# CHAPTER 2

# THE MATHEMATICS OF ANCIENT GREECE

THE CULTURE OF ANCIENT GREECE (600 B.C. – 300 A.D.) still astonishes today by its originality, depth and diversity. The works of the Greek philosophers Plato and Aristotle, of the tragedians Euripides and Sophocles, of the poets Homer and Pindar, of the historians Herodotus and Thucydides—and many others—have had an incalculable influence on the development of the arts. No less influential were the scientists and mathematicians of ancient Greece: for centuries the name "Euclid" has been inseparable from the term "geometry" and mathematics known as "the Greek science".

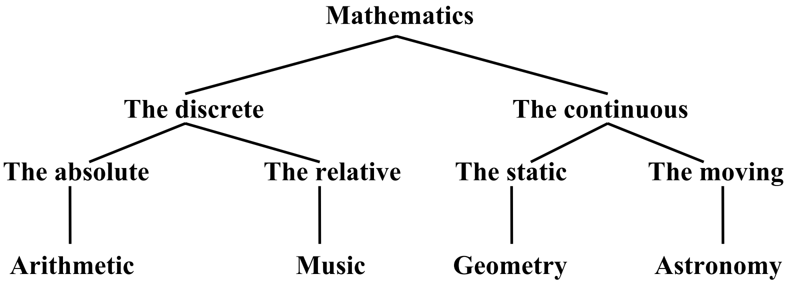
Mathematics—or to be more precise, *geometry*, a word which derives from the Greek *geometrein*, "earth measurement"—begins in Greece with *Thales of Miletus* (*c.*624 – 547 B.C.), whose travels in Egypt are believed to have furnished the original sources of his geometric knowledge. Egyptian geometry itself seems to have arisen from the practical requirement of fixing accurately the boundaries of landed property subject to periodic flooding by the river Nile. As attested by the Rhind papyrus (*c.*1700 B.C.) and other documents, Egyptian geometry consisted in the main of practical rules for measuring areas of figures such as squares, triangles, circles, etc. and volumes of measures of corn, grain, etc. of various shapes. It may be inferred from the fact that the Egyptians constructed pyramids of definite slope that they were also in possession of the idea of similarity of figures, and certainly of triangles. Moreover, the Egyptians knew that a triangle with its sides in the ratio 3:4:5 is right-angled, and made use of this fact as a means of drawing right angles. However, nothing indicates that they were aware of the general property linking the sides of a right-angled triangle, or that they proved any general geometric theorem.

It may be plausibly surmised that the diagrams illustrating the measurement of various plane figures that Thales would have seen during his Egyptian travels suggested to him the idea of isolating the general principles behind their construction. This in turn probably led to the discovery of the following theorems with which tradition associates him:

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. If two straight lines cut one another, the vertically opposite angles are equal.
4. The angle in a semicircle is a right angle.

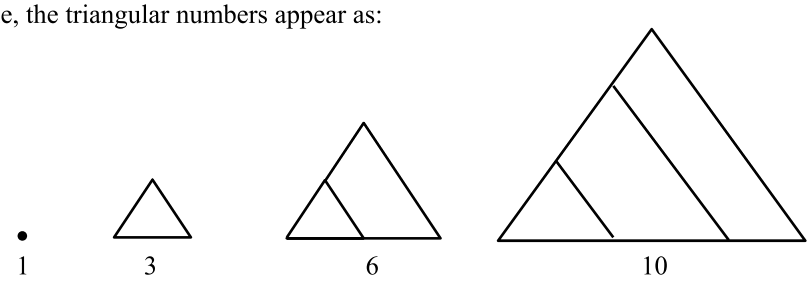
Elementary as these assertions are, their *generality* already places them far beyond their Egyptian origins, and indeed marks them as the first steps towards a systematic *theory* of geometry.

After Thales came *Pythagoras of Samos* (572–497 B.C.) and his school, who are said to have coined the term “mathematics” from a root meaning “learning” or “knowledge”. The remarkable advances in mathematics made by the Pythagoreans led them to the belief that mathematics, and more especially *number*, lies at the heart of all existence -- the first “mathematical” philosophy. For the Pythagoreans the structure of mathematics took the form of a bifurcating scheme of oppositions:

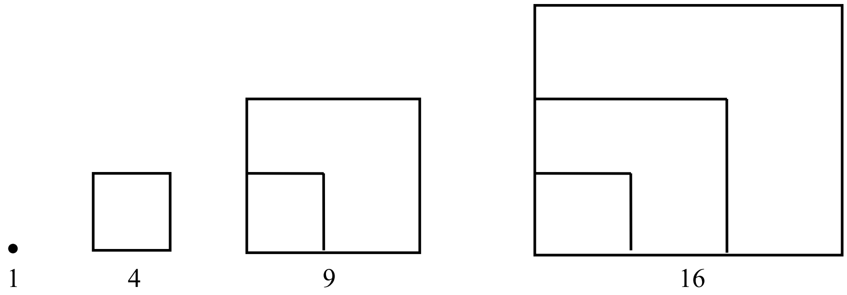


This scheme is the source of the so-called *quadrivium*, which served as the basis for Western pedagogy until the end of the Middle Ages. The further addition of the *trivium*, comprising grammar, rhetoric and logic, formed what came to be known as the *seven liberal arts.*

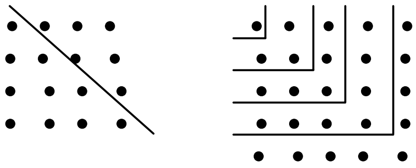
The Pythagoreans elevated the idea of number above the realm of practical utility and transformed it into a pure concept, creating what we may call the *Pythagorean theory of numbers*. This theory included definitions of the various classes of numbers: odd, even, prime, composite, odd times even, even times even, etc. Also ascribed to the Pythagoreans is the concept of *perfect* number, i.e. a number such as 6 or 28 which is the sum of its proper divisors. They were certainly the originators of the idea of *figurate* number, a link between number and geometry which must have played an important role in the creation of their numerical philosophy. Figurate numbers arise as the number of dots in certain geometric figurations such as triangles, squares, etc. For example, the triangular numbers appear as:



and the square numbers as:



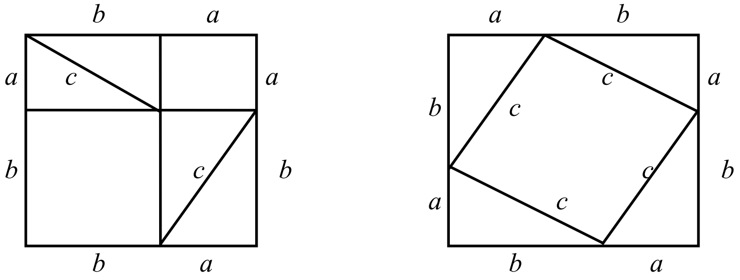
The Pythagoreans established many striking properties of figurate numbers in a purely geometric fashion: this is probably the first instance of an application of one area of mathematics to obtain results in another. For example, any square number is the sum of two successive triangular numbers and the sum of arbitrarily many consecutive odd numbers, beginning with 1, is a perfect square. These facts follow immediately from a glance at the figures:



The fourth triangular number 10, the so-called "tetraktys", played an important role in the Pythagorean scheme as the sum of the possible geometric dimensions: 1 point, the generator; 2 points, a line; 3 points, a plane; 4 points, a (tetrahedric) volume.

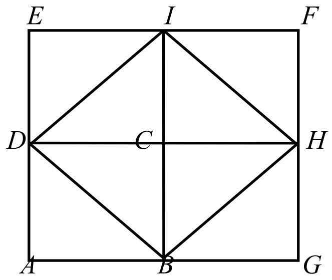
The Pythagoreans are also credited with discovering the correspondence between musical intervals and simple arithmetic ratios of lengths of stretched strings: the octave corresponding to the ratio 2:1, the fifth to 3:2 and the fourth to 4:3. This discovery, which later led to the construction of musical modes and scales, is particularly striking in being both a contribution to *mathematical physics*, the mathematical description of the natural world, and to *aesthetics,* the analysis of the beautiful.

The most famous proposition attributed to Pythagoras himself is the geometric theorem that the square on the hypotenuse of any right-angled triangle is equal to the sum of the squares on the two remaining sides. This fact was already known to the Babylonians more than a thousand years earlier, but it seems to have been Pythagoras who formulated its first general proof. It is believed that Pythagoras's proof was *dissective* in nature, as suggested by the equal square figures below. The left hand figure is the sum of four copies of the original right triangle with the sum of the squares on the legs, and the right hand figure is the sum of four such copies with the square on the hypotenuse.



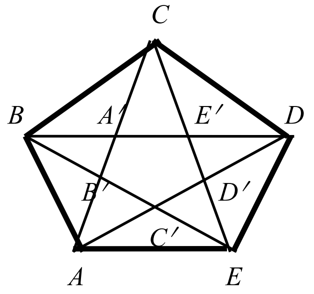
As we have said, it was a fundamental principle of Pythagoreanism that all is explicable in terms of properties of, and relations between, whole numbers—that number, indeed, forms the very essence of the real. It must, accordingly, have come as a great shock to the Pythagoreans to find, as they did, that this principle *cannot be upheld within geometry itself*. This followed upon the shattering discovery, probably made by the later Pythagoreans before 410 B.C., that ratios of whole numbers do not suffice to enable the diagonal of a square or a pentagon to be compared in length with its side. They showed that these line segments are *incommensurable*, that is, it is not possible to choose a unit of length sufficiently small so as to enable both the diagonal of a square (or a pentagon) and its side to be measured by an integral number of the chosen units. Pythagorean geometry, depending as it did on the assumption that all line segments are *commensurable* (i.e., measurable by sufficiently small units), and with it the whole Pythagorean philosophy, was dealt a devastating blow by this discovery. It may be remarked that the Pythagorean catastrophe shows how risky it can be to base a world outlook on a literal interpretation of mathematics: the exactness of mathematics may be used to undermine it!

The initial steps of what may have been the Pythagorean proof of the incommensurability of the side and diagonal of a square are presented in Plato's dialogue the *Meno,* in which Socrates asks an uneducated slave how to double the area of a square while maintaining its shape. Eventually Socrates solves the problem by producing the figure below, in which the inscribed square *BDIH* has double the area of the given square *ABCD:*



Having achieved his purpose, Socrates stops here, but it is likely that the Pythagoreans continued the argument along the following lines. Suppose *AB* and *BD* are commensurable; then we may take a unit of measure which divides both *AB* and *DB* an integral number of times, and, by successively doubling the length of the unit, if necessary, we may assume that at least one of these numbers is odd. Then (the area of) □*AGFE* is divisible by 4. But □*AGFE* = 2 □*BHID*, so □*BHID* is even, and so $BD^2$ and *BD* are even. Therefore □*BHID* is divisible by 4. But □*BHID* = 2 □*ABCD* so □*ABCD* is even. Hence $AB^2$, and *AB* are even. Contradiction.

It is possible that the actual *discovery* of the phenomenon of incommensurability was a by-product of the procedure—known to the Pythagoreans —of *subdividing a regular pentagon*. Starting with a regular pentagon *ABCDE*, and then drawing all five diagonals, we find that these intersect in points $A'B'C'D'E'$ which form another regular pentagon, whose diagonals in turn form a still smaller regular pentagon, and so on. Clearly this process of subdivision never stops, and there is no "smallest" pentagon whose side could serve as a unit of measure. To be precise, let $AB = a$, $AC = d$, $A'B' = a'$, $A'C' = d'$. Then $d' = d - a < \frac{1}{2}d$, $a' = 2a - d$. Therefore, if *AB* and *AC* were



commensurable by some unit of length *u*, the side and diagonal of all smaller pentagons obtained by subdivision as above would be commensurable by the same unit *u*. Since these pentagons can be made arbitrarily small, and, in particular, so small that their sides are shorter than *u*, we have a contradiction.

In modern terms, the Pythagorean incommensurabilities are understood as asserting that the "numbers" representing the lengths of certain lines are *irrational*, that is, are not rational fractions of the form $m/n$ with integral $m,n$. Thus, for example, the diagonal of a square of side 1 has length $\sqrt{2}$, and we may establish the irrationality of the latter by means of the following argument, which is essentially an algebraic version of one given above. Suppose that $\sqrt{2} = m/n$ with $m/n$ in lowest terms so that at least one of $m$, $n$ is odd. Then $2 = m^2/n^2$ so that $m^2 = 2n^2$. Hence $m^2$ is even, and so therefore is $m$. It follows that $n$ must be odd. But since $m$ is even, $m^2$ is divisible by 4, so that $n^2 = m^2/2$ is even and so therefore is $n$. Thus $n$ is both even and odd, a contradiction. A similar argument shows that $\sqrt{p}$ is irrational whenever $p$ is not a perfect square.

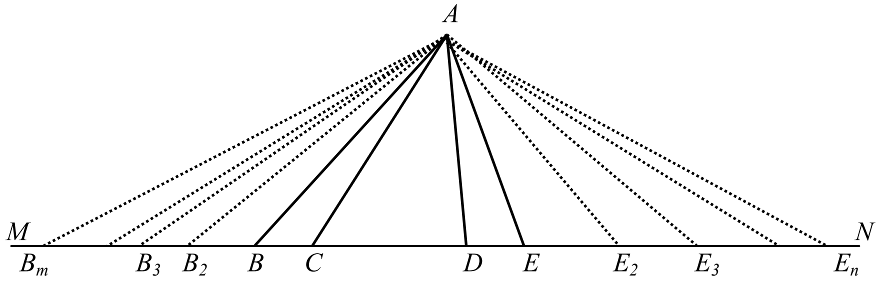
The discovery of incommensurables undermined the Pythagorean concept of "ratio" of magnitudes, which could no longer be represented in all cases in terms of whole numbers. It was *Eudoxus of Cnidus* (*c*.400–350 B.C.) who, by formulating an *axiomatic* theory of ratios, or proportions, resolved the problem of incommensurables as it affected geometry and in so doing anticipated the modern theory of real numbers.

In Eudoxus's theory, we are given a collection of *similar magnitudes*, e.g. line segments, or planes , or volumes, or angles, etc., together with the notion of *ratio* of similar magnitudes satisfying certain *axioms*, of which the following, expressed in modern terms, are the most important:

**P$_1$**. Given any two ratios, there is an integral multiple of the one which exceeds the other.
**P$_2$.** The ratio $a{:}b$ of two magnitudes $a$, $b$ is equal to the ratio $c{:}d$ of two other magnitudes $c$, $d$ if and only if, for any natural numbers $m,n$, $na$ is greater than, equal to, or less than $mb$ according as $nc$ is greater than, equal to, or less than $md$.

Axiom **P$_1$**, which is also known as *Archimedes' principle*, has the effect of excluding infinitely small or infinitely large quantities, and axiom **P$_2$** is, in essence, a *definition of equality* for ratios.

Let us use these principles to prove the familiar proposition that *the areas of triangles having the same altitude are to one another as their bases*. The Pythagorean proof of this theorem assumed that the bases of the triangles are commensurable, but the Eudoxan proof succeeds in avoiding this assumption, as we shall see. Thus suppose given two triangles $ABC$ and $ADE$, whose bases $BC$ and $DE$ lie on the same line $MN$. On $CB$ produced, mark off, successively from $B$, $m - 1$ segments equal to $CB$, and connect the points of division $B_2, B_3, \ldots, B_m$ with vertex $A$ as in the figure below. Similarly, on $DE$ produced, mark off, successively from $E$, $n - 1$ segments equal to $DE$, and connect the points of division $E_2, E_3, \ldots, E_n$ with vertex $A$. Then $B_mC = m.BC$, $\Delta AB_mC = m.\Delta ABC$, $DE_n = n.DE$, $\Delta ADE_n = n.\Delta ADE$. Also, since the triangles $AB_mC$ and $ADE_n$ have the same altitude, their areas will be greater, smaller, or coincide according as the lengths of their bases are greater, smaller, or coincide. It follows that $m.\Delta ABC$ is greater than, less than, or equal to $n.\Delta ADE$ according as $m.BC$ is greater



than, less than, or equal to, *n.DE.* So, by $\mathbf{P_2}$, $\Delta ABC : \Delta ADE = BC : DE$, and the proposition is established.

In addition to its major function of providing a rigorous foundation for the theory of similar figures in geometry, the Eudoxan theory of proportion played an important role in the justification of arguments based on the so-called *method of exhaustion*—in essence, an anticipation of the *integral calculus*—for computing the areas of regions bounded by curves, e.g. the circle. One of the earliest contributions to the problem of *squaring the circle*, that is, of constructing a square equal in area to a given circle, was made by *Antiphon the Sophist* (480–411B.C.) and *Bryson of Heraclea* (b. *c.*450 B.C.) , both contemporaries of Socrates. It is believed that their idea was to double successively the sides of a regular polygon inscribed in the circle, so that the difference in area between the polygon and the circle will eventually be exhausted. Since a square can be constructed equal in area to any given polygon, the squaring of the circle will be effected. This procedure met with the objection that, since magnitudes are divisible without limit, the area of the inscribed polygon will never in fact coincide with that of the circle. Nevertheless, it contains the germ of the method of exhaustion.

As an illustration of the method we give Eudoxus's proof that the area of a circle is directly proportional to the square on its diameter. Eudoxus bases his argument on the following assumptions:

$\mathbf{M_1}$. Magnitudes are divisible without limit.

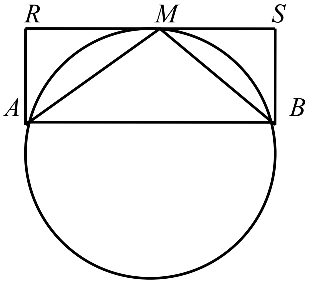
$\mathbf{M_2}$. If from any magnitude there be subtracted a part not less than its half, from the remainder a part not less than its half, etc., then there will remain eventually a magnitude less than any given magnitude of the same kind.

Axiom $\mathbf{M_1}$ expresses the idea that magnitudes are *continuous* entities and are not built from atoms. Axiom $\mathbf{M_2}$ is known as *Eudoxus's principle of convergence*.

Now it must be shown that, if $A_1$ and $A_2$ are the areas of two circles with diameters $d_1$ and $d_2$, then

$$A_1 : A_2 = d_1^2 : d_2^2.$$

To do this we first show that the difference in area between a circle and an inscribed regular polygon can be made as small as desired.



Let *AB* be a side of a regular inscribed polygon and let *M* be the midpoint of the arc *AB*. Since $\Delta AMB = \frac{1}{2}\ \square ARSB > \frac{1}{2} \underline{\frown} AMB$, it follows that by doubling the number of sides of the regular inscribed polygon we increase the area of the polygon by more than half the difference in area between the polygon and the circle. Consequently, using $M_1$ and $M_2$ we see that by doubling the number of sides of the polygon sufficiently often, we can make the difference in area between it and the circle less than any assigned area, however small.

Now suppose, for contradiction's sake, that we had

$$A_1 : A_2 > d_1^2 : d_2^2.$$

Then, by the argument above, we can inscribe in the first circle a regular polygon whose area $P_1$ differs so little from $A_1$ that

$$P_1 : A_2 > d_1^2 : d_2^2.$$

Let $P_2$ be the area of a regular polygon similar to the original regular polygon, only inscribed in the second circle. Then, by elementary geometry,

$$P_1 : P_2 = d_1^2 : d_2^2,$$

so that

$$P_1 : A_2 > P_1 : P_2,$$

whence $P_2 > A_2$, an absurdity. A similar argument shows that we cannot have $A_1 : A_2 < d_1^2 : d_2^2$, and so we conclude that

$$A_1 : A_2 = d_1^2 : d_2^2,$$

as required.

It follows that, if $A$ is the area and $d$ the diameter of a circle, then $A = kd^2$, where $k$ is a constant (by definition, equal to $\pi/4$) for all circles.

The best known Greek mathematical text is the celebrated *Elements* of *Euclid*, written about 300 B.C. This definitive geometric treatise appears to have quickly and completely superseded all previous works of its kind: in fact, no trace whatsoever remains of its predecessors. With the exception of the Bible, no work has been circulated more widely or studied more assiduously, and no work, without exception, has exercised a greater influence on scientific thinking. Over a thousand editions of the *Elements* have appeared since the first was printed in 1482, and for more than two millenia it has dominated the teaching of geometry.

Euclid's *Elements* is not devoted to geometry alone, but also contains considerable number theory and elementary algebra. Although some of its proofs and propositions were doubtless invented by Euclid himself, the work is, in the main, a systematic compilation of the works of his predecessors. Euclid's achievement lies in his skilful selection of propositions and their arrangement in a logical sequence, creating in the process a standard of rigorous presentation that was not to be surpassed until modern times. Through its use of the *postulational-deductive*, or *axiomatic,* method the *Elements* became the universally accepted model for rigorous mathematical demonstration: it may be said without exaggeration that it is to the influence of this work that we owe the pervasiveness of the axiomatic method in the mathematics of today.

Applied to a given discourse, the essence of the axiomatic method is to begin by laying down a number of *definitions*, and then to introduce a group of statements — the *axioms* or *postulates* — which are accepted without proof. From these latter all the remaining true statements of the discourse are to be derived in a rigorous manner. Axioms and postulates were distinguished by Greek mathematicians and philosophers in at least three different ways:

(1) an axiom is a self-evident assertion about, and a postulate a construction of, a certain thing;
(2) an axiom is a universal assumption, whereas a postulate is an assumption germane only to the subject under study;
(3) an axiom is obvious and acceptable, while a postulate is not necessarily so.

It seems likely that in his *Elements* Euclid inclined toward the second of these distinctions.

As it has come down to us, the *Elements* opens with 23 definitions, of which the following may be taken as typical:

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. A *straight line* is a line which lies evenly with the points in itself.
8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
10. When a straight line set up on a straight line makes adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called *perpendicular* to that on which it stands.
15. A *circle* is a plane figure enclosed by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.
23. *Parallel* straight lines are those which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

It continues with five axioms and five geometric postulates:

A1. Things which are equal to the same thing are equal to each other.
A2. If equals be added to equals, the wholes are equal.
A3. If equals be subtracted from equals, the remainders are equal.

A4. Things which coincide with one another are equal to one another.

A5. The whole is greater than the part..

P1. It is possible to draw a straight line from any point to any other point.

P2. It is possible to produce a straight line indefinitely in that straight line.

P3. It is possible to describe a circle with any point as center and with a radius equal to any finite straight line drawn from the center.

P4. All right angles are equal to one another.

P5. If a straight line intersects two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will intersect, if indefinitely produced, on the side on which are the angles which are together less than two right angles.

From these ten statements Euclid proceeds to derive 465 geometric propositions, assembled into thirteen books. In the first few books are developed the basic properties of familiar plane figures such as triangles, squares and circles, and the construction, using straightedge and compasses, of regular polygons. In Book V we find a masterly exposition of Eudoxus's theory of proportions, which in Book VI is applied to plane geometry, yielding the fundamental theorems on similar triangles. Books VII, VIII and IX deal with elementary number theory. Book VII contains in particular an account of the process, known today as the *Euclidean algorithm*, for finding the greatest common divisor of a pair of numbers. (That is, divide the larger of the two numbers by the smaller. Then divide the divisor by the remainder. Continue the process of dividing the last divisor by the last remainder until the division is exact. The final divisor is the sought greatest common divisor.[1]) Book IX includes a number of significant propositions, in particular one equivalent to the so-called *fundamental theorem of arithmetic*, namely that any integer greater than 1 can be uniquely expressed as a product of *prime numbers*, that is, numbers not possessing any smaller divisors apart from 1). Another is a proof of the remarkable proposition that there are *infinitely many prime numbers*. (Given any number $n$, consider the number $(1.2.3....n) + 1$. This number is not divisible by $n$ or any smaller number, so its prime factors must all exceed $n$. Thus, given any number, we can find a prime number which exceeds it, and so there must be infinitely many prime numbers.) Book X deals with incommensurable line segments and Eudoxus's method of exhaustion. The last three books are devoted to solid geometry and the determination of volumes. The final proposition is the beautiful result that there are exactly five *regular solids*, that is, polyhedra all of whose faces are congruent and all of whose vertex angles are equal, namely, the regular tetrahedron (with 4 faces), cube (6), octahedron (8), dodecahedron (12), and icosahedron (20).

The mode of presentation of the *Elements* is *synthetic* in that it passes from the known and simpler to the unknown and more complex, a procedure which has become standard in mathematical exposition. But the reverse process—*analysis*—must have played a significant role in the *discovery* of the theorems, even if scarcely any trace of it is to be found in the text itself.

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[1] The Euclidean algorithm is discussed in more detail in Chapter 4.

While the *Elements* was regarded as the *ne plus ultra* of rigour well into the nineteenth century, a few mathematicians recognized that the work is marred by a number of logical deficiencies. One of these—of which Euclid himself seems to have been aware—is the use of superposition to establish the congruence of figures. Against this two objections may be raised. First, it involves the idea of *motion* for which no logical basis has been provided, and secondly, it presupposes without justification that a figure's properties remain unaltered when moved from one position to another. This is a very strong assumption concerning physical space. Another deficiency is the vagueness of many of the Definitions, for example, those of point, line and surface. Other shortcomings stem from assumptions concerning the *continuity* of lines and circles which, although evident from the properties of figures, are not furnished with logical justification. Such is the case, for instance, in the proof of the very first proposition, which shows that an equilateral triangle can be erected on any line segment. It requires that two particular circles have a point in common, a claim which, while intuitively obvious on grounds of continuity (since the two curves "cross"), is not logically derivable from the initial assumptions. Euclid also makes frequent use of the assumption that straight lines crossing one another must have a common point.

It is implicit in Euclid's postulates that in constructing straight lines and circles the straightedge and compasses are to be employed in strict accordance with certain rules, namely: *with the straightedge one is permitted to draw a straight line of arbitrary length through any two distinct points; with the compasses one is permitted to draw a circle with any given point as centre and passing through any second given point.* These instruments, used in accordance with the rules just stated, have become known as *Euclidean tools*. Note that the straightedge is taken to be *unmarked*, and the compasses are not to be employed as dividers, i.e. for transferring distances[2].

The ancient Greek mathematicians knew that many geometric constructions, for example, bisecting angles, erecting perpendiculars, and extracting square roots, could be carried out using Euclidean tools. However, try as they might, they were unable to devise constructions, using Euclidean tools alone, for resolving the following problems—the *Three Famous Problems of Antiquity*—whose refractory nature has given them great mathematical notoriety.

1. *Doubling the cube*: the problem of constructing the edge of a cube having twice the volume of a given cube.
2. *Trisecting the angle*: the problem of dividing an arbitrary given angle into three equal parts.
3. *Squaring the circle:* the problem of constructing a square having an area equal to that of a given circle.

In modern terms, the first of these problems may be stated as that of constructing a line of length $\sqrt[3]{2}$, and the third as that of constructing a line of length $\pi$. Many attempts were made to solve them, but it was not until the nineteenth century, more than two thousand years after their formulation, that they were finally shown to be

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[2] Remarkably, it is possible to show that dividers can be "reconstructed" using Euclidean tools alone.

insoluble by Euclidean means[3]. The unremitting search by Greek mathematicians for solutions to these problems had a profound influence on Greek geometry and led to many important discoveries, of conic sections (see below) to name but one. In our era the analysis of these problems has also had a fruitful influence, especially on the development of algebra.

Tradition has it that the problem of doubling the cube arose as the result of attempting to carry out the instructions of the oracle to Apollo at Delos. It is reported that an Athenian delegation was despatched to this oracle to enquire how a plague then raging could be curbed, to which the oracle responded that the cubic altar to Apollo must be doubled in size. The Athenians are said to have then dutifully doubled the dimensions of the altar, with no perceptible effect on the plague. Their belated realization that the oracle had meant doubling the *volume*, and not the *dimensions*, of the altar is supposed to have led to the problem of doubling the cube.

It seems to have been *Hippocrates* (*c.*450 B.C.) who made the first real progress in the problem of doubling the cube. He recognized that the problem could be reduced to the construction of a pair of *mean proportionals* between two given line segments of arbitrary given lengths $a, b$, that is, the construction of two line segments of lengths $x, y$ standing in the ratios $a : x = x : y = y : b$. For if $b = 2a$, then, upon eliminating $y$, we obtain $x^3 = 2a^3$, so that $x$ is the length of the side of a cube of double the volume of one of side $a$.

While the problem of doubling the cube continued to elude solution using only Euclidean tools, Hippocrates's breakthrough made it possible for his successors to elaborate constructions which "solved" the problem in a manner *not strictly Euclidean*. Two such "solutions" were furnished by *Menaechmus*, a pupil of Eudoxus, around 350 B.C. His arguments are based on the idea of a *conic section*, a concept he is reputed to have introduced for the express purpose of solving the problem. Menaechmus defined a conic section to be a curve bounding a section of a right circular cone cut off by a plane perpendicular to the cone's surface. According as the vertex angle of the cone is less than, equal to, or greater than, a right angle, the curve obtained is an *ellipse*, a *parabola*, or a *hyperbola.* (These terms are later coinages: Menaechmus himself would have used the descriptive words *oxytome*, "sharp cut", *orthotome*, "right cut", and *amblytome*, "dull cut".) Menaechmus showed that a solution to the problem of constructing two mean proportionals, and hence to that of doubling the cube, could be achieved by constructing a pair of parabolas, or by constructing a parabola and a hyperbola. In modern terms, his argument went as follows: to find two mean proportionals between $a, b,$ we seek $x, y$ so that $a/x = x/y = y/b$, that is, to satisfy the equations

$$x^2 = ay, \quad y^2 = bx, \quad xy = ab.$$

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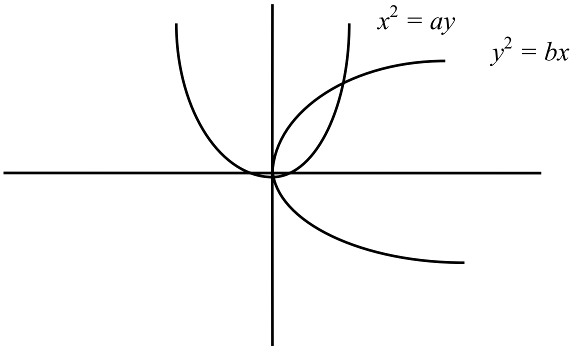
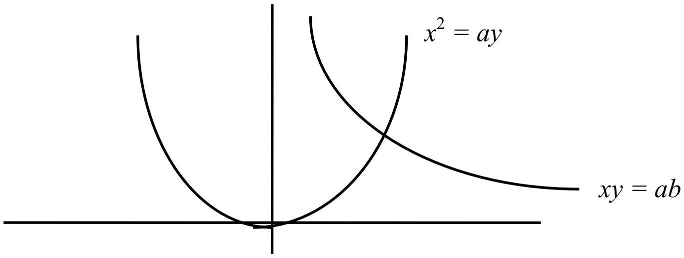
[3] The insolubility of problem 3 is epitomized in the following word palindrome: *You can circle the square, can't you, but you can't square the circle, can you?*

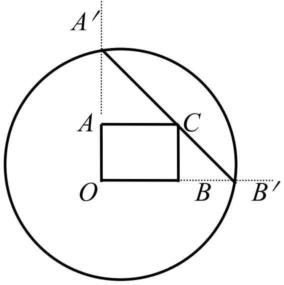
The required values can thus be obtained from the points of intersection of the parabolas $x^2 = ay$, $y^2 = bx$:



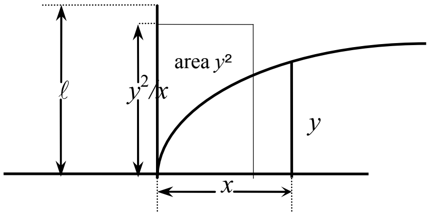
or of the parabola $x^2 = ay$ with the hyperbola $xy = ab$:



Another such "solution" to the problem of doubling the cube was devised by *Apollonius of Perga* (*c.*262–190 B.C.), an outstanding mathematician of antiquity. Draw a rectangle *OACB*, where *OB* is twice the length of *OA*, and then a circle, concentric with *OACB*, cutting *OA* and *OB* produced in $A'$ and $B'$ such that $A', C, B'$ are collinear. It can then be shown that $(BB')^3 = 2.(OA)^3$. Actually, it is impossible to construct the specified circle with Euclidean tools, but Apollonius did manage to formulate a mechanical way of describing it.



To his contemporaries it was Apollonius, and not Euclid, who was known as “the Great Geometer,” a title which was deservedly bestowed on him in recognition of his authorship of the masterly treatise, the *Conics*, one of only two of his works which have come down to us. In this work Apollonius assigns the names *ellipse, parabola,* and *hyperbola* to the conic sections, terms which had originally been employed, probably by the Pythagoreans, in the areal solution of quadratic equations. This method had involved the placing of the base of a given rectangle along a line segment so that one end of each coincided. According as the other end of the rectangle’s base fell short of, coincided with, or fell beyond the other end of the line segment, it was said that one had *ellipsis* (“deficiency”), *parabole* (“thrown alongside”) or *hyperbole* (“thrown beyond”). Using the modern representation of conic sections in coordinate geometry, it is not difficult to see why Apollonius applied these terms in this new context. The equation of a parabola with its vertex at the origin is $y^2 = \ell x$, so that, at any point on it, the square on the ordinate $y$ is precisely equal to the rectangle on the abscissa $x$ and the parameter $\ell$. The equations of the hyperbola and the ellipse, again with a vertex as origin, are $(x \pm a)^2/a^2 + y^2/b^2 = 1$, or $y^2 = \ell x \pm b^2x^2/a^2$, where $\ell = 2b^2/a$. Thus, for the ellipse, $y^2 < \ell x$ and for the hyperbola $y^2 > \ell x$. So for the ellipse, hyperbola and parabola, we have $y^2/x <, =$, or $> \ell$, respectively, that is, the base of the rectangle with area $y^2$ and width $x$ falls short of, coincides with, or exceeds the line segment of length $\ell$.



In the *Conics* Apollonius shows that all three varieties of conic section can be obtained from a single cone by varying the inclination of the cutting plane, thus establishing the important fact that all three are *projections* of each other, and showing also that the circle is a special type of ellipse (obtained when the cutting plane is parallel to the base of the cone). Apollonius also shows that sectioning an oblique or scalene cone yields the same curves as does a right cone. Finally he took the significant step of replacing the single cone by the double cone obtained by allowing a straight line, with one point fixed, to move around the circumference of a given circle. In this way he revealed that the hyperbola has two branches, corresponding to sections of both sheets of the double cone.

Of all the famous problems of mathematics, that of *squaring the circle* has exerted the greatest fascination through the ages[4]. This problem also eluded solution with

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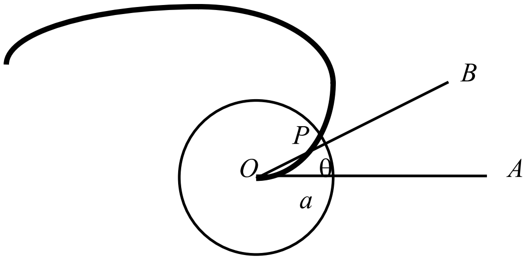
[4] This problem was apparently sufficiently familiar to the Athenian public of the 4th century B.C. for Aristophanes (c.450–*c*.385 B.C.) to allude to it in one of his comedies, *The Birds*.

Euclidean tools, but, again, Greek mathematicians produced "solutions" which, although not Euclidean, possessed great ingenuity and elegance. One such solution is due to *Archimedes* (287–212 B.C.), regarded as the greatest mathematician of antiquity, and one of the greatest of all time. His solution of the problem was achieved by means of a special curve known as the *spiral of Archimedes*. We may define the spiral in kinematic terms as the locus of a point $P$ moving uniformly along a ray which, in turn, is uniformly rotating about its end point in a plane. (More concretely, the spiral is the path traced out by a bug crawling along a rotating rod.) If we write $r$ for the distance of $P$ from the end point of the ray when it makes an angle $\theta$ with its initial position (i.e., when $P$ is at the end point $O$), then $r$ and $\theta$ are related by an equation of the form $r = a\theta$, where $a$ is some constant. Now draw the circle with centre at $O$ and radius $a$. Then $OP$ and the arc of this circle between the lines $OA$ and $OP$ are equal, since each is given by $a\theta$. Accordingly, if we take $OP$ perpendicular to $OA$, then $OP$ will have a length equal to one quarter of the circumference of the circle. Since—as the Greeks knew—the area $K$ of the circle is half the product of its radius and its circumference, we get



$$K = (a/2).(4OP) = (2a).OP.$$

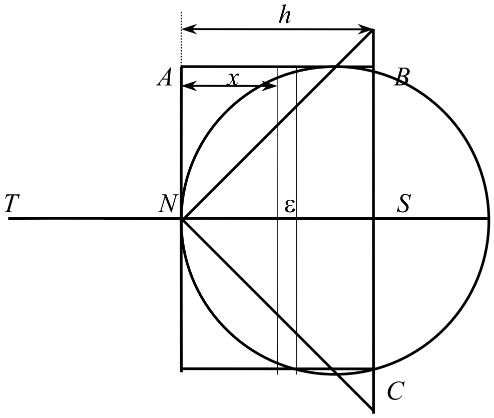
In other words, the area of the circle is the same as the area of the rectangle whose sides are the diameter of the circle and the radius vector of the spiral which is perpendicular to $OA$. Since it is an easy Euclidean construction to produce the side of a square equal in area to that of a given rectangle, we get a solution to the problem of squaring the circle.

Archimedes was also a master of the method of exhaustion, and used it to give rigorous proofs of the formulas for areas and volumes of quite complex figures, for example parabolic segments, and ellipsoids and paraboloids of revolution. Observing that the method of exhaustion is essentially a tool for *proof* but not for *discovery*, one may ask how Archimedes actually *found* his formulas. This only became known with the discovery in Constantinople in 1906 of a copy of Archimedes' long-lost treatise *The Method*, addressed to the mathematician *Eratosthenes* (*c.*230 B.C., famous for his remarkably accurate estimate of the diameter of the earth). The manuscript was found in what is known as a palimpsest: it had been copied onto parchment in the tenth century which was later—in the thirteenth century—washed off and reused for inscribing a religious text. This, it may be remarked, provides an excellent illustration of the intellectual priorities of the times! Fortunately, most of the original text was restorable from beneath the later one.

In *The Method,* Archimedes reveals that in discovering some of his remarkable area and volume formulas he employed ideas from the physical science of statics which was itself another product of his fertile mind. The fundamental idea was this: to find the area or volume of a given figure, (mentally) cut it up into a very large number of parallel thin strips or layers, and (again mentally) hang these pieces at one end of a given lever in such a way as to be in equilibrium with a figure whose content and center of gravity are known. We illustrate this idea by using it to obtain the formula for the *volume of a spherical segment*. In the derivation we shall make use of the *law of the lever*, which states that two bodies on a lever balance when their weights are inversely proportional to their distances from the fulcrum. Accordingly two bodies will balance if they have equal moments about the fulcrum, where the *moment* of a body about a point is defined to be the product of the mass of the body with the distance of its centre of gravity from the point.

Thus consider a segment of height $h$ of a sphere of radius $r$, positioned with its polar diameter along the $x$-axis with its north pole $N$ at the origin. By rotating the rectangle $NABS$ and the triangle $NCS$ about the $x$-axis, we obtain a cylinder and a cone. We assume that sphere, cone and cylinder are homogeneous solids of unit density. Now cut from these three solids thin vertical slices (assuming that these are, approximately, flat cylinders) at distance $x$ from $N$ and thickness $\varepsilon$. The volumes of these slices are, approximately,[5]

sphere: $\pi x(2r-x)\varepsilon$
cylinder: $\pi r^2$
cone: $\pi x^2\varepsilon$.



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[5] In making these computations we assume the usual formula $A = \pi r^2$ for the area $A$ of a circle of radius $r$.

Now let $T$ be a point on the $x$-axis at distance $r$ from the origin in the opposite direction from $S$. If the slices of the sphere and cone are suspended at $T$, then their combined moment about $N$ is

$$[\pi x(2r - x)\varepsilon + \pi x^2\varepsilon].r = 2\pi r^2 x\varepsilon.$$

This is clearly twice the moment about $N$ of the slice cut from the cylinder when that slice is left where it is. In other words, the slices of the sphere and cone placed at $T$ exactly balance —with fulcrum placed at $N$— twice the slice of the cylinder left where it is. Adding all these slices together, the results will still balance, so that, about $N$,

moment of mass of spherical segment + mass of cone concentrated at $N$
= 2.moment of cylinder.

Assuming it is known that the volume of a cone is $\frac{1}{3}$ base area × height, and that of a cylinder is base area × height (facts both familiar to Archimedes), we get

$$r.\ [\text{volume of spherical segment} + \pi h^3/3] = 2.h/2.\pi r^2 h,$$

from which we deduce that the volume of the spherical segment is

$$\pi r h^2 - \pi h^3/3 = \pi h^2(3r - h)/3.$$

From this it is easy to deduce Proposition 2 of *The Method*, which Archimedes states in the following way:

*Any segment of a sphere has to the cone with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment.*

Archimedes wrote many masterly mathematical treatises. Those that have come down to us include *On the Sphere and Circle, The Measurement of the Circle, On Conoids and Spheroids, On Spirals, On the Quadrature of the Parabola, On the Equilibrium of Planes, On Floating Bodies,* and *The Sand-Reckoner.* In the first of these Archimedes obtains, by strictly geometric means, the volumes and surface areas of spherical segments, cones, and cylinders, establishing in particular that the ratio of the volume of a right circular cylinder to that of an inscribed sphere is 3:2 (a representation of which was inscribed on his tomb), and that the area of a sphere is exactly 4 times that of a great circle. In *The Measurement of a Circle* he shows, using inscribed and circumscribed polygons of 96 sides, that the circumference of a circle is bounded between 22/7 and 223/71 times its diameter. *On Conoids and Spheroids* contains an investigation of the properties of solids obtained by the revolution of conic sections about their axes, the main results being the comparison of the volume of a segment of such a solid cut off by a plane with that of a cone having the same base and axis: for a paraboloid of revolution he shows that the ratio is 3:2. Archimedes also determines the area of an ellipse: as he puts the matter, “the areas of ellipses are as to the rectangles under their axes.” Most of the proofs in this work employ the method of exhaustion. *On Spirals*, regarded by Archimedes’ successors as his most recondite work, contains, in addition to the argument for squaring the circle described above, a number of propositions concerning the areas swept out by the radius vector of the spiral. In *On the Quadrature of the Parabola*, said to have been Archimedes’ most popular work, he shows that the area of a parabolic segment is $\frac{4}{3}$ times that of the inscribed triangle. This was the first quadrature of a conic section. *On the Equilibrium of Planes* contains Archimedes’ pioneering work on statics. Here he derives the law of the lever and locates the centres of gravity of various figures, including triangles, trapezoids, and parabolic segments. In *On Floating Bodies* Archimedes starts from a simple postulate concerning fluid pressure and proceeds to create the science of hydrostatics. He derives the now familiar principle of buoyancy that a floating solid displaces its own weight of fluid—a discovery which is, famously, reputed to have induced him to leap from the bath and run naked through the streets shouting *Eureka!*, “I have found it!”. In this work he also obtains complete determinations of the positions of equilibrium of floating paraboloids of revolution, a problem possibly arising in connection with the design of ship’s hulls.

*The Sand-Reckoner* occupies a special place in Archimedes’s output, for here he is concerned not with a geometric problem, but with a notational one. In this short work he shows that any finite set *A*, however large, can always be enumerated in the strong sense that a number exceeding that of the number of elements of *A* can always be *named*. Taking *A* to be a set of grains of sand filling the entire universe, he elaborates a system of notation, based on powers of a *myriad myriads*, i.e., $10^8$, in which he is able to express an upper bound for the number of elements of *A*. By “universe” Archimedes meant the (finite) cosmos as conceived by *Aristarchus of Samos* (*c.* 310 – 230 B.C.), famous for his calculations of the distances of the sun and moon from the earth.[6] Archimedes calculated that the number of grains of sand required to fill a sphere the size of Aristarchus’s universe is less than a number we would write as $10^{63}$. Archimedes actually extended his notation to embrace numbers up to 10 to the power $8 \times 10^{15}$.

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The idea of a *deductive* or *demonstrative* science, with mathematics as a prime example, seems to be original with the Greeks. But, one may ask, what prompted them to transform mathematics into such a science, an idea which appears to have been unknown to their predecessors? The pre-Greek mathematical documents that have come down to us are all concerned with practical or empirical questions; in none of them do we find a *theorem* or a *demonstration*, a *definition*, a *postulate*, or an *axiom*. As far as is known, these concepts originate with Greek mathematicians. But why were they introduced? Why, in other words, did Greek mathematicians come to put more trust in theoretical demonstration than in what could be verified by practice alone?

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[6] Aristarchus had also put forward a heliocentric theory of the solar system, but this was not accepted by most of the leading Greek astronomers, Archimedes included.

This turning-point in Greek thought has been attributed to several sources: the advanced sociopolitical development of the Greek city-states; the effort to isolate the general principles underlying the diverse methods for solving mathematical problems that had been inherited from the Egyptians and Babylonians; the influence of the art of disputation. Perhaps each of these played some role. An intriguing theory has been put forward by the historian of mathematics *Arpad Szabó,* who believes that Greek mathematics became a deductive science through the introduction by the Pythagoreans of the technique of *indirect demonstration*, an idea which he thinks was inherited from the *Eleatic* school of philosophy (fl. 550–400 B.C.), whose members included Parmenides and Zeno. The principal doctrines of the Eleatics were developed in opposition to the physical theories of the early Greek materialist philosophers, such as Thales and *Anaximander* (fl. *c.*560 B.C.) who explained all existence in terms of primary substance, and also to the theory of *Heraclitus* (fl. *c.*500 B.C.) that existence is perpetual flux. As against these theories the Eleatics maintained that the world is an unchanging unity and that the impression to the contrary arising through the senses is an illusion. In arguing for these claims they made constant use of the technique of indirect demonstration or *reductio ad absurdum*, that is, establishing the truth of a proposition by demonstrating the absurdity of its contrary[7].

Szabó conjectures that the Pythagoreans were moved to adopt the indirect form of reasoning after all efforts to generate a common measure for the side and diagonal of a square had failed. Beginning to suspect that such a common measure does not exist, they saw that the only possible way in which this nonexistence could be conclusively established was by means of an indirect argument in the Eleatic manner. Thus they started by assuming the contrary possibility, that is, the commensurability of the two segments. From this, as we have seen above, they derived the contradiction that a number can be both even and odd. In this way they demonstrated the absurdity of the original supposition, so conclusively establishing that no common measure could exist.

Szabó believes that initially Greek mathematics was, like its near-eastern predecessors, of a primarily illustrative character, its arguments built on concrete visualization, making essential use of figures. But, with the assimilation of the abstract and anti-illustrative Eleatic modes of reasoning, Greek mathematicians could no longer appeal to visual intuition to justify their arguments and were instead forced to make explicit the principles and assumptions on which their reasoning rested. It is this fact, Szabó suggests, which ultimately led the Greeks to cast their mathematics in deductive form.

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[7] The most famous Eleatic arguments of this kind are *Zeno's paradoxes* which purport to demonstrate the impossibility of motion: see Ch. 10