

## CHAPTER 8

### THE DEVELOPMENT OF GEOMETRY, II

#### PROJECTIVE GEOMETRY

THE LAWS OF PERSPECTIVE DRAWING—the technique used to portray three dimensions on a two dimensional surface—have been studied by artists since the Stone Age. For example, a fifteen thousand year old etching of a herd of reindeer on a bone fragment discovered by archaeologists creates the impression of distance by displaying the legs and antlers as if seen beyond the fully sketched animals of the foreground. The main perspective problem encountered by *Egyptian* artists was the portrayal of a single important object with the necessary dimension of depth: this was achieved in an ingenious manner by drawing a combination of horizontal and side view. Thus, for instance, in drawing a Pharaoh carrying a circular tray of sacrificial offerings, the top view of the tray is shown in half display by means of a semicircle, and on this half-tray is presented the sacrificial food as it would appear from above. This stylized method of expressing a third dimension persisted in Egyptian drawing for three thousand years. In *America*, an arresting method of achieving this effect was created by northwest Indian artists who, in their drawings of persons or animals, present views of both front and left- and right hand sides. The figures are drawn as if split down the back and flattened like a hide, with the result that each side of the head and body becomes a profile facing the other. Landscapes drawn by *Chinese* artists create the impression of space and distance by skillful arrangement of land, water and foliage. In drawing buildings, however, it was necessary to display the parallel horizontal lines of the construction, and for this the technique of *isometric drawing* was used. This is a simulation of perspective drawing in which parallel lines are drawn parallel, instead of converging as in true perspective.

In *Europe*, it was not until the first half of the fifteenth century that Italian painters, through the introduction of the *horizon line* and *vanishing point*, transformed perspective drawing into an exact science. The fundamental principles were worked out by the Florentine architect *Filippo Brunelleschi* (1377–1446) and developed by the painter *Paolo Uccello* (1397–1475). The formulation of the laws of perspective quickly transformed painting in Renaissance Italy, and the technique of perspective became an essential constituent in the works of later Italian masters. In his great wall-painting *The*

*Last Supper*<sup>1</sup>, for example, *Leonardo da Vinci* (1542–1519) employs perspective in a subtle way to draw the viewer’s eye to the composition’s centre.

The origins of *projective geometry* lie in the study of perspective. A painter’s picture, or a photograph, can be regarded as a *projection* of the depicted scene onto the canvas or photographic film, with the painter’s eye or the focal point of the camera’s lens acting as the centre of projection. For example, suppose we take a photograph of a straight railway track, with equally spaced ties, going directly away from us. In the photograph the parallel lines of the rails appear to converge, meeting at a vanishing point or “point at infinity”; the equal spaces between the ties appear as unequal; and the right angles between the rails and the ties appear as acute. A circular pond in the landscape would appear as an ellipse. Nevertheless the geometric structure of the original landscape can still be discerned in the photograph. This is possible only if the original scene and its image have certain geometric properties in common, properties which, unlike lengths and angles, are preserved under the passage from the one to the other. The object of projective geometry is to identify and investigate these properties in a general setting.

The transformation from scene to image may be described in the following way. Given any plane geometric figure (confining ourselves to plane figures for the sake of simplicity), let  $O$  be any point in the plane of the figure and imagine straight lines drawn from  $O$  to every point of the figure. Now allow this bundle of straight lines to be cut by any plane not passing through  $O$ . The resulting plane section of the lines through  $O$  yields a new figure in the cutting plane called the *projection* of the original figure. (In the case of our photograph, the original scene is the first figure, the photographic film is the cutting plane, and the focal point of the camera lens is the point  $O$ .) With each point or straight line of the original figure this process associates a definite point or straight line in the new figure; if a point  $P$  in the original figure lies on a straight line  $\ell$ , in the new figure the associated point  $P'$  lies on the associated line  $\ell'$ . Any mapping of one figure onto another by a projection of the kind we have just described, or the result of a finite sequence of such projections, is called a *projective transformation*. A projective transformation carries points into points and straight lines into straight lines in such a way as to preserve the incidence of points and lines. It does *not* preserve lengths or angles.

Two figures are *homologous* or *projectively equivalent* if each can be obtained from the other by a projective transformation. It is a fundamental fact of projective geometry that all the conic sections are projectively equivalent.

The set of all projective transformations furnishes another example of a *group*, since the result of successively performing any two of the operations in the set is still a transformation in the set; the set contains an identity transformation (“projection” onto the original plane); and each member of the set has an inverse whose composite with it in either order is the identity transformation (obtained by “reversing” the given projection). Projective geometry may be characterized completely as the study of those properties of figures which remain unchanged—are *invariant*—when acted upon by the elements of the group of projective transformations. Similarly, the set of all *rigid*

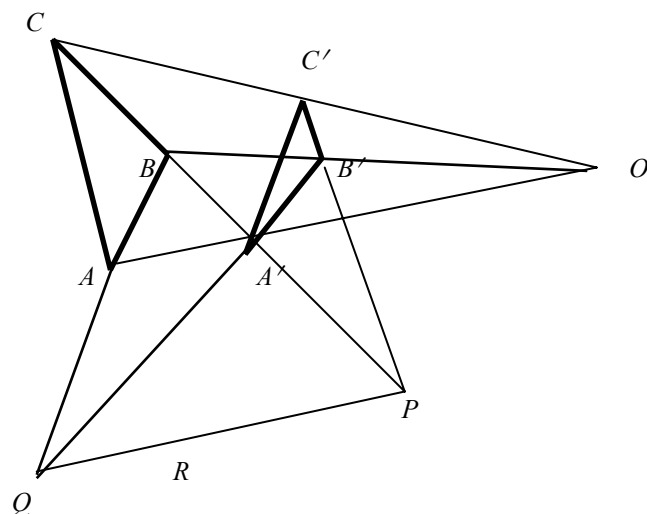
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<sup>1</sup> Happily, this masterpiece, in a sadly deteriorated state for as long as anyone can recall, has recently (1999) been restored.

*motions* in space forms a group: ordinary Euclidean geometry may then be identified as the geometry in which are studied those properties of figures, e.g., lengths and angles, which are invariant under the group of all such motions. As another example, the set of *conformal* (i.e., angle-preserving) *transformations* forms a group; *conformal geometry*—which arose in connection with cartography, a science rich in geometric possibilities—is the study of properties of figures which are invariant under all the transformations in this group. These are all special instances of a general principle—enunciated by Klein in his *Erlangen Program* of 1872—which asserts that corresponding to every group of transformations in space there is a “geometry” comprising those properties of figures which are invariant under all the transformations in the given group. This principle establishes a systematic link between geometry and algebra, and enables geometries to be classified by the properties of their corresponding groups. Thus, for example, since the group of rigid motions may be identified as a subgroup of the group of conformal transformations, which is in turn a subgroup of the group of projective transformations, ordinary Euclidean geometry is implicitly contained in conformal geometry, which is in turn contained in projective geometry. Each of the first two geometries may accordingly be obtained by specialization from the more general geometry containing it.

The first mathematician to take up the geometric problems suggested by perspective drawing was the Frenchman *Girard Desargues* (1591–1661) who was chiefly concerned to develop new methods for establishing properties of conics. In his major work on the subject, published in 1636, Desargues introduces, for each set of parallel lines, a *ideal point*—a *point at infinity* corresponding to the vanishing point of perspective drawing—which all the lines in the set have in common. He also introduces an *ideal line*—a *line at infinity* corresponding to the horizon line—and makes the assumption that all the new ideal points lie on that line. Thus in Desargues’ geometry each “ordinary” line contains exactly one “ideal” point—its “point at infinity”; each pair of lines meets at, or determines, a unique point; and each pair of points determines a unique line.

Desargues’ most famous result is his *Triangle Theorem* (see figure on following page). The theorem states: *If two triangles  $ABC$  and  $A'B'C'$  are situated in the same plane so that the straight lines joining corresponding vertices are concurrent in a point  $O$ , then the corresponding sides, if extended, will intersect in three collinear points.* The two triangles  $ABC$ ,  $A'B'C'$  are said to be *in perspective* from  $O$ . This theorem clearly belongs to projective geometry, since the whole figure can be projected onto any other plane without affecting any of the features mentioned in the theorem’s formulation. Its proof is based on this fact. For the points  $Q$  and  $R$  can then be “projected to infinity”, causing the pairs of lines  $AB$ ,  $A'B'$  and  $AC$ ,  $A'C'$  to meet at ideal points, so rendering each pair parallel. It is not difficult to show that, as a result,  $BC$  and  $B'C'$  become parallel, so that their point of intersection  $P$  is now also an ideal point. Thus under projection the points  $P$ ,  $Q$ , and  $R$  all lie on the line at infinity, and are, accordingly, collinear. Projecting back to the original plane,  $P$ ,  $Q$ , and  $R$  return to their original positions and remain collinear. So they must have been collinear in the first place. This proves the theorem.

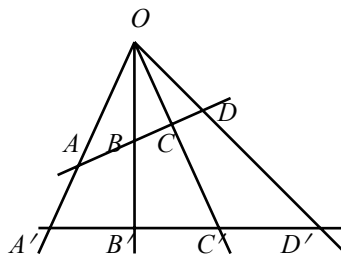


As noted by Desargues, the theorem remains true—and is, surprisingly perhaps, more easily proved—in three dimensions, when the triangles lie in different, nonparallel planes.

Desargues also established the fundamental fact that *cross-ratio* (a concept originally introduced by *Pappus of Alexandria* c.300 B.C.) is *invariant under projection*. The cross-ratio of four points  $A, B, C, D$  on a line  $\ell$  is defined to be the quantity

$$(ABCD) = \frac{CA/CB}{DA/DB}$$

where a fixed direction on  $\ell$  is taken as positive. Referring to the figure below, Desargues proves that  $(ABCD) = (A'B'C'D')$ . Pappus had established this fact only when  $A$  and  $A'$  coincide.



The next important contributor to projective geometry, *Blaise Pascal* (1623–1662) was at the same time a major literary figure and religious philosopher: his *Pensées* and *Lettres Provinciales* are classic works of French literature. Pascal is known for the famous theorem that bears his name: *if a hexagon is inscribed in a conic, the three points of intersection of the pairs of opposite sides are collinear*. Pascal did not supply an explicit proof of his theorem, but asserts that, since he knows it to be true for a circle, it must, by projection and section, be true of all conics.

During the latter half of the seventeenth century and throughout the eighteenth, coordinate geometry, with its quantitative and “analytic” methods, underwent rapid development. Projective geometry, on the other hand, with its emphasis on purely “synthetic” properties of geometric figures, remained becalmed. The stimulus that led to the revival of the subject in the nineteenth century was due in large measure to the French geometer *Gaspard Monge* (1746–1818), who, as an enthusiastic supporter of the French revolution, served as a technical advisor to Napoleon. In his *Traité de Géométrie* of 1799 Monge introduces the chief ideas of what was to become known as *descriptive geometry*, in which a three dimensional object is projected orthogonally onto two planes, one horizontal, the other vertical, so as to enable mathematical features of the object to be inferred.

The revival of projective geometry was carried out chiefly by Monge’s pupils *Charles-Julien Brianchon* (1765–1864), *Lazare M. N. Carnot* (1753–1823), a major figure in the French revolution, and *Jean-Victor Poncelet* (1788–1867). Brianchon is remembered for the important theorem bearing his name: *the three diagonals joining opposite vertices of a hexagon circumscribed about a conic are concurrent*. Poncelet’s work of 1822, *Traité des Propriétés Projectives des Figures*, is really the first systematic work on projective geometry, in which the subject is treated as an independent discipline with methods and goals of its own. It was Poncelet who established projective geometry in its modern sense as the study of properties of figures which are invariant under projection.

Poncelet was also the first to recognize what later became known—through the work of *Joseph-Diez Gergonne* (1771–1859) and *Jacob Steiner* (1796–1863)—as the *Principle of Duality*. This is best illustrated by writing the statements of Pascal’s and Brianchon’s theorems side by side:

*Pascal’s Theorem*

If the *vertices* of a hexagon lie alternately on two straight lines, the *points* where opposite sides meet are *collinear*.

*Brianchon’s Theorem*

If the *sides* of a hexagon pass alternately through two *points*, the *lines* joining opposite vertices are *concurrent*.

Here we observe that each of these statements may be obtained from the other by interchanging terms such as “vertices” and “collinear” which make reference to *points*, with corresponding terms such as “sides” and “concurrent” which make reference to *lines*. The Principle of Duality is the assertion that the dual of any theorem of projective geometry is likewise a theorem of projective geometry. (Note, incidentally, that the dual of Desargues’ triangle theorem is precisely its converse!) The idea of duality, first discerned in projective geometry, has since been extended to other branches of mathematics, notably algebra.

The advances made in projective geometry in the first part of the nineteenth century also excited the interest of geometers whose work was primarily in the algebraic spirit. From their efforts issued the subject of *algebraic projective geometry*. Here a leading idea is that of a *homogeneous coordinate system*. The first such system was introduced by the German mathematician *August Ferdinand Möbius* (1790–1868) in his work *Der Barycentrische Calcul* of 1827. Möbius's idea was to start with a fixed triangle and to take as coordinates of any point  $P$  in the plane the amounts of mass which must be placed at each of the three vertices so as to make  $P$  the centre of mass or *barycentre* of the three masses (points lying outside the triangle being assigned some negative coordinates). Since multiplication of all three masses by the same nonzero constant does not change their barycentre, the coordinates are not unique, only their ratios being uniquely determined. Thus, as  $t$  varies over all nonzero real numbers, the triples  $(ta, tb, tc)$  all represent the same point, which justifies the use of the term “homogeneous” for the coordinate system.

The system of homogeneous coordinates normally employed by geometers today was invented by the German mathematician *Julius Plücker* (1801–1868). In his *Analytisch-geometrische Entwicklungen* of 1831, he takes the coordinates of any point  $P$  in the plane to be the signed distances  $(a, b, c)$  of  $P$  from the sides of a fixed triangle of reference. Since, again, the coordinates of a point can be multiplied by any nonzero constant without changing the point, Plücker went on to formulate what has become the standard definition of homogeneous coordinates in the plane, namely *ordered triples*  $(a, b, c)$  of real numbers not all zero, where the triples  $(ta, tb, tc)$  are taken to represent the same point as  $t$  varies over the nonzero real numbers. This is equivalent to replacing the usual rectangular coordinates  $(X, Y)$  by  $(x/t, y/t)$ , so that the equations of curves become *homogeneous* in  $x, y$ , and  $t$ , that is, all terms have the same degree. In Plücker's homogeneous coordinate system, ordinary points in the plane are represented by triples  $(a, b, c)$  with  $c \neq 0$ , ideal points lying on ordinary lines by triples  $(a, b, 0)$  with  $a \neq 0$ , and points (necessarily ideal) lying on the line at infinity by triples  $(0, b, 0)$  with  $b \neq 0$ . The *projective plane*, one of the most important concepts in projective geometry, is defined to be the set of points determined in this way.

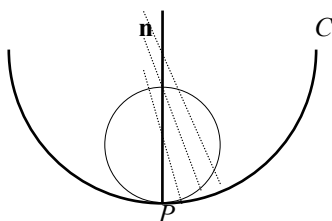
## DIFFERENTIAL GEOMETRY.

The introduction of the methods of the *calculus* into coordinate geometry during the seventeenth century issued in a branch of mathematics now known as *differential geometry*<sup>2</sup>. In contrast with Euclidean or projective geometry, which is concerned with the whole of a diagram or a figure, that is to say, with its *global* properties, differential geometry concentrates on its *local* properties, that is, those arising in the immediate neighbourhoods of points of the figure, and which may vary from point to point. Thus, for example, one of the central concepts of differential geometry is the *curvature* of a

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<sup>2</sup> The term “differential geometry” was introduced in 1894 by the Italian mathematician *Luigi Bianchi* (1856–1928).

(plane) curve at a point, which is a quantity measuring the “sharpness of bending” of the curve in the neighbourhood of the point. More suggestively, if we think of the curve as being traced out by a moving particle, then the curvature at a given point is (proportional to) the rate at which the object’s path is changing direction at that point. The curvature of a straight line at any point on it is then zero, as is, more generally, that at a point of inflection of any curve, while that of a circle is constantly equal to the reciprocal of the length of its radius. This latter fact suggests that we define the *radius of curvature* of a curve  $C$  at a point  $P$  to be the *reciprocal* of the curvature at  $P$ . The radius of curvature is then the radius of the so-called *osculating*, or “kissing”, circle—



a term first introduced by Leibniz—to the curve at  $P$ . This circle is the one which approximates best to the curve at  $P$ ; its centre is called the *centre of curvature* of  $C$  at  $P$ .

Many of the basic concepts of differential geometry were introduced by *Christiaan Huygens* (1629–1675) in his *Horologium Oscillatorium* of 1673, a work largely devoted, as its title suggests, to the theoretical design of accurate pendulum clocks. Huygens obtained his results by means of purely geometrical methods and it was in fact *Newton* who, in his *Geometria Analytica* of 1736, first employs the methods of the calculus in this area, obtaining results essentially identical to those of Huygens. Thus, for example, both Huygens and Newton show that the centre of curvature of a curve at a point  $P$  on it is the limiting position of the point of intersection of a fixed normal<sup>3</sup>  $\mathbf{n}$  to a curve at  $P$  with an adjacent normal moving toward  $\mathbf{n}$ , as in the figure above. They both show that the radius of curvature of the curve at the point  $(x, y)$  is, using the notation of the calculus,

$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

The first steps in *three-dimensional* differential geometry were taken by *Clairault* in his *Recherche sur les Courbes à Double Courbure* of 1731, a work treating of both curves and surfaces. Clairault termed space curves “curves of double curvature” because he recognized that the projection of a space curve onto each of two perpendicular planes would give rise to a pair of curves each possessing an

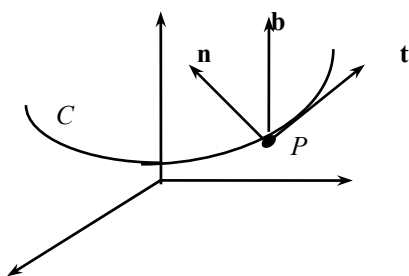
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<sup>3</sup> A *normal* to a curve at a point  $P$  on it is a straight line passing through  $P$  perpendicular to the tangent to the curve at  $P$ .

independent curvature. He also saw that a space curve can have an infinity of normals in a plane perpendicular to the tangent at a point.

The next important advances in the theory of space curves were made in 1774 and 1775 by *Leonhard Euler* (1707–1783). He introduces the concept of *osculating plane* to a space curve at a point on it, that is, the plane lying closest to the curve in the neighbourhood of the point. Both the tangent and the osculating circle to the curve at the point lie in this plane. Euler defines the *curvature* of the curve at the point to be the reciprocal of the length of the radius of the osculating circle to the curve at the point.

In 1806 *Michel-Ange Lancret* (1774–1807), a student of Monge, introduced the scheme, now known as the *moving trihedron*, by means of which space curves are analyzed today. Lancret singled out three principal directions at each point  $P$  of a space



curve  $C$ : the tangent  $\mathbf{t}$ , the normal  $\mathbf{n}$  in the osculating plane, and the common perpendicular to these, the *binormal*  $\mathbf{b}$ . It is helpful to think of these as constituting a rigid frame of rectangular axes, with  $P$  as origin, moving forward and rotating as  $P$  traverses the curve with unit speed. The angular velocity of the frame about the binormal is then the curvature at  $P$ . The angular velocity of the frame about the tangent—the *torsion* of the curve at  $P$ —represents the rate at which the curve deviates from a plane at the point. The significance of the curvature and torsion of a curve is that together they provide a complete description of the curve: if we are given both at each point of the curve, then the curve is uniquely determined except for spatial position. This fact became clear when the theory of space curves was brought into essentially its modern form by *Augustin-Louis Cauchy* (1789–1857) in his *Leçons sur les Applications du Calcul Infinitésimal à la Géométrie* of 1826.

### *The Theory of Surfaces*

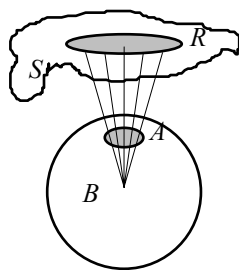
The origins of the differential geometry of surfaces began in the seventeenth century with the study of *geodesics*, that is, curves of least length on a surface (initially, in accordance with the needs of navigation, that of the earth). In 1697 *Jean Bernoulli* (1667–1748) posed the problem of finding the shortest curve joining two points on a convex surface, and in 1698 observed that the osculating plane at any point of such a curve on a surface is normal to the surface at that point. Also in 1698 *Jacques Bernoulli* (1655–1705) determined the geodesics on cylinders, cones, and surfaces of revolution.

The general form of the equations for geodesics on surfaces was obtained by Euler in 1728.

The theory of surfaces was placed on a sound basis by Euler in his *Recherches sur la Courbure des Surfaces* of 1760. In this work he introduces the *principal curvatures* of a surface at a point. These are defined as follows. Given a surface  $S$  and a point  $P$  on it, the assemblage of planes through a normal  $\mathbf{n}$  to  $S$  at  $P$  cuts  $S$  in a family of plane curves, each of which has a curvature at  $P$ . The two planes containing the curves with the largest and smallest curvatures can then be shown to be perpendicular to each other (assuming they are distinct). These two curvatures are then the principal curvatures of the surface  $S$  at  $P$ . It was shown in 1776 by *Jean-Baptiste Meunier* (1754–1793) that the only surfaces for which the two principal curvatures everywhere coincide are planes or spheres.

Euler was the first to study *developable* surfaces, that is, surfaces which can be flattened out onto a plane without distortion. This study had its origins in cartography, whose practitioners had come to recognize the awkward fact that a sphere cannot be cut and so flattened, thus making the task of mapping the earth's surface a subtle one. In his work *De Solidis quorum Superficium in Planum Explicare Licet* of 1771, Euler formulates necessary and sufficient conditions for a surface to be developable, and proves the striking result that the family of tangents to a space curve constitutes a developable surface.

In 1827 Gauss published his definitive paper on surfaces, *Disquisitiones Generales circa Superficies Curvas*, in which the study of the curvature of surfaces is raised to a new level. Gauss introduces a new measure of curvature, the *total* or *Gaussian curvature*, of a surface. To define Gaussian curvature, we consider a small region  $R$  on a surface  $S$ , and at each point of  $R$  we erect a normal to  $S$ . These normals



will form a cone, or solid angle, whose size is measured by the area of the region  $A$  in which a sphere  $B$  of unit radius intersects it. This size will depend both on the area of  $R$  and, crucially, on the extent to which it is curved. Thus the curvature of  $R$  may be characterized as the ratio of the size of the solid angle to the area of  $R$ . The total or Gaussian curvature of  $S$  at  $P$  is then defined to be the limit of this ratio as  $R$  shrinks to the point  $P$ .

Gauss proved that the total curvature of a surface at a point is the product of Euler's principal curvatures at that point. More significantly still, he established the following remarkable property of total curvature. Suppose that the surface has been stamped out from some flexible but inextensible material, a thin sheet of tin, for example, so that it

can be bent into various shapes without stretching or tearing it. During this process the principal curvatures will change but, as was shown by Gauss, their product, and hence the total curvature, will *remain unchanged at every point*. This shows that two surfaces with different Gaussian curvatures are *intrinsically distinct*, the distinction consisting in the fact that the surfaces can never be deformed without stretching or tearing in such a way as to enable them to be superposed on one another. Thus, for example, a segment of the surface of a sphere can never be distorted in this manner so as to lie flat on a plane or on a sphere of different radius. Total curvature accordingly furnishes a measure of the curvature of a surface which is *intrinsic*, that is, not dependent on the fact that the surface is part of three-dimensional space. The idea that surfaces have intrinsic geometries was to serve as the basis for Riemann's far-reaching generalization of geometry, to which we now turn.

### *Riemann's Conception of Geometry*

In his famous lecture of 1854, published as a paper in 1868, "On the Hypotheses which Lie at the Foundations of Geometry," Riemann introduces the idea of an intrinsic geometry for an arbitrary "space" which he terms a *multiply extended manifold*. Riemann conceives of a manifold as being the domain over which varies what he terms a *multiply extended magnitude*. Such a magnitude  $M$  is called *n-fold extended*, and the associated manifold *n-dimensional*, if  $n$  quantities—called *coordinates*—need to be specified in order to fix the value of  $M$ . For example, the position of a rigid body is a 6-fold extended magnitude because three quantities are required to specify its location and another three to specify its orientation in space. Similarly, the fact that pure musical tones are determined by giving intensity and pitch show these to be 2-fold extended magnitudes. In both of these cases the associated manifold is *continuous* in so far as each magnitude is capable of varying continuously with no "gaps". By contrast, Riemann terms *discrete* a manifold whose associated magnitude jumps discontinuously from one value to another, such as, for example, the number of leaves on the branches of a tree. Of discrete manifolds Riemann remarks:

Concepts whose modes of determination form a discrete manifold are so numerous, that for things arbitrarily given there can always be found a concept...under which they are comprehended, and mathematicians have been able therefore in the doctrine of discrete quantities to set out without scruple from the postulate that given things are to be considered as being all of one kind. On the other hand there are in common life only such infrequent occasions to form concepts whose modes of determination form a continuous manifold, that the positions of objects of sense, and the colours, are probably the only simple notions whose modes of determination form a continuous manifold. More frequent occasion for the birth and development of these notions is first found in higher mathematics.

The size of parts of discrete manifolds can be compared, says Riemann, by straightforward counting, and the matter ends there. In the case of continuous manifolds, on the other hand, such comparisons must be made by *measurement*. Measurement, however, involves superposition, and consequently requires the positing of some magnitude—not a pure number—independent of its place in the manifold. Moreover, in a continuous manifold, as we pass from one element to another in a

necessarily continuous manner, the series of intermediate terms passed through itself forms a one-dimensional manifold. If this whole manifold is now induced to pass over into another, each of its elements passes through a one-dimensional manifold, so generating a two-dimensional manifold. Iterating this procedure yields  $n$ -dimensional manifolds for an arbitrary integer  $n$ . Inversely, a manifold of  $n$  dimensions can be analyzed into one of one dimension and one of  $n - 1$  dimensions. Repeating this process finally resolves the position of an element into  $n$  magnitudes. These ideas are not dissimilar to those put forward by Grassmann in his *Ausdehnungslehre*.

Riemann thinks of a continuous manifold as a generalization of the three-dimensional space of experience, and refers to the coordinates of the associated continuous magnitudes as *points*. He was convinced that our acquaintance with physical space arises only *locally*, that is, through the experience of phenomena arising in our immediate neighbourhood. Thus it was natural for him to look to differential geometry to provide a suitable language in which to develop his conceptions. In particular, the *distance* between two points in a manifold is defined in the first instance only between points which are at *infinitesimal* distance from one another. This distance is calculated according to a natural generalization of the distance formula in Euclidean space. In  $n$ -dimensional Euclidean space, we recall that the distance  $\varepsilon$  between two points  $P$  and  $Q$  with coordinates  $(x_1, \dots, x_n)$  and  $(x_1 + \varepsilon_1, \dots, x_n + \varepsilon_n)$  is given by

$$\varepsilon^2 = \varepsilon_1^2 + \dots + \varepsilon_n^2. \quad (1)$$

In an  $n$ -dimensional manifold, the distance between the points  $P$  and  $Q$ —assuming that the quantities  $\varepsilon_i$  are infinitesimally small—is given by Riemann as the following generalization of (1):

$$\varepsilon^2 = \sum g_{ij} \varepsilon_i \varepsilon_j,$$

where the  $g_{ij}$  are functions of the coordinates  $x_1, \dots, x_n$ ,  $g_{ij} = g_{ji}$  and the sum on the right side, taken over all  $i, j$  such that  $1 \leq i, j \leq n$ , is always positive. The array of functions  $g_{ij}$  is called the *metric* of the manifold. In allowing the  $g_{ij}$  to be functions of the coordinates Riemann allows for the possibility that the nature of the manifold or “space” may vary from point to point, just as the curvature of a surface may so vary.

Riemann also extends Gauss’s concept of total curvature of a surface to manifolds. In doing so his goal was to characterize Euclidean space, and, more generally, spaces which are *homogeneous* in that within them figures can be moved about without change of size or shape. Like total curvature of a surface, Riemann’s notion of curvature of a manifold is an intrinsic property of the manifold (or, more precisely, of its metric); it is not required to think of the manifold as being situated in some manifold of higher dimension. Riemann introduces manifolds of *constant curvature*: by definition, in these spaces all measures of curvature are equal and remain unchanged from point to point, so yielding the required homogeneity property. In his paper Riemann states, but does not prove, that in a manifold of constant curvature the metric is given by

$$\varepsilon^2 = \frac{\sqrt{\sum \varepsilon_i^2}}{1 + \frac{1}{4}\alpha \sum x_i^2}.$$

Riemann observes that when  $\alpha$  is positive we obtain a spherical space, when  $\alpha = 0$ , a Euclidean (flat) space, and when  $\alpha$  is negative, a surface resembling the inside of a torus.

In the final section of his paper, Riemann applies his ideas to the problem of determining the structure of physical space. He points out that, as regards physical space, *infinitude* must be carefully distinguished from *boundlessness*. For example, the surface of a sphere is finite but unbounded and, for all we know, the same may be true of physical space. In any case it is the boundlessness, rather than the infinitude of space that is required for maneuvering in the external world, so that the former has a far greater empirical certainty than the latter. Moreover, if space has a constant positive measure of curvature, however small, then it would take the form of a spherical surface and would accordingly be finite and boundless.

Riemann concludes his discussion with the following words, the last sentence of which proved to be prophetic:

While in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in the case of a continuous manifold. Either then the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relations must be sought for outside that actuality, in colligating forces that operate on it. A decision on these questions can only be found by starting from the structure of phenomena that has been confirmed in experience hitherto...and by modifying the structure gradually under the compulsion of facts which it cannot explain...This path leads out into the domain of another science, into the realm of physics.

Riemann is saying, in other words, that if physical space is a continuous manifold, then its geometry cannot be derived *a priori*—as claimed, famously, by Kant—but can only be determined by experience. In particular, and again in opposition to Kant, who held that the axioms of Euclidean geometry were necessarily and exactly true of our conception of space, these axioms may have no more than approximate truth.

Riemann's final sentence proved to be prophetic because in 1916 his geometry—*Riemannian geometry*—was to provide the basis for a landmark development in physics, Einstein's celebrated *General Theory of Relativity*. In Einstein's theory, the geometry of space is determined by the gravitational influence of the matter contained in it, thus perfectly realizing Riemann's contention that this geometry must come from "somewhere else", to wit, from physics.

## TOPOLOGY

### *Combinatorial Topology*

The set of all *continuous (reversible) transformations* in space, i.e., deformations of figures or objects which occur without tearing anything apart, also forms a *group*, and the corresponding "geometry" is called *topology* (from Greek *topos*, "place"). By what

we have already said, topology must then involve the consideration of properties of figures which are invariant under any continuous transformation. Calling two figures *topologically equivalent* if each one can be continuously and reversibly deformed into the other, we say that a property of a figure is *topological* if, when possessed by a given figure, it is also possessed by all figures topologically equivalent to the given one. *Topology* may now be broadly defined as the study of topological properties.<sup>4</sup> We see immediately that, in general, projective properties such as, e.g., being a triangle or a straight line are not topological, since a triangle is evidently topologically equivalent to any simple closed curve and a straight line to any open curve. (To see this, imagine both triangle and line made from cooked pasta or modelling clay.) Topological properties are grosser than projective ones, since they must stand up under arbitrary continuous deformations. So, for example, a topological property of the triangle is not its triangularity but the property of dividing the plane into two—“inside” or “outside”—regions, as well as the property that, if two points are removed, it falls into two pieces, while if only a single point is removed, one piece remains. As another example we may consider the properties of *one-sidedness* or *two-sidedness* of a surface. The standard one-sided surface—the so-called *Möbius strip* (or band), discovered independently in 1858 by Möbius and *J. B. Listing*<sup>5</sup> (1806–1882)—may be constructed by gluing together the two ends of a strip of paper after giving one of the ends a half twist. Both one- and two-sidedness are topological properties.

The historical record shows that the first theorem of an indisputably topological nature is that proved by Euler<sup>6</sup> in 1752. This is the assertion that, for any simple polyhedron (i.e., closed, convex, and without holes), the numbers  $v$  of vertices,  $e$  of edges and  $f$  of faces are related by the equation

$$v - e + f = 2. \quad (1)$$

Here is a proof of this assertion, based on that given by Cauchy in 1811. Imagine the given simple polyhedron to be hollow and made of thin rubber. Now cut out one of its faces and deform the remaining surface so as to stretch it out flat on a plane. In this way the network of vertices and edges of the original polyhedron is flattened out into a similar network on the plane, without changing the number of either. However, the number of polygons in the plane is one less than in the original polyhedron, since one face has been removed. We show that, for the plane network,  $v - e + f = 1$ , so that, when the excised face is counted,  $v - e + f = 2$  for the original polyhedron.

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<sup>4</sup>In view of the fact that a doughnut and a coffee cup (with a handle) are topologically equivalent, John Kelley famously defined a *topologist* to be someone who cannot tell the two apart. On this light-hearted note, the published phrase containing the maximum number of references to topology and geometry must surely be

*On the analytic and algebraic topology of locally Euclidean metrization of infinitely differentiable Riemannian manifolds,*

part of a line in Tom Lehrer’s wittily irreverent “mathematical” song *Lobachevsky*.

<sup>5</sup> It was also Listing who, in his work *Vorstudien zur Topologie* of 1848, first uses the term “topology”; the subject being known prior to this as *analysis situs*, “positional analysis”.

<sup>6</sup>The content of the theorem appears already to have been known to Descartes a century earlier.

First, we convert the plane network into a number of linked triangles by drawing all the diagonals in polygons which are not already triangles. This does not affect the value of  $v - e + f$ , because each time a diagonal is drawn, the values of  $e$  and  $f$  are both increased by 1, while that of  $v$  is unaffected. Next, from any triangle lying on the boundary of the network we remove that portion which does not belong to any other triangle. Any such triangle has either one or two edges on the boundary. In the first case, just the outer edge and the face of the triangle are removed, so that  $e$  and  $f$  are both decreased by 1, while  $v$  is unchanged, and accordingly  $v - e + f$  remains the same. In the second case, the face, the two outer edges and one vertex of the triangle are removed, so that  $v$  and  $f$  are both diminished by 1, and  $e$  by 2, again leaving  $v - e + f$  unaffected. Continuing in this way we can remove triangles on the boundary (which of course changes each time a triangle is removed) until finally only a single triangle remains, with its three edges, three vertices, and one face. For this simple network,  $v - e + f$  clearly has value 1. Since the procedure of erasing triangles has not altered the value of  $v - e + f$ , this quantity must have value 1 for the original plane network, and hence also for the polyhedron with one face missing. So  $v - e + f = 2$  for the original polyhedron, as claimed.

The *topological* nature of Euler's relationship (although Euler himself failed to recognize it) may be discerned through the fact that equation (1) will continue to hold when the polyhedron is subjected to an arbitrary continuous deformation. Under such a deformation, the edges will, in general, cease to be straight, and the faces cease to be flat, but, nevertheless, they will remain edges and faces. Thus the deformation will preserve both the number of edges and the number of faces, as well as, of course, the number of vertices, so that the relationship above will remain valid, even though the procedure will cause the surface of the polyhedron to become curved. The vertices, edges and faces of the original polyhedron may be considered to constitute a *map* drawn on the resulting (curved) surface. When all the faces of the map so constituted are *triangles*—curved or rectilinear—it is known as a *triangulation* of the surface. As we saw in the proof of Euler's theorem, the number  $v - e + f$  for an arbitrary map is the same as for the triangulation obtained from it by drawing all the diagonals in its faces. Accordingly, as far as the value of  $v - e + f$  is concerned, we lose nothing by confining our attention to triangulations. This fact is the basis of the *combinatorial method* in topology, in which the properties of the surface are investigated by means of one of its triangulations<sup>7</sup>. Of course, for this purpose one only considers properties of the triangulation which are *independent* of its particular identity and thus, being common to all triangulations of the given surface, express some property of the surface itself.

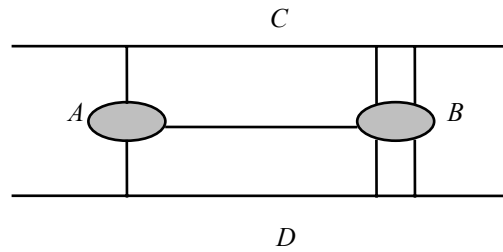
Euler's relationship (1) in fact yields one such property. Let us call the expression  $v - e + f$ , where  $v$ ,  $e$ , and  $f$  are, respectively, the number of vertices, edges and triangles of the given triangulation its *Euler characteristic*. Then Euler's theorem can be taken as asserting that for all triangulations of a surface topologically equivalent to a sphere the Euler characteristic is *two*. As it turns out, all triangulations of any given surface have the same Euler characteristic, which we call the *Euler characteristic of the given*

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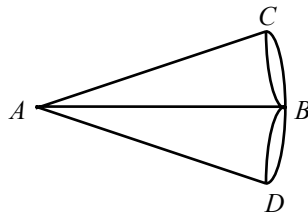
<sup>7</sup>This is an instance of the reduction of the *continuous* (in this case, a surface) to the *discrete* (in this case, the finite configuration provided by a triangulation). See Chapter 10.

surface. This number is a *topological invariant* in the sense that its value is the same for all topologically equivalent surfaces. For example, the Euler characteristic of a cylinder or a torus (doughnut) is zero, and that of a figure eight shaped pretzel is  $-2$ . In general, the Euler characteristic of a surface with  $k$  “holes” is  $2 - 2k$ .

Another famous problem whose topological nature came later to be recognized is the *Königsberg Bridge Problem*. Königsberg lies on the banks of the river Pregel, which contains two islands linked to each other and to the banks in the following way:



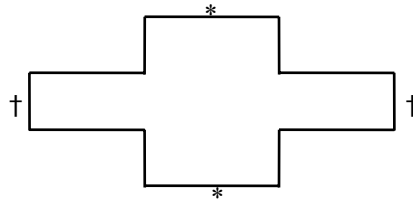
For many years the residents of Königsberg had tried without success to find a way of crossing all seven bridges exactly once on a continuous walk. Euler got wind of the problem and, in 1735, solved it in the negative. He achieved this by simplifying the formulation of the problem, replacing the land by points  $A, B, C, D$  and the linking bridges by arcs or lines as in the figure:



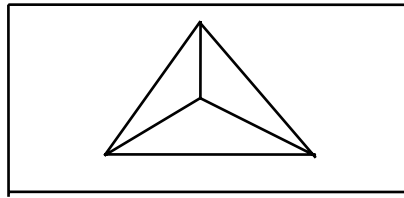
The entire configuration is called a *graph*; its points are called *vertices*, and its lines and arcs *edges*. The problem of crossing the bridges is thus reduced to that of traversing the graph in one continuous sweep of the pencil without lifting it from the paper. Calling a vertex *odd* or *even* according as the number of edges leading from it is odd or even, Euler discovered that a graph can be continuously traversed in the manner specified if all its vertices are even. If the graph contains at least one, but no more than two odd vertices, it can be traversed in one journey, but it is not possible to return to the starting point. In general, if the graph contains  $2n$  odd vertices, it will require exactly  $n$  journeys to traverse it. In the case of the Königsberg bridges  $n = 2$ .

It was Möbius who gave topology its first explicit formulation. In his *Theory of Elementary Relationships* of 1863 he proposed studying the relationship between two figures whose points can be placed in biunique correspondence in such a way that neighbouring points correspond to neighbouring points, that is, continuously. It is in this work that the technique of triangulation is first employed in a systematic way and used to show that any polyhedron may be systematically reduced to an assemblage of

triangles. Möbius also showed that certain curved surfaces could be dissected and displayed as polygons with sides properly identified. Thus, for example, a double torus can be represented as the polygon below, where edges marked alike are to be identified:



Another example of a topological property derives from the so-called *four-colour theorem*—first suggested by Möbius in 1840—which states that any map in the above sense drawn on the surface of a sphere (or a plane) can be coloured with at most four colours, in such a way that any two regions with a common boundary line are assigned different colours. The simplest map in the plane requiring four colours looks like this:

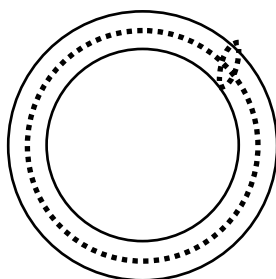


The problem of proving that four colours suffice for any map has an entertaining, if comparatively brief history, and includes what is probably the most infamous fallacious “proof” in the whole of mathematics. This was announced in 1879 by *A. B. Kempe* and for more than a decade was believed to be sound until *P. J. Heawood* spotted the flaw in 1890. By revising Kempe’s proof Heawood was, however, able to establish that *five* colours are sufficient. In 1976, *W. Haken* and *K. Appel* finally established that four are enough, but only by resorting to the use of a computer program to check several hundred basic maps. It is with a certain reluctance that mathematicians have come to accept this “proof”, since it cannot be examined in the traditional line-by-line manner that has always been taken for granted. The question of the legitimacy of “computer proofs” in mathematics has sparked a lively debate in recent years.

A major stimulus to the development of topology resulted from Riemann’s work in the theory of functions of a complex variable in the middle decades of the nineteenth century. He introduced the concept of *connectivity* of a surface, defining it in the following way:

If upon a surface there can be drawn  $n$  closed curves which neither individually nor in combination completely bound a part of the surface, but with whose aid every other closed curve forms the complete boundary of a part of the surface, then the surface is said to be  $(n + 1)$ -fold connected.

Riemann observes that the connectivity of a surface can be reduced by cutting it, giving as an example the torus, which is triply connected but can be reduced to a singly, or *simply* connected surface by means of one latitudinal and one longitudinal cut:



Riemann recognized that connectivity is a topological property and that it could be used to classify surfaces. In the latter half of the nineteenth century the term *genus* was introduced in connection with this classification. The genus of a surface is the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it: thus a sphere has genus 0, a torus genus 1 and, in general, a surface with  $p$  holes genus  $p$ . Riemann regarded it as intuitively evident that topologically equivalent surfaces have the same genus and observed that surfaces of genus 0 are topologically equivalent, each being mappable onto the surface of a sphere.

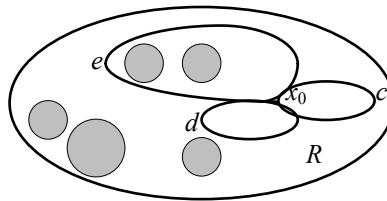
In 1882 Klein suggested a topological model for a surface of genus  $p$  which has become standard, namely, a sphere with  $p$  "handles" attached. It is clear that a closed surface of genus  $p$  can be continuously deformed into such an object, and so is topologically equivalent to it. Using this model it is quite easy to show that *the Euler characteristic of a surface  $S$  of genus  $p$  is  $2 - 2p$* . For  $S$  may be taken to be a sphere with  $p$  handles, and each handle is attached to the sphere at two bounding curves. Cut the sphere along one of these bounding curves for each handle, and straighten the handles out. Each handle will now have a free edge bounded by a new curve having the same number of edges and vertices. Thus the additional edges and vertices exactly counterbalance one another, and so, since no new faces have been created, the Euler characteristic remains unchanged. Now flatten out the protruding handles, so that the resulting surface is simply a sphere from which  $2p$  regions have been removed. Since the Euler characteristic of a sphere is 2, that of a sphere with  $2p$  holes is clearly  $2 - 2p$ , hence also for the sphere with  $p$  handles, and any surface of genus  $p$ .

The problem of the topological equivalence of closed surfaces was investigated intensively throughout the latter half of the nineteenth century. In 1874 a major advance was made by Klein, who showed that two orientable closed surfaces of the same genus are topologically equivalent. Here by an orientable surface is meant one equipped with a triangulation whose constituent triangles can be oriented in such a way that any side common to two triangles has opposite orientations induced on it. Thus any

two-sided surface such as a sphere or a plane is orientable, but a one-sided surface such as a Möbius strip is not.

By the end of the nineteenth century the only area of topology to have been more or less worked out was the topology of closed surfaces. Beginning in 1895 Poincaré began to develop the topological theory of spaces more general than surfaces, thereby establishing combinatorial topology as an autonomous branch of mathematics. In this general theory, later elaborated by the Dutch mathematician *L. E. J. Brouwer* (1882–1966), triangles are replaced by higher-dimensional objects known as *simplexes*—a one-dimensional simplex is a line segment, a two-dimensional simplex a triangle, a three-dimensional simplex a tetrahedron, and an  $n$ -dimensional simplex a generalized tetrahedron with  $n + 1$  vertices. The role of triangulations is assumed by *complexes*, where a complex is defined to be an assemblage  $K$  of simplexes any two of which meet, if at all, in a common face, and every face of a simplex in  $K$  is also in  $K$ . Using these ideas the concept of connectivity and the Euler characteristic can be extended to spaces of higher dimensions.

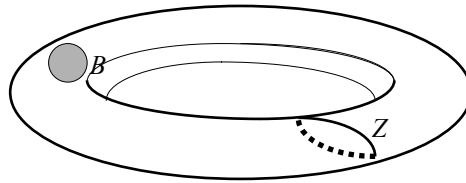
Another important idea introduced by Poincaré is the so-called *fundamental group* of a space, the initial concept of what was to evolve into the subject known today as *algebraic topology*. The idea arises from considering the distinction between simply and multiply connected plane regions. In the interior of a circle, for instance, every closed curve can be contracted to a point, while in an annulus or circular ring a closed curve enclosing the inner circular boundary cannot be so contracted. Now consider an arbitrary region  $R$  in the plane—which we take as a typical “space”—and the closed curves that begin and end at a given point  $x_0$  of  $R$ . Curves such as  $c$  and  $d$  in the figure below which can be continuously deformed into one another while remaining entirely within  $R$  are called *homotopic* and are considered to form one class. Roughly speaking,



a class in this sense—a *homotopy class*—comprises those curves going round the same holes the same number of times. Thus the closed curves starting and ending at  $x_0$  and not enclosing any holes form one homotopy class, while those such as  $e$  enclosing the two upper holes will form another. Two curves can be “merged” into the curve obtained by traversing first the one and then the other. This gives rise to an operation of *multiplication* on homotopy classes: the product of two homotopy classes  $C$ ,  $D$  is to consist of all the curves obtained by merging a member of  $C$  with a member of  $D$  (in the order specified). It can now be shown that with this operation the collection of all homotopy classes forms a group, known as the *fundamental group* of the space  $R$  with base point  $x_0$ . Usually the structure of the fundamental group is independent of the choice of base point, and the group is simply called the *fundamental group* of the space in question. In this way an algebraic structure—a group—is associated with each space.

The properties of the fundamental group reflect the connectivity of the space. Any simply connected space has a trivial fundamental group consisting of just the identity element, while the fundamental group of an annular region in the plane is isomorphic to the additive group of integers.

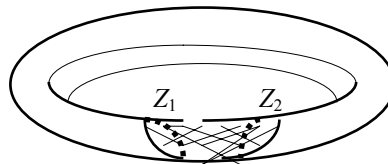
A further concept of algebraic topology which can be traced back to Poincaré is that of a *homology group*, a somewhat different device for charting the connectivity properties of the space. To explain the idea, consider the difference between the surface of a torus  $T$  and that of a sphere. It is intuitively obvious that any closed curve on the sphere cuts the surface into two separate regions. But this is not true in the case of the torus:



For cutting along the curve  $Z$  here does not disconnect the torus, which means that  $Z$  is not the boundary of a portion of  $T$ . But of course there are closed curves such as  $B$  which are boundaries.

Because the intuitive idea of a closed curve involves the notion that it “goes around something”, a closed curve such as  $B$  or  $Z$  is called a *cycle* on  $T$ . While  $B$  and  $Z$  are *simple* curves, i.e., do not cross themselves, we do not insist that all cycles be simple in this sense. The special cycles such as  $B$ —the *bounding* cycles—do not tell us very much about the structure of  $T$ , so for the moment we ignore them. It is the nonbounding cycles such as  $Z$  which are of interest.

Obviously there are infinitely many nonbounding cycles on the torus. But we shall see that these admit a simple description. First, the two cycles  $Z_1$  and  $Z_2$  in the figure on the page following are not intrinsically different since they both go round the torus once latitudinally. More significantly, however, taken together they form



the boundary of a portion of the torus, e.g. the shaded cylinder. The idea of taking cycles together suggests that they can be “added”. So let us extend our definition of cycle to include all unions of finitely many closed curves. Thus the operation of union gives a well-defined sum of two cycles, and we see that the sum  $Z_1 + Z_2$  of the two

cycles considered above is a bounding cycle. This leads to the idea of calling two cycles  $Z$  and  $Z'$  *homologous* if  $Z + Z'$  is a bounding cycle. This relation partitions cycles into mutually disjoint classes—the *homology classes*. The purely latitudinal cycles such as  $Z$  form one such homology class, the purely longitudinal ones another, and the bounding cycles still another. It is not difficult to show that every nonbounding cycle is homologous to a sum of a purely latitudinal cycle and a purely longitudinal one.

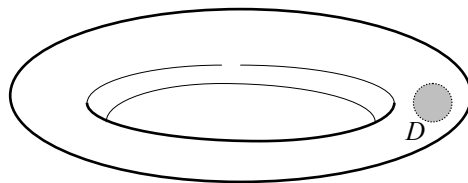
Now we can define an operation of *addition* on homology classes by stipulating that the sum of two such classes is to be the class consisting of the cycles obtained by adding a cycle of the first class to one of the second. With this operation the set of homology classes becomes a group—the *homology group* of the torus. It has 4 elements, namely the homology classes  $O, A, B, C$  consisting respectively of bounding cycles, purely latitudinal cycles, purely longitudinal cycles, and sums of cycles from  $A$  and  $B$ . It has neutral element  $O$ , and its addition table is:

$$A + A = B + B = C + C = O$$

$$A + B = B + A = C, \quad A + C = C + A = B, \quad B + C = C + B = A.$$

In a natural sense the two elements  $A$  and  $B$  may be said to *generate* the homology group of the torus; this corresponds to the fact that the torus has two “holes” around which curves can go latitudinally or longitudinally. Topologists express this by saying that the Betti number—named after the Italian mathematician *Enrico Betti* (1823–1892)—of the torus is 2.

Using the notions of simplex and complex already mentioned, homology groups can be constructed for each space. The number of generators of this group—the Betti number of the space—is the number of “holes” in the space. Both homology and fundamental groups thus reflect the connectivity of the space. In general, homology groups are considerably easier to determine than fundamental groups. But the latter are more sensitive, as can be seen by considering the surface—depicted below—obtained from a torus by removing an open disc  $D$ . While the fundamental group of this surface is clearly different from that of the torus, the homology group remains unchanged: the new “hole” is invisible as far as the latter is concerned.



To conclude our account of combinatorial topology, we mention its most outstanding open problem, the so-called *Poincaré conjecture*. In 1904 Poincaré conjectured that every simply connected, closed orientable three-dimensional space is topologically equivalent to the three-dimensional sphere. Informally, the conjecture

says that the three-dimensional sphere is the only possible type of bounded three-dimensional space containing no holes. This famous claim remains unresolved. Remarkably, however, the corresponding assertion for dimensions  $\geq 4$  has been established.

### *Point-set Topology*

The key feature which distinguishes topology from other branches of geometry is its concern with *continuity*, a notion which in turn involves the idea of *proximity*. Thus a continuous function is, intuitively, one which does not pull “neighbouring” points of its domain “too far” apart. But what exactly do these terms mean? In the first decades of the twentieth century the use of *set theory* in developing *point-set* or *general topology* enabled these concepts to be rigorously defined, thereby placing the whole subject of topology on a solid basis.

The central concept of point-set topology is that of *topological space*. This concept may be defined in a variety of equivalent ways; the one given here—due originally to *Felix Hausdorff* (1868–1942)—is framed in terms of the concept of *neighbourhood*.

Suppose we are given a set  $X$  of elements which may be such entities as points in  $n$ -dimensional space, plane or space curves, complex numbers or quaternions, although we make no assumptions as to their exact nature. The members of  $X$  will be called *points*. Now suppose in addition that corresponding to each point  $x$  of  $X$  we are given a collection  $N_x$  of subsets of  $X$ , which we call the *neighbourhoods* of  $x$ , subject to the following conditions:

- (i) each member of  $N_x$  contains  $x$ ;
- (ii) the intersection of any pair of members of  $N_x$  includes a member of  $N_x$ ;
- (iii) for any  $U$  in  $N_x$  and any  $y$  in  $U$ , there is  $V$  in  $N_y$  such that  $V \subseteq U$ .

We may think of each  $U$  in  $N_x$  as determining a notion of proximity to  $x$ : the points of  $U$  are accordingly said to be  *$U$ -close* to  $x$ . Using this terminology, (i) may be construed as saying that, for any neighbourhood  $U$  of  $x$ ,  $x$  is  $U$ -close to itself; (ii) that, for any neighbourhoods  $U$  and  $V$  of  $x$ , there is a neighbourhood  $W$  of  $x$  such that any point  $W$ -close to  $x$  is both  $U$ -close and  $V$ -close to  $x$ ; and (iii) that, for any neighbourhood  $V$  of  $x$ , any point which is  $U$ -close to  $x$  itself has a neighbourhood all of whose points are  $U$ -close to  $x$ .

A *topological space* may now be defined to be a set  $X$  together with an assignment, to each point  $x$  of  $X$ , of a collection  $N_x$  of subsets of  $X$  satisfying conditions (i) – (iii). As examples of topological spaces we have: the real line  $\mathbb{R}$  with the neighbourhood system at a point consisting of all open intervals with rational radii centred on that point; the Euclidean plane with the neighbourhood system at a point consisting of all open discs with rational radii centred on that point; and, in general,  $n$ -dimensional Euclidean space with the neighbourhood system at a point consisting of all open  $n$ -spheres centred on that point. Any subset  $X$  of any of these spaces becomes a

topological space by taking as neighbourhoods the intersections with  $X$  of the neighbourhoods in the containing space.

Most topological notions can be defined entirely in terms of neighbourhoods. Thus a *limit point* of a set of points in a topological space is one each of whose neighbourhoods contains points of the set: a limit point of a set is thus a point which, while not necessarily in the set, is nevertheless “arbitrarily close” to it (lying on its boundary, for instance). A set is *open* if it includes a neighbourhood of each of its points, *closed* if it contains all its limit points, and *connected* if no matter how it is split into two disjoint sets, at least one of these contains limit points of the other. For example, any interval on the real line, and any Euclidean space, is connected in this sense.

The notions of continuous function and topological equivalence can also be defined in terms of neighbourhoods. Thus a function  $f: X \rightarrow Y$  between two topological spaces  $X$  and  $Y$  is said to be *continuous*, if, for any point  $x$  in  $X$ , and any neighbourhood  $V$  of the image  $f(x)$  of  $x$ , a neighbourhood  $U$  of  $x$  can be found whose image under  $f$  is included in  $V$ , in other words, such that the images of any point  $U$ -close to  $x$  is  $V$ -close to  $f(x)$ . In the case when  $X$  and  $Y$  are both the real line  $\mathbb{R}$ , this condition translates into the famous  $\varepsilon, \delta$  condition for continuity: for any  $x$  and for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  whenever  $|x - y| < \delta$ . A *topological equivalence* or *homeomorphism* between topological spaces is a biunique function which is continuous in both directions, that is, both it and its inverse are continuous.

Metric spaces constitute an important class of topological spaces. A *metric* on a set  $X$  is a function  $d$  which assigns to each pair  $(x, y)$  of elements of  $X$  a nonnegative real number  $d(x, y)$  in such a way that:

$$d(x, y) = d(y, x), \quad d(x, y) + d(y, z) \geq d(x, z), \quad d(x, y) = 0 \text{ if and only if } x = y.$$

We think of  $d(x, y)$  as the *distance* between  $x$  and  $y$ . The first of these conditions then expresses the symmetry of distance, and the second—the *triangle inequality*—corresponds to the familiar fact that the sum of the lengths of two sides of a triangle is never exceeded by that of the remaining side. A set equipped with a metric is called a *metric space*. Each metric space may be considered a topological space in which a typical neighbourhood of a point  $x$  is the subset of  $X$  consisting of all points  $y$  at distance  $< r$ , for positive  $r$ . All of the examples of topological spaces given above are metric spaces, but not every topological space is a metric space. Metric spaces are the topological generalizations of Euclidean spaces.

In Euclidean spaces one has a natural definition of *dimension*, namely, the number of coordinates required to identify each point of the space. In a general topological space there is no mention of coordinates and so this definition is not applicable. In 1912 Poincaré formulated the first definition of dimension for a topological space; this was later refined by Brouwer, *Paul Urysohn* (1898–1924) and *Karl Menger* (1902–1985). The definition in general use today is formulated in terms of the *boundary* of a subset of a topological space, which is defined to be the set of all points of the space which are limit points both of the subset and its complement. Now the topological dimension of a subset  $M$  of a topological space is defined inductively as follows: the empty set is assigned dimension  $-1$ ; and  $M$  is said to be  $n$ -dimensional at a point  $P$  if  $n$  is the least number for which there are arbitrarily small neighbourhoods of  $P$  whose

boundaries in  $M$  all have dimension  $< n$ . The set  $M$  has topological dimension  $n$  if its dimension at all of its points is  $\leq n$  but is equal to  $n$  at one point at least.

Dimension thus defined is a topological property. Menger and Urysohn defined a curve as a one-dimensional closed connected set of points, so rendering the property of being a curve a topological property. In 1911 Brouwer proved the important result that  $n$ -dimensional Euclidean space has topological dimension  $n$ . This means that Euclidean spaces of different dimensions cannot be topologically equivalent, even though—as Cantor showed (see Ch. 11)—they have the same “number” of points.