THE CONTINUUM IN SMOOTH INFINITESIMAL ANALYSIS

by

John L. Bell

As presented in [1], *smooth infinitesimal analysis*, **SIA**, is a theory formulated within higher-order intuitionistic logic and based on the following axioms:

Axioms for the continuum, or smooth real line R. These are the usual axioms for a(n) (intuitionistic) field expressed in terms of two operations + and •, and two distinguished elements 0, 1.

Axioms for the strict order relation < on R. These are:

1. $a < b$ and $b < c$ implies $a < c$.
2. $\neg (a < a)$
3. $a < b$ implies $a + c < b + c$ for any c.
4. $a < b$ and $0 < c$ implies $a.c < b.c$.
5. either $0 < a$ or $a < 1$.
6. $a \neq b$[1] implies $a < b$ or $b < a$.

The relation $\leq$ on **R** is defined by $a \leq b \Leftrightarrow \neg b < a$. The open interval (a, b) and closed interval $[a, b]$ are defined as usual, viz. $(a, b) = \{x\colon a < x < b\}$ and $[a, b] = \{x\colon a \leq x \leq b\}$; similarly for half-open, half-closed, and unbounded intervals.

---

[1] Here $a \neq b$ stands for $\neg a = b$. It should be pointed out that axiom 6 is omitted in some presentations of **SIA**, e.g. those in [3] and [4].

Write Δ for the subset $\{x: x^2 = 0\}$ of **R**; we use the letter ε as a variable ranging over Δ. Elements of Δ are called (nilsquare) *infinitesimals* or *microquantities.* Since, clearly, $0 \in \Delta$**,** Δ may be regarded as an *infinitesimal neighbourhood of* 0. Δ is subject to the

Microaffineness Principle. *For any map g: $\Delta \to$* **R** *there exist unique a, $b \in$* **R** *such that, for all ε, we have*

$$g(\varepsilon) = a + b.\varepsilon.$$

Notice that then $a = g(0)$.

From these three axioms it follows that the continuum in **SIA** differs in certain key respects from its counterpart in *constructive analysis* **CA**, which is furnished with an elegant axiomatization in [2].

To begin with, the third basic property of the strict ordering relation < in **CA**, given as axiom R2(3) on p.102 of [2], and which may be written

$$(*) \qquad \neg(x < y \vee y < x) \to x = y$$

is incompatible with the axioms of **SIA**. For (*) implies

$$(**) \qquad \forall x \neg(x < 0 \vee 0 < x) \to x = 0.$$

But in **SIA** we have by Exercise 1.6 and Thm. 1.1 (i) of [1],

$$\forall x \in \Delta \neg(x < 0 \vee 0 < x) \wedge \Delta \neq \{0\},$$

which clearly contradicts (**).

Thus in **CA** the set Δ of infinitesimals would be degenerate (i.e., identical with $\{0\}$), while the nondegeneracy of Δ in **SIA** is one of its characteristic features.

Next, call a binary relation S on **R** *stable* if it satisfies

$$\forall x \forall y\ (\neg\neg xRy \to xRy).$$

In **CA**, the equality relation is stable, a fact which again follows from principle $R2(3)$ referred to above. But in **SIA** it is not stable, for, as shown in Thm. 1.1(ii) of [1], there we have $\forall x \in \Delta \neg\neg x = 0$. If = were stable, it would follow that $\forall x \in \Delta\ x = 0$, in other words, that Δ is degenerate, which is not the case in **SIA.**

Axiom 6 of **SIA**, together with the transitivity and irreflexivity of $<$, implies that $<$ is stable. This may be seen as follows. Suppose $\neg\neg a < b$. Then certainly $a \neq b$, since $a = b \to \neg a < b$ by irreflexivity. Therefore $a < b$ or $b < a$. The second disjunct together with $\neg\neg a < b$ and transitivity gives $\neg\neg a < a$, which contradicts $\neg a < a$. Accordingly we are left with $a < b$. As can be deduced from assertion 8 on p. 103 of [2], the stability of $<$ implies *Markov's principle*, which is not affirmed in **CA.**[2]

A subset $A \subseteq \mathbf{R}$ is *indecomposable* if it admits only trivial partitionings, that is, if $A = U \cup V$ and $U \cap V = \varnothing$, then $U = \varnothing$ or $V = \varnothing$. Clearly A is indecomposable iff any map $f\colon A \to 2 = \{0, 1\}$ is constant.

In **SIA** one also assumes the

Constancy Principle. *If* $A \subseteq \mathbf{R}$ *is any closed interval on* **R,** *or* **R** *itself, and* $f\colon A \to \mathbf{R}$ *satisfies* $f(a + \varepsilon) = f(a)$ *for all* $a \in A$ *and* $\varepsilon \in \Delta$, *then* f *is constant.*

As shown in Thm. 2.1.of [1], it follows in **SIA** from the Constancy Principle that **R** *itself and each of its closed intervals is indecomposable.* From this we can deduce that in **SIA** *all intervals in* **R** *are indecomposable.* To do this we employ the following

Lemma. Suppose that A is an inhabited subset of **R** satisfying

(*) for any $x, y \in A$ there is an indecomposable set B such that $\{x, y\} \subseteq B \subseteq A$.

Then A is indecomposable.

Proof. Suppose A satisfies (*) and $A = U \cup V$ with $U \cap V = \varnothing$. Since A is inhabited, we may choose $a \in A$. Then $a \in U$ or $a \in V$. Suppose $a \in U$; then if $y \in V$ there is an indecomposable B for which $\{a, y\} \subseteq B \subseteq A = U \cup V$. It follows that $B = (B \cap U) \cup (B \cap V)$, whence $B \cap U = \varnothing$ or

---

[2] In versions of **SIA** that omit axiom 6 neither the stability of $<$, nor Markov's principle, can be derived.

$B \cap V = \varnothing$. The former possibility is ruled out by the fact that $a \in B \cap U$, so $B \cap V = \varnothing$, contradicting $y \in B \cap V$. Therefore $y \in V$ is impossible; since this is the case for arbitrary y, we conclude that $V = \varnothing$. Similarly, if $a \in V$, then $U = \varnothing$, so that A is indecomposable as claimed.

We use this lemma to show that the open interval (0, 1) is indecomposable; similar arguments work for arbitrary intervals. In fact, if $\{x, y\} \subseteq (0, 1)$, it is easy to verify that

$$\{x, y\} \subseteq [xy/x+y,\ 1-xy/2-x-y] \subseteq (0, 1).$$

Thus, in view of the indecomposability of closed intervals, (0, 1) satisfies condition (*) of the lemma, and so is indecomposable.

Aside from certain infinitesimal subsets to be discussed below, in **SIA** indecomposable subsets of **R** correspond to connected subsets of **R** in classical analysis, that is, to intervals. In particular, any puncturing of **R** is *decomposable*, for it follows immediately from Axiom 6 that

$$\mathbf{R} - \{a\} = \{x\colon x > a\} \cup \{x\colon x < a\}.$$

Similarly, the set $\mathbf{R} - \mathbb{Q}$ of irrational numbers is decomposable as

$$\mathbf{R} - \mathbb{Q} = [\{x\colon x > 0\} - \mathbb{Q}] \cup [\{x\colon x < 0\} - \mathbb{Q}\}.$$

This is in sharp contrast with the situation in *intuitionistic analysis* **IA**, that is, **CA** augmented by Kripke's scheme, the continuity principle, and bar induction. For it is shown in [5] that in **IA** not only is any puncturing of **R** indecomposable, but that this is even the case for the set of irrational numbers (further indecomposabilty results for **IA** may be found in [6].) This would seem to indicate that in some sense the continuum in **SIA** is considerably less "syrupy" [3] than its counterpart in **IA.**

It can be shown that the various "infinitesimal" subsets of **R** introduced in [1] are indecomposable. For example, the indecomposability of Δ can be established as follows. Suppose $f\colon \Delta \to \{0, 1\}$. Then by Microaffineness there are unique $a, b \in \mathbf{R}$ such that $f(\varepsilon) =$

---

[3] It should be emphasized that this phenomenon is a consequence of axiom 6: it cannot necessarily be

$a + b.\varepsilon$ for all ε. Now $a = f(0) = 0$ or 1; if $a = 0$, then $b.\varepsilon = f(\varepsilon) = 0$ or 1, and clearly $b.\varepsilon \neq 1$. So in this case $f(\varepsilon) = 0$ for all ε. If on the other hand $a = 1$, then $1 + b.\varepsilon = f(\varepsilon) = 0$ or1; but $1 + b.\varepsilon = 0$ would imply $b.\varepsilon = -1$ which is again impossible. So in this case $f(\varepsilon) = 1$ for all ε. Therefore f is constant and Δ indecomposable.

In **SIA** *nilpotent infinitesimals* are defined to be the members of the sets

$$\Delta_k = \{x \in \mathbf{R}: x^{k+1} = 0\},$$

for $k = 1, 2, \ldots$, each of which may be considered an infinitesimal neighbourhood of 0. These are subject to the

Micropolynomiality Principle. *For any* $k \geq 1$ *and any* $g: \Delta_k \to \mathbf{R}$, *there exist unique* $a, b_1, \ldots, b_k \in \mathbf{R}$ *such that for all* $\delta \in \Delta_k$ *we have*

$$g(\delta) = a + b_1\delta + b_2\delta^2 + \ldots + \mathrm{b_k}\,\delta^{\mathrm{k}}.$$

Micropolynomiality implies that no Δ_k coincides with $\{0\}$.

An argument similar to that establishing the indecomposability of Δ does the same for each Δ_k. Thus let $f: \Delta_k \to \{0, 1\}$; Micropolynomiality implies the existence of $a, b_1, \ldots, b_k \in \mathbf{R}$ such that $f(\delta) = a + \zeta(\delta)$, where $\zeta(\delta) = b_1\delta + b_2\delta^2 + \ldots + b_k\delta^k$. Notice that $\zeta(\delta) \in \Delta_k$, that is, $\zeta(\delta)$ is nilpotent. Now $a = f(0) = 0$ or 1; if $a = 0$ then $\zeta(\delta) = f(\delta) = 0$ or 1, but since $\zeta(\delta)$ is nilpotent it cannot =1. Accordingly in this case $f(\delta) = 0$ for all $\delta \in \Delta_k$. If on the other hand $a = 1$, then $1 + \zeta(\delta) = f(\delta) = 0$ or 1, but $1 + \zeta(\delta) = 0$ would imply $\zeta(\delta) = -1$ which is again impossible. Accordingly f is constant and Δ_k indecomposable.

The union **D** of all the Δ_k is the *set of nilpotent infinitesimals,* another infinitesimal neighbourhood of 0. The indecomposability of **D** follows immediately by applying the Lemma above.

---

affirmed in versions of **SIA** not including this axiom.

The next infinitesimal neighbourhood of 0 is the closed interval [0, 0], which, as a closed interval, is indecomposable. It is easily shown that [0, 0] includes **D,** so that it does not coincide with {0}.

It is also easily shown, using axioms 2 and 6, that [0, 0] coincides with the set

$$\mathbf{I} = \{x \in \mathbf{R}: \neg\neg x = 0\}.$$

So **I** is indecomposable. (In fact the indecomposability of I can be proved independently of axioms 1-6 through the general observation that, if A is indecomposable, then so is the set $A^* = \{x: \neg\neg x \in A\}$.)

Finally, we observe that the sequence of infinitesimal neighbourhoods of 0 generates a strictly ascending sequence of decomposable subsets containing **R** – {0}, namely:

$$\mathbf{R} - \{0\} \subset (\mathbf{R} - \{0\}) \cup \{0\} \subset (\mathbf{R} - \{0\}) \cup \Delta_1 \subset (\mathbf{R} - \{0\}) \cup \Delta_2 \subset \ldots (\mathbf{R} - \{0\}) \cup \mathbf{D} \subset (\mathbf{R} - \{0\}) \cup [0, 0].$$

Note

It was hearing Dirk van Dalen's stimulating talk at the conference that got me thinking about the topic discussed in this paper. I would like to thank Peter Schuster for encouraging me to write it, and the referee for helpful suggestions.

References.

[1] Bell, John L. *A Primer of Infinitesimal Analysis.* Cambridge University Press, 1998.

[2] Bridges, Douglas. *Constructive mathematics: a foundation for computable analysis.* Theoretical Computer Science 219 (1999), 95-109.

[3] Kock, A. *Synthetic Differential Geometry*. Cambridge University Press, 1981.

[4] McLarty, C. *Elementary Categories, Elementary Toposes.* Oxford University Press, 1992.

[5] van Dalen, Dirk. *How connected is the intuitionistic continuum?* Journal of Symbolic Logic, 62, no. 4, 1997, 1147-1150.

[6] van Dalen, D. *From Brouwerian counterexamples to the creating subject.* Studia Logica **62** (1999), no. 2, 305-314.

Department of Philosophy
University of Western Ontario.