he hoped to be perfect rigour for all of mathematics. To this end Hilbert elaborated a subtle philosophy of mathematics, later to become known as *formalism*, which differs in certain important respects from the logicism of Frege and Russell and betrays certain Kantian features. Its flavour is well captured by the following quotation from an address he made in 1927:

> No more than any other science can mathematics be founded on logic alone; rather, as a condition

[13]In this connection one recalls his famous remark: *one must be able to say at all times, instead of points, lines and planes—tables, chairs, and beer mugs.*

> for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else, nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed, no scientific thinker can dispense with it, and therefore everyone must maintain it, consciously or not.

Thus, at bottom, Hilbert, like Kant, wanted to ground mathematics on the description of concrete spatiotemporal configurations, only Hilbert restricts these configurations to *concrete signs* (such as inscriptions on paper). No inconsistencies can arise within the realm of concrete signs, since precise descriptions of concrete objects are always mutually compatible. In particular, within the mathematics of concrete signs, actual infinity cannot generate inconsistencies since, as for Kant, this concept cannot describe any concrete object. On this reckoning, the soundness of mathematics thus issues ultimately, not from a *logical* source, but from a *concrete* one, in much the same way as the consistency of truly reported empirical statements is guaranteed by the concreteness of the external world.

Yet Hilbert also thought that adopting this position would not require the abandonment of the infinitistic mathematics of Cantor and others which had emerged in the nineteenth century and which had enabled mathematics to make such spectacular strides. He accordingly set himself the task of accommodating infinitistic mathematics within a mathematics restricted to the deployment of finite concrete objects. Thus *Hilbert's program*, as it came to be called, had as its aim the provision of a new foundation for mathematics not by reducing it to logic, but instead by *representing its essential form within the realm of concrete symbols.* As the quotation above indicates, Hilbert considered that, in the last analysis, the completely reliable, irreducibly self-evident constituents of mathematics are *finitistic,* that is, concerned just with finite manipulation of surveyable domains of concrete objects, in particular, mathematical symbols presented as marks on paper. Mathematical propositions referring only to concrete objects in this sense he called *real*, or *concrete*, propositions, and all other mathematical propositions he considered as possessing an *ideal*, or *abstract* character. Thus, for example, $2 + 2 = 4$ would count as a real proposition, while *there exists an odd perfect number* would count as an ideal one.

Hilbert viewed ideal propositions as akin to the ideal lines and points "at infinity" of projective geometry. Just as the use of these does not violate any truths of the "concrete" geometry of the usual Cartesian plane, so he hoped to show that the use of ideal propositions—in particular, those of Cantor's set theory—would never lead to falsehoods among the real propositions, that, in other words, such use *would never contradict any self-evident fact about concrete objects.* Establishing this by strictly concrete, and so unimpeachable means was the central aim of Hilbert's program. In short, its objective was to prove classical mathematical reasoning *consistent.* With the attainment of this goal, mathematicians would be free to roam unconstrained within

"Cantor's Paradise" (in Hilbert's memorable phrase[14]). This was to be achieved by setting it out as a purely formal system of symbols, devoid of meaning[15], and then showing that no proof in the system can lead to a false assertion, e.g. 0 = 1. This, in turn, was to be done by employing the *metamathematical* technique of replacing each abstract classical proof of a real proposition by a concrete, finitistic proof. Since, plainly, there can be no concrete proof of the real proposition 0 = 1, there can be no classical proof of this proposition either, and so classical mathematical reasoning is consistent.

In 1931, however, Gödel rocked Hilbert's program by demonstrating, through his celebrated *Incompleteness Theorems*, that *there would always be real propositions provable by ideal means which cannot be proved by concrete means.* He achieved this by means of an ingenious modification of the ancient *Liar paradox*. To obtain the liar paradox in its most transparent form, one considers the sentence *this sentence is false*. Calling this sentence *A,* it is clear that *A* is true if and only if it is false, that is, *A asserts its own falsehood.* Now Gödel showed that, if in *A* one replaces the word *false* by the phrase *not concretely provable*, then the resulting statement *B* is *true* —i.e., provable by ideal means—but *not concretely provable*. This is so because, as is easily seen, *B* actually asserts its own concrete unprovability in just the same way as *A* asserts its own falsehood. And by extending these arguments Gödel also succeeded in showing that the *consistency of arithmetic* cannot be proved by concrete means.[16]

There seems to be no doubt that Hilbert's program for establishing the consistency of mathematics (and in particular, of arithmetic) *in its original, strict form* was shown by Gödel to be unrealizable. However, Gödel himself thought that the program for establishing the consistency of arithmetic might be salvageable through an enlargement of the domain of objects admitted into finitistic metamathematics. That is, by allowing finite manipulations of suitably chosen *abstract* objects in addition to the concrete ones Gödel hoped to strengthen finitistic metamathematics sufficiently to enable the consistency of arithmetic to be demonstrable within it. In 1958 he achieved his goal, constructing a consistency proof for arithmetic within a finitistic, but not strictly concrete, metamathematical system admitting, in addition to concrete objects (numbers), abstract objects such as functions, functions of functions, etc., over finite objects. So, although Hilbert's program cannot be carried out in its original form, for arithmetic at least Gödel showed that it can be carried out in a weakened form by countenancing the use of suitably chosen abstract objects.

As for the doctrine of "formalism" itself, this was for Hilbert (who did not use the term, incidentally) not the claim that mathematics could be *identified* with formal axiomatic systems. On the contrary, he seems to have regarded the role of formal systems as being to provide distillations of mathematical practice of a sufficient degree of precision to enable their formal features to be brought into sharp focus. The fact that Gödel succeeded in showing that certain features (e.g. consistency) of these logical

[14]Hilbert actually asserted that "no one will ever be able to expel us from the paradise that Cantor has created for us." There is no question that "Cantor's Paradise" furnishes the ideal site on which to build Hilbert's hotel (see the previous chapter.)

[15]It should be emphasized that Hilbert was *not* claiming that (classical) mathematics *itself* was meaningless, only that the formal system representing it was to be so regarded.

[16]A sketch of Gödel's arguments is given in Appendix 2.

distillations could be *expressed,* but *not demonstrated* by finitistic means does not undermine the essential cogency of Hilbert's program.

Intuitionism.

A third tendency in the philosophy of mathematics to emerge in the twentieth century, *intuitionism*, is largely the creation of Brouwer, already mentioned in connection with his contibutions to topology. Like Kant, Brouwer believed that mathematical concepts are admissible only if they are adequately grounded in *intuition*, that mathematical theories are significant only if they concern entities which are constructed out of something given immediately in intuition, that mathematical definitions must always be constructive, and that the completed infinite is to be rejected. Thus, like Kant, Brouwer held that mathematical theorems are synthetic *a priori* truths. In *Intuitionism and Formalism* (1912), while admitting that the emergence of noneuclidean geometry had discredited Kant's view of space, he maintained, in opposition to the logicists (whom he called "formalists") that arithmetic, and so all mathematics, must derive from the *intuition of time*. In his own words:

> Neointuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal ω . Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between", which is not exhaustible by the interposition of new units and which can therefore never be thought of as a mere collection of units. In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry, and not only for elementary two- and three-dimensional geometry, but for non-euclidean and *n*-dimensional geometries as well. For since Descartes we have learned to reduce all these geometries to arithmetic by means of coordinates.

For Brouwer, intuition meant essentially what it did to Kant, namely, the mind's apprehension of what it has itself constructed; on this view, the only acceptable mathematical proofs are *constructive*. A constructive proof may be thought of as a kind of "thought experiment" —the performance, that is, of an experiment in imagination. According to *Arend Heyting* (1898–1980), a leading member of the intuitionist school,

> Intuitionistic mathematics consists ... in mental constructions; a mathematical theorem expresses a purely empirical fact, namely, the success of a certain construction. "2 + 2 = 3 + 1" must be read as an abbreviation for the statement "I have effected the mental construction indicated by '2 + 2' and '3 + 1' and I have found that they lead to the same result."

From passages such as these one might infer that for intuitionists mathematics is a purely subjective activity, a kind of introspective reportage, and that a mathematician, as such, is necessarily a solipsist. Certainly they reject the idea that mathematical thought is dependent on any special sort of language, even, occasionally, claiming that, at bottom, mathematics is a "languageless activity". Nevertheless, the fact that intuitionists evidently regard mathematical theorems as being valid for all intelligent beings indicates that for them mathematics has, if not an objective character, then at least a *transsubjective* one.

The major impact of the intuitionists' program of constructive proof has been in the realm of *logic*. Brouwer maintained, in fact, that the applicability of traditional logic to mathematics

> was caused historically by the fact that, first, classical logic was abstracted from the mathematics of the subsets of a definite finite set, that, secondly, an a priori existence independent of mathematics was ascribed to the logic, and that, finally, on the basis of this supposed apriority it was unjustifiably applied to the mathematics of infinite sets.

Thus Brouwer held that much of modern mathematics is based, not on sound reasoning, but on an illicit extension of procedures valid only in the restricted domain of the finite. He therefore embarked on the heroic course of setting the whole of existing mathematics aside and starting afresh, using only concepts and modes of inference that could be given clear intuitive justification. He hoped that, once enough of the program had been carried out, one could discern the logical laws that intuitive, or constructive, mathematical reasoning actually obeys, and so be able to compare the resulting *intuitionistic,* or *constructive, logic*[17] with classical logic.

The most important features of constructive mathematical reasoning are that *an existential statement can be considered affirmed only when an instance is produced*,[18] and—as a consequence—*a disjunction can be considered affirmed only when an explicit one of the disjuncts is demonstrated.* A striking consequence of this is that, as far as properties of (potentially) infinite domains are concerned, *neither the classical law of excluded middle*[19] *nor the law of strong reductio ad absurdum*[20] *can be accepted without qualification.* To see this, consider for example the existential statement *there exists an odd perfect number* (i.e., an odd number equal to the sum of its proper divisors) which we shall write as $\exists nP(n)$. Its contradictory is the statement $\forall n\neg P(n)$. Classically, the law of excluded middle then allows us to affirm the disjunction

$$\exists nP(n) \vee \forall n\neg P(n) \qquad (1)$$

Constructively, however, in order to affirm this disjunction we must *either* be in a position to affirm the first disjunct $\exists nP(n)$, i.e., to possess, or have the means of obtaining, an odd perfect number, *or* to affirm the second disjunct $\forall n\neg P(n)$, i.e. to possess a demonstration that no odd number is perfect. Since at the present time mathematicians have neither of these[21], the disjunction (1), and *a fortiori* the law of excluded middle is not constructively admissible.

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[17]This is not to say that Brouwer was primarily interested in *logic,* far from it: indeed, his distaste for formalization led him not to take very seriously subsequent codifications of intuitionistic logic.

[18] Hermann Weyl said of nonconstructive existence proofs that "they inform the world that a treasure exists without disclosing its location."

[19]This is the assertion that, for any proposition *p,* either *p* or its negation $\neg p$ holds.

[20]This is the assertion that, for any proposition *p,* $\neg\neg p$ implies *p.*

[21]And indeed may never have; as observed in Chapter 3, little if any progress has been made on the ancient problem of the existence of odd perfect numbers.

It might be thought that, if in fact the second disjunct in (1) is *false*, that is, not every number falsifies P, then we can actually find a number satisfying P by the familiar procedure of testing successively each number 0, 1, 2, 3,... and breaking off when we find one that does: in other words, that from $\neg \forall n \neg P(n)$ we can infer $\exists n P(n)$. Classically, this is perfectly correct, because the *classical* meaning of $\neg\forall n\neg P(n)$ is "$P(n)$ will not as a matter of *fact* be found to fail for every number *n.*" But *constructively* this latter statement has no meaning, because it presupposes that every natural number *has already been constructed* (and checked for whether it satisfies P). Constructively, the statement must be taken to mean something like "we can derive a contradiction from the supposition that we could prove that $P(n)$ failed for every n." From this, however, we clearly cannot extract a guarantee that, by testing each number in turn, we shall eventually find one that satisfies P. So we see that the law of strong reductio ad absurdum also fails to be constructively admissible.

As a simple example of a classical existence proof which fails to meet constructive standards, consider the assertion

there exists a pair of irrational real numbers a,b such that a^b is rational.

Classically, this can be proved as follows: let $b = \sqrt{2}$; then b is irrational. If b^b is rational, let $a = b$; then we are through. If b^b is irrational, put $a = b^b$; then $a^b = 2$, which is rational. But in this proof we have not *explicitly identified a*; we do not know, in fact, whether $a = 2$ or[22] $a = \sqrt{2}^{\sqrt{2}}$, and it is therefore constructively unacceptable.

Thus we see that constructive reasoning differs from its classical counterpart in that it attaches a stronger meaning to some of the logical operators. It has become customary, following Heyting, to explain this stronger meaning in terms of the primitive relation *a is a proof of p,* between mathematical constructions a and mathematical assertions p. To assert the *truth* of p is to assert that one has a construction a such that a is a proof of p[23]. The meaning of the various logical operators in this scheme is spelt out by specifying how proofs of composite statements depend on proofs of their constituents. Thus, for example,

a is a proof of $p \wedge q$ means: a is a pair (b, c) consisting of a proof b of p and c of q;
a is a proof of $p \vee q$ means: a is a pair (b, c) consisting of a natural number b and a construction c such that, if $b = 0$, then c is a proof of p, and if $b \neq 0$, then c is a proof of q;
a is a proof of $p \rightarrow q$ means: a is a construction that converts any proof of p into a proof of q;
a is a proof of $\neg p$ means: a is a construction that shows that no proof of p is possible.

[22]In fact a much deeper argument shows that 2^2 is irrational, and is therefore the correct value of *a.*

[23]Here by *proof* we are to understand a mathematical construction that establishes the assertion in question, *not* a derivation in some formal system. For example, a proof of 2 + 3 = 5 in this sense consists of successive constructions of 2, 3 and 5, followed by a construction that adds 2 and 3, finishing up with a construction that compares the result of this addition with 5.

It is readily seen that, for example, the law of excluded middle is not generally true under this ascription of meaning to the logical operators. For a proof of $p \vee \neg p$ is a pair (b,c) in which c is either a proof of p or a construction showing that no proof of p is possible, and there is nothing inherent in the concept of mathematical construction that guarantees, for an arbitrary proposition p, that either will ever be produced.

As shown by Gödel in the 1930s, it is possible to represent the strengthened meaning of the constructive logical operators in a classical system augmented by the concept of *provability*. If we write $\Box p$ for "*p is provable*", then the scheme below correlates constructive statements with their classical translates.

Constructive	*Classical*
$\neg p$	$\Box\neg\Box p$
$p \wedge q$	$\Box p \wedge \Box q$
$p \vee q$	$\Box p \vee \Box q$
$p \rightarrow q$	$\Box(\Box p \rightarrow \Box q)$

The translate of the sentence $p \vee \neg p$ is then $\Box p \vee \Box\Box\neg\Box p$, which is (assuming $\Box\Box p \leftrightarrow \Box p$) equivalent to $\neg\Box p \rightarrow \Box\neg\Box p$, that is, to the assertion

if p is not provable, then it is provable that p is not provable.

The fact that there is no *a priori* reason to accept this "solubility" principle lends further support to the intuitionists' rejection of the law of excluded middle.

Another interpretation of constructive reasoning is provided by *Kolmogorov's calculus of problems* (*A. N. Kolmogorov,* 1903–1987). If we denote problems by letters and $a \wedge b$, $a \vee b$, $a \rightarrow b$, $\neg a$ are construed respectively as the problems

to solve both a and b
to solve at least one of a and b
to solve b, given a solution of a
to deduce a contradiction from the hypothesis that a is solved,

then a formal calculus can be set up which coincides with the constructive logic of propositions.

Although intuitionism in Brouwer's original sense has not been widely adopted as a philosophy of mathematics, the constructive viewpoint associated with it has been very influential[24]. The intuitionistic logical calculus has also come under intensive investigation. If we compare the law of excluded middle with Euclid's fifth postulate,

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[24] Remarkably, it is also the logic of *smooth infinitesimal analysis*: see Appendix 3.

then intuitionistic logic may be compared with *neutral* geometry—geometry, that is, without the fifth postulate—and classical logic to Euclidean geometry. Just as noneuclidean geometry revealed a "strange new universe", so intuitionistic logic has allowed new features of the logico-mathematical landscape—invisible through the lens of classical logic—to be discerned. Intuitionistic logic has proved to be a subtle instrument, more delicate, more discriminating than classical logic, for investigating the mathematical world.[25]

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Despite the fact that Logicism, Intuitionism and Formalism cannot be held to provide complete accounts of the nature of mathematics, each gives expression to an important partial truth about that nature: *Logicism,* that mathematical truth and logical demonstration go hand in hand; *Intuitionism*, that mathematical activity proceeds by the performance of mental constructions, and finally *Formalism,* that the results of these constructions are presented through the medium of formal symbols.

[25] Famously, Hilbert remarked, in opposition to intuitionism, that "to deny the mathematician the use of the law of excluded middle would be to deny the astronomer the use of a telescope or the boxer the use of his fists." But with experience in using the refined machinery of intuitionistic logic one comes to regard Hilbert's simile as inappropriate. A better one might be: to deny the mathematician the use of the law of excluded middle would be to deny the surgeon the use of a butcher knife, or the general the use of nuclear weapons.