Precovers, Modalities and Universal Closure Operators in a Topos

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Abstract. In this paper we develop the notion of formal precover in a topos by defining a relation between elements and sets in a local set theory. We show that such relations are equivalent to modalities and to universal closure operators. Finally we prove that these relations are well characterized by a convenient restriction to a particular set.

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1 Introduction

The concept of precover is an essential constituent of that of pretopology, which was formulated by Giovanni Sambin in [4] in order to provide a semantics for linear logic. The concept of pretopology may itself be regarded as a generalization of the notion of formal topology, which was introduced by Sambin as part of a program for developing an intuitionistic approach to pointless topology, that is, topology in which the basic concept is that of neighbourhood and points are defined as particular filters of neighbourhoods (see [3] and [4]).

In this paper we develop the notion of precover in an elementary topos. Our arguments will in fact be formulated within the so-called local set theories and the linguistic toposes associated with them (see [1]). After a brief review of the relevant definitions, we show how each modality within a local set theory induces a relation possessing the reflexivity and transitivity properties of a precover. This relation, defined by John Bell in [2] and called universal precover, extends the notion of precover to local set theories, and hence to elementary toposes. We show that the concept of universal precover is in fact equivalent to that of modality, and hence to that of universal closure operation in a local set theory. We also observe that the notion of universal precover is a natural generalization of the membership relation within a local set theory.

1) We would like to express our gratitude to Giovanni Sambin for his assistance.
To define a universal precover in a local set theory we have to assign a relation to each "set" (or to each object of the associated topos). We show that, when the set is a monoid, the assigned relation actually becomes a precover in the original sense of SAMBIN. Finally, we show that a universal precover may be obtained by assigning such a relation just to the truth-value set of the local set theory.

2 Precovers and pretopologies

Definition. Let \( (S, \cdot, 1) \) be a commutative monoid. A precovering relation, or precover, on \( S \) is a relation \( \cdot \) between elements and subsets of \( S \), satisfying the following conditions:

(i) If \( a \in U \), then \( a \cdot U \) (reflexivity);
(ii) if \( a \cdot U \) and \( U \cdot V \), then \( a \cdot V \), where \( U \cdot V \equiv \forall b \in U (b \cdot V) \) (transitivity);
(iii) if \( a \cdot U \) and \( b \cdot V \), then \( a \cdot b \cdot U \cdot V \) (stability).

A pretopology is a quadruple \( \mathcal{F} \equiv (S, \cdot, 1, \mathcal{G}_\mathcal{F}) \), where \( (S, \cdot, 1) \) is a commutative monoid and \( \mathcal{G}_\mathcal{F} \) is a precovering relation on it.

A precover on \( S \) can equivalently be presented as an operator \( \mathcal{F} : P(S) \rightarrow P(S) \) defined by

\[ \mathcal{F}(U) = \{ a \in S : a \cdot \mathcal{G}_\mathcal{F} U \} . \]

It follows from properties (i) and (ii) that \( \mathcal{F} \) is a closure operator on \( S \) (see [4]). \( \mathcal{F}(U) \) is called the \( \mathcal{F} \)-saturation of \( U \) and \( U \) is called \( \mathcal{F} \)-saturated if \( \mathcal{F}(U) = U \).

The collection \( \text{Sat}(\mathcal{F}) \) of \( \mathcal{F} \)-saturated subsets of \( S \) is then a complete lattice with respect to \( \subseteq \) in which arbitrary infima and suprema are defined by

\[ \bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i \quad \text{and} \quad \bigvee_{i \in I} U_i = \mathcal{F}(\bigcup_{i \in I} U_i) . \]

Furthermore, if we define the operation \( \cdot \mathcal{F} \) by

\[ U \cdot \mathcal{F} V = \mathcal{F}(U \cdot V) , \]

for any \( U, V \in \text{Sat}(\mathcal{F}) \),

\( (\text{Sat}(\mathcal{F}), \cdot \mathcal{F}) \) becomes a commutative monoid. The only link between the algebraic and the lattice structure is infinite distributivity, that is to say

\[ V \cdot \mathcal{F} \bigvee_{i \in I} U_i = \bigvee_{i \in I} (V \cdot \mathcal{F} U_i) . \]

If \( \cdot \mathcal{F} \) coincides with \( \wedge \) or when we require \( (\text{Sat}(\mathcal{F}), \cdot \mathcal{F}) \) to be a semilattice with ordering \( \subseteq \), \( \text{Sat}(\mathcal{F}) \) is a complete Heyting algebra [4].

3 Local set theories and linguistic toposes

A linguistic topos is a topos built up from a language in a way resembling the construction of Lindenbaum algebras. It may be regarded as a "model" for a local set theory, which is a type theoretic system built up from the primitive symbols \( \in, =, \) and \{ : \} in a language whose basic axioms and rules are formulated so as to yield exactly the theorems of intuitionistic logic (see [1]).

To sketch the construction of a linguistic topos, we begin by describing a local language. A local language is a typed language with no logical operations among
its primitive symbols, in which are specified two particular type symbols: $1$ and $\Omega$, called the unity type and the truth value type, respectively. Its terms must include the following:

1) $\ast$ of type $1$;
2) $\{x_A : \alpha\}$ of type $P(A)$, where $x_A$ is a variable of any type $A$ and $\alpha$ a term of type $\Omega$;
3) $\sigma = \tau$ of type $\Omega$, where $\sigma$ and $\tau$ are terms of the same type;
4) $\sigma \in \tau$ of type $\Omega$, where $\sigma$ and $\tau$ are terms of type $A$ and $P(A)$, respectively;
5) $(\tau_1, \ldots, \tau_n)$ of type $A_1 \times \cdots \times A_n$, where $\tau_i$ is of type $A_i$ ($i = 1, \ldots, n$);
6) $(\tau)_i$ of type $A_i$, where $\tau$ is of type $A_1 \times \cdots \times A_n$.

An occurrence of a variable $x$ in a term is bound if it appears within a context of the form $\{x : \alpha\}$, otherwise it is free. A term of type $\Omega$ is called a formula.

The logical operations among the terms in the language can be defined as follows, where $\alpha, \beta$ are terms of type $\Omega$.

\[
\begin{align*}
\alpha \equiv \beta & \quad \text{for} \quad \alpha = \beta, \\
true & \quad \text{for} \quad \ast = \ast, \\
\alpha \land \beta & \quad \text{for} \quad \langle \alpha, \beta \rangle = \langle true, true \rangle, \\
\alpha \rightarrow \beta & \quad \text{for} \quad (\alpha \land \beta) = \alpha, \\
\forall x \alpha & \quad \text{for} \quad \{x : \alpha\} = \{x : true\}, \\
false & \quad \text{for} \quad \forall \omega, \omega \text{ a variable of type } \Omega, \\
\neg \alpha & \quad \text{for} \quad \alpha \rightarrow false, \\
\alpha \lor \beta & \quad \text{for} \quad \forall \omega[\langle (\alpha \rightarrow \omega) \land (\beta \rightarrow \omega) \rangle \rightarrow \omega], \\
& \quad \omega \text{ a variable of type } \Omega \text{ not occurring in } \alpha \text{ or } \beta, \\
\exists x \alpha & \quad \text{for} \quad \forall \omega[\forall z (\alpha \rightarrow \omega) \rightarrow \omega], \\
& \quad \omega \text{ a variable of type } \Omega \text{ not occurring in } \alpha.
\end{align*}
\]

To describe the axioms and the inference rules of a local set theory we use the sequent notation $\Gamma : \alpha$ in which $\alpha$ is a formula and $\Gamma$ is a (possibly empty) finite set of formulas.

The basic axioms of a local set theory are the following:

- tautology $\emptyset : \alpha = \alpha$,
- unity $\emptyset : x_1 = \ast$,
- equality $x = y, \alpha(z/x) : \alpha(z/y)$, with $x$ and $y$ free for $z$ in $\alpha$,
- products $\emptyset : ((x_1, \ldots, x_n))_i = x_i$ and $\emptyset : x = ((x_1), \ldots, (x)_n)$,
- comprehension $\emptyset : x \in \{x : \alpha\} \leftrightarrow \alpha$.

The rules of inference of a local set theory are the following: thinning, cut (with the limitation that any free variable of the cut-formula is free in the antecedents or
in the succedents of the sequents, different from $\alpha$), and

- **substitution**: $\Gamma : \alpha \quad \frac{}{\Gamma(x/\tau) : \alpha(x/\tau)}$, $\tau$ free for $x$ in $\Gamma$ and $\alpha$,

- **extensionality**: $\frac{\Gamma : x \in \sigma \quad x \notin \Gamma, \sigma, \tau, \Gamma : \sigma = \tau}{\Gamma : \alpha}$, $x$ not free in $\Gamma, \sigma, \tau$,

- **equivalence**: $\frac{\alpha, \Gamma : \beta \quad \beta, \Gamma : \alpha}{\Gamma : \alpha \iff \beta}$.

A *proof* from $S$ is defined in the same way as in the sequent calculus and a sequent $\Gamma : \alpha$ is said to be *derivable* from $S$ if there is a proof from $S$ of which $\Gamma : \alpha$ is the conclusion. We will write $\Gamma \vdash_S \alpha$ to indicate that $\Gamma : \alpha$ is derivable from $S$, and $\vdash_S \alpha$ for $\emptyset \vdash_S \alpha$. A *local set theory* is a theory in a local language, that is to say a collection of sequents, $S$, which is closed under derivability.

Linguistic toposes are constructed as follows. In a local language the terms of power type (if $A$ is a type symbol $P(A)$ is called the *power type of A*) are called *set-like terms* and closed set-like terms are called $L$-sets or simply *sets*. We consider the equivalence classes, called $S$-*sets*, of $L$-sets under the relation $\sim_S$ defined in the following way:

$X \sim_S Y \quad \text{iff} \quad \vdash_S X = Y$.

An *$S$-map* $f : X \to Y$ is a triple $(f, X, Y)$ of $S$-sets such that $\vdash_S f \in Y^X$, where $Y^X$ is defined as

$\{ u : u \subseteq X \times Y \quad \forall x (x \in X \to \exists y (y \in Y \land (x, y) \in u)) \}$.

The collection of $S$-sets and maps then forms a topos which is called a *linguistic topos*. It can be shown (see [1]) that any elementary topos is equivalent to a linguistic topos.

### 4 Modalities

In this paragraph we will show how, given a local set theory $S$ and a modality on $S$ (defined below), we can obtain a new relation between elements and $S$-sets which generalizes the membership relation. We will also see that saturating the $S$-sets with respect this new relation yields a universal closure operation.

**Definition 4.1.** Let $S$ be a local set theory and $\Omega$ the $S$-set $\{ \omega : \text{true} \}$. A *modality* is an $S$-map $\mu : \Omega \to \Omega$ such that

1. $\alpha \vdash_S \mu(\alpha)$;
2. if $\alpha \vdash_S \beta$, then $\mu(\alpha) \vdash_S \mu(\beta)$;
3. $\mu(\mu(\alpha)) \vdash_S \mu(\alpha)$.

We shall usually write $\alpha^*$ for $\mu(\alpha)$.

Note that this definition does not specify any relation between a modality and the logical operators, but it is consistent with them as we state in the next proposition, which is Corollary 5.3 of [1].
Proposition 4.2. If \( \mu \) is a modality in a local set theory \( S \), we have:

1. \( \vdash_S (\alpha \land \beta)^* = \alpha^* \land \beta^* \);
2. \( (\alpha \rightarrow \beta)^* \vdash_S \alpha^* \rightarrow \beta^* \);
3. \( \alpha \rightarrow \beta \vdash_S \alpha^* \rightarrow \beta^* \);
4. \( \exists x \alpha^* \vdash_S (\exists x \alpha)^* \), whence \( (\exists x \alpha)^* \vdash_S (\exists x \alpha)^* \);
5. \( (\forall x \alpha)^* \vdash_S \forall x \alpha^* \).

\[\Box\]

Modalities enable us to generalize the membership relation. Thinking of \( \mu(\alpha) \) as meaning "\( \alpha \) is possibly true", \( \mu(x \in U) \) means "it is possible that \( x \) is in \( U \)". Thus, given a modality \( \mu \) on a local set theory \( S \), for each \( S \)-set \( X \) we define a relation \( \alpha_X \subseteq X \times P(X) \) by

\[ x \circ x U \equiv \mu(x \in U), \]

for any \( S \)-set \( U \) such that \( \vdash_S U \subseteq X \).

Theorem 4.3. The relation \( \circ \) has the following properties:

1. If \( \vdash_S U \subseteq X \), then \( x \in U \vdash_S x \circ x U \).
2. If \( \vdash_S U \subseteq X \) and \( \vdash_S V \subseteq X \), then \( \forall x \in U, \forall x \in V \vdash_S (x \circ x V) \equiv x \circ x V \).
3. If \( f : X \rightarrow Y \) is an \( S \)-function and \( \vdash_S U \subseteq Y \), then

\[ x \in X \vdash_S f(x) \circ y U \equiv x \circ x f^{-1}[U], \]

where \( f^{-1}[U] \) is the term \( \{ x : x \in X \land f(x) \in U \} \).

Proof.

1. This follows immediately from the first property of modalities.
2. According to the definition of \( \circ \), we need to show:

\[ \forall x [x \in U \rightarrow (x \in V)^*] \vdash_S (y \in U)^* \rightarrow (y \in V)^*. \]

Now \( y \in U \rightarrow (y \in V)^* \vdash_S (y \in U)^* \rightarrow (y \in V)^* \) by 4.2.3, so, since \( (y \in V)^* \vdash_S (y \in V)^* \), it follows that \( y \in U \rightarrow (y \in V)^* \vdash_S (y \in U)^* \rightarrow (y \in V)^* \), from which (*) follows immediately.

3. As \( x \in X \vdash_S (x \in f^{-1}[U])^* \leftrightarrow (f(x) \in U)^* \) follows from

\[ x \in X, (x \in f^{-1}[U])^* : (f(x) \in U)^* \text{ and } x \in X, (f(x) \in U)^* : (x \in f^{-1}[U])^* \]

by the equivalence rule, we need to infer the last two sequents. Regarding the first one, we have, by substituting into the equality axiom,

\[ x \in f^{-1}[U] = (f(x) \in U \land x \in X), (x \in f^{-1}[U])^* \vdash_S (f(x) \in U \land x \in X)^* \]

so that, using 4.2.1,

\[ x \in f^{-1}[U] = (f(x) \in U \land x \in X), (x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^* \land (x \in X)^*. \]

Hence, by the comprehension axiom we get \( (x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^* \land (x \in X)^* \) so that \( (x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^* \). Hence, \( x \in X, (x \in f^{-1}[U])^* \vdash_S (f(x) \in U)^* \).

Similarly, we show
From the first property of modalities we have \( x \in X \vdash_S (x \in X)^* \) so concluding that 
\( (\lambda x \in X, (f(x) \in U)^* \vdash_S (x \in f^{-1}[U])^* \).

The three properties in Theorem 4.3 embody the minimal conditions which a
putative membership-relation has to satisfy. We also note that when the \( S \)-set is a
monoid, the reflexive and transitive properties correspond to those of a precovers.

The relation \( \alpha_X \) enables us to generalize the subset-relation by writing \( U \sqsubseteq_X V \)
for \( \forall x(x \in U \rightarrow x \sqsubseteq_X V) \). With this new definition we can rewrite Theorem 4.3.2 as
follows:

\[
2'. \text{ If } \vdash_S U \subseteq X \text{ and } \vdash_S V \subseteq X, \text{ then } x \sqsubseteq_X U, \text{ } U \sqsubseteq_X V \vdash_S x \sqsubseteq_X V.
\]

In particular, we can "saturate" an \( S \)-set \( U \), \( \vdash_S U \subseteq X \), with respect to the relation
\( \alpha_X \) writing \( U^S \) for \( \{ x : x \sqsubseteq_X U \} \) and calling it the saturation of \( U \). It follows easily
from Theorem 4.3 that the saturation operation \( \Delta \) is a universal closure operation in
the sense of [1].

5 Universal precovers

We are now going to define a relation between elements and \( S \)-sets in a local set
theory which will be characterized by the properties established in Theorem 4.3.

**Definition 5.1.** A universal precovers on a local set theory \( S \) is an assignment
of an \( S \)-set \( \alpha_X \) to each \( S \)-set \( X \) such that \( \vdash_S \alpha_X \subseteq X \times P(X) \) and

(i) if \( \vdash_S U \subseteq X \), then \( x \in U \vdash_S x \sqsubseteq_X U \),

(ii) if \( \vdash_S U \subseteq X \) and \( \vdash_S V \subseteq X \), then \( x \sqsubseteq_X U, \text{ } U \sqsubseteq_X V \vdash_S x \sqsubseteq_X V \),

(iii) if \( f : X \rightarrow Y \) is an \( S \)-function and \( \vdash_S U \subseteq Y \), then

\[
x \in X \vdash_S f(x) \sqsubseteq_Y U \iff x \sqsubseteq_X f^{-1}[U].
\]

We will show that, given a type \( A \), a universal precovers is "uniform" on every
\( S \)-set of type \( P(A) \), and so also on every subset of a given \( S \)-set \( X \). For this reason
we can write \( \alpha_A \) in place of \( \alpha_X \) if \( A \) is the type of \( X \), and we can omit the subscript
whenever it is clear from the context.

**Proposition 5.2.** If \( X \) is an \( S \)-set of type \( P(A) \) and \( \vdash_S U \subseteq X \), then

\[
x \in X \vdash_S x \sqsubseteq_X U \iff x \sqsubseteq_A U,
\]

where we write \( A \) for the \( S \)-set \( \{ x_A : \text{true} \} \).

**Proof.** Consider the insertion \( j : X \rightarrow A \) defined as \( j(x) \equiv x \). Since \( \vdash_S U \subseteq A \)

it follows from the third property of universal precovers that

\[
x \in X \vdash_S x \sqsubseteq_X j^{-1}[U] \iff j(x) \sqsubseteq_A U.
\]

It remains to prove \( \vdash_S j^{-1}[U] = U \), yielding \( x \in X \vdash_S x \sqsubseteq_X U \iff x \sqsubseteq_A U \). Noting

that \( j^{-1}[U] = \{ x : x \in X \wedge x \in U \} \) we infer \( x \in X, x \in U \vdash_S x \in j^{-1}[U] \), whence

\( x \in U \vdash_S x \in j^{-1}[U] \), since \( \vdash_S U \subseteq X \). From \( x \in j^{-1}[U] \vdash_S x \in X \wedge x \in U \) we
obtain \( x \in j^{-1}[U] \vdash_S x \in U \), and \( \vdash_S j^{-1}[U] = U \) follows.

From this we easily obtain

**Corollary 5.3.** If \( X \) is an \( S \)-set of type \( P(A) \) and \( U \) is an \( S \)-set such that
\( \vdash_S U \subseteq X \), then \( x \in U \vdash_S x \sqsubseteq_U U \iff x \sqsubseteq_X U \).
We want next to show that universal precovers respect the operations of function image and cartesian product. To do this we require the following

Lemma 5.4.
(a) $x \circ X, y \in Y \vdash_S (x, y) \triangleleft X \times Y$, $x \in X, y \circ Y \vdash_S (x, y) \triangleleft X \times Y$.
(b) $x \circ X, y \circ Y \vdash_S (x, y) \triangleleft X^\Delta \times Y^\Delta$, $x \in X, y \circ Y \vdash_S (x, y) \triangleleft X \times Y^\Delta$.
(c) $\vdash_S X^\Delta \times Y \triangleleft X \times Y^\Delta$.

Proof.
(a) Let $X$ be of type $A$ and consider $\sigma : A \times Y \rightarrow A$ defined by $\sigma((x, y)) \equiv x$.
The third property of universal precovers yields
$X \subseteq A, (x, y) \in A \times Y, x \circ X \vdash_S (x, y) \triangleleft X \times Y$
and so, since the cartesian product is defined by $A \times Y \equiv \{(x, y) : x \in A \land y \in Y\}$, we obtain $X \subseteq A, x \in A, y \in Y, x \circ X \vdash_S (x, y) \triangleleft X \times Y$. From the hypothesis $\vdash_S X \subseteq A$, we obtain $x \in A, y \in Y, x \circ X \vdash_S (x, y) \triangleleft X \times Y$, i.e. $y \in Y, x \circ X \vdash_S (x, y) \triangleleft X \times Y$, recalling that $A = \{x_A : true\}$ so that $\vdash_S x \in A$. The second sequent in (a) is an easy consequence.

(b) This follows from (a) by substituting $X^\Delta$ for $X$ and observing that
$\vdash_S x \in X^\Delta \dashv x \circ X$
follows from the comprehension axiom, so that $x \circ X \vdash_S x \in X^\Delta$. In the same way we deduce the second sequence in (b).

(c) The comprehension axiom implies $(x, y) \in X^\Delta \times Y \vdash_S x \in X^\Delta \land y \in Y$, and by (a), $x \circ X, y \in Y \vdash_S (x, y) \triangleleft X \times Y$, whence $(x, y) \in X^\Delta \times Y \vdash_S (x, y) \triangleleft X \times Y$.
Hence $\vdash_S \forall x, y((x, y) \in X^\Delta \times Y \rightarrow (x, y) \triangleleft X \times Y)$ which may be written in the form $\vdash_S X^\Delta \times Y \triangleleft X \times Y$. In the same way we deduce $\vdash_S X \times Y^\Delta \triangleleft X \times Y$.

Proposition 5.5. If $\sigma$ is a universal precover on a local set theory $S$, then

(i) if $f : X \rightarrow Y$ is an $S$-function and $\vdash_S U \subseteq X$, then
$\vdash_S x \circ X, U \vdash_S f(x) \triangleleft Y f[U]$;
(ii) $\vdash_S (x, y) \triangleleft X \times Y \rightarrow (x \circ X \land y \circ Y)$.

Proof.

(i) Let $f : X \rightarrow Y$ be an $S$-function. Since $U \subseteq X \vdash_S U \subseteq f^{-1}[f[U]]$ it follows that $U \subseteq X \vdash_S U \circ f^{-1}[f[U]]$. By the second property of universal precovers, we have $x \circ X, U \vdash f^{-1}[f[U]] \vdash_S x \circ X f^{-1}[f[U]]$ which yields
$\vdash_S x \in X, f[U] \subseteq Y, x \circ X f^{-1}[f[U]] \vdash_S f(x) \triangleleft Y f[U]$.

The third property of universal precovers gives
$x \in X, f[U] \subseteq Y, x \circ X f^{-1}[f[U]] \vdash_S f(x) \triangleleft Y f[U]\text{(1)}$
so that, since $U \subseteq X \vdash_S f[U] \subseteq Y$, we infer
$\vdash_S U \subseteq X, x \in X, x \circ X f^{-1}[f[U]] \vdash_S f(x) \triangleleft Y f[U]$.\text{(2)}

From (1) and (2) follows

$\vdash_S U \subseteq X, x \in X, x \circ X U \vdash_S f(x) \triangleleft Y f[U]$,
whence $x \in X, x \circ X U \vdash_S f(x) \triangleleft Y f[U]$.
(ii) Let \( X, Y \) be of types \( P(A), P(B) \), respectively, write \( A, B \) for \( \{ x_A : \text{true} \}, \{ x_B : \text{true} \} \), respectively, and \( \pi_1 : A \times B \rightarrow A, \pi_2 : A \times B \rightarrow B \) for the \( S \)-functions \( (x, y) \mapsto x, (x, y) \mapsto y \), respectively. We first show \( (x, y) \triangleleft X \times Y \triangleleft S (x \triangleleft X \land y \triangleleft Y) \). As we saw in (i) we can get
\[
X \times Y \subseteq A \times B, (x, y) \in A \times B, (x, y) \triangleleft X \times Y \vDash S \pi_1((x, y)) \triangleleft \pi_1(X \times Y)
\]
so that
\[
(3) \quad X \times Y \subseteq A \times B, (x, y) \triangleleft X \times Y \vDash S \pi_1((x, y)) \triangleleft \pi_1(X \times Y),
\]
recalling the definitions of \( A \) and \( B \) and observing that \( \vDash_S (x, y) \in A \times B \). From the two hypotheses \( \vDash_S X \subseteq A \) and \( \vDash_S Y \subseteq B \) we infer \( \vDash_S X \times Y \subseteq A \times B \), and \( (x, y) \triangleleft X \times Y \vDash S x \triangleleft X \) follows from (3). In the same way, using \( \pi_2 \) in place of \( \pi_1 \), we have \( (x, y) \triangleleft X \times Y \vDash S y \triangleleft Y \), whence \( (x, y) \triangleleft X \times Y \vDash S x \triangleleft X \land y \triangleleft Y \). Lastly, we have \( (x, y) \triangleleft X \times Y \vDash S (x, y) \triangleleft X \times Y \). From for Lemma 5.4.(c), we obtain
\[
\vDash_S X \triangleleft X \land Y \triangleleft Y \triangleleft X \times Y
\]
from which follows
\[
(x, y) \in X \triangleleft X \times Y \vDash S (x, y) \triangleleft X \times Y
\]
so that \( x \triangleleft X, y \triangleleft Y \vDash S (x, y) \triangleleft X \times Y \).
\( \Box \)

We finally show that a universal precover is uniquely determined by the relation between formulas and the \( S \)-set \( Tr \equiv \{ \omega : \omega = \text{true} \} \).

**Proposition 5.6.** If \( U \) is an \( S \)-set of type \( P(A) \) and \( \vDash_S U \subseteq X \), then
\[
\vDash_S x \triangleleft A U \mapsto (x \in U) \triangleleft q_1 Tr.
\]

**Proof.** Let \( \mathcal{X} : A \rightarrow \Omega \) be the \( S \)-function defined by \( \mathcal{X}(x) = x \in U \). Then
\[
\vDash_S \mathcal{X}^{-1}[Tr] = U
\]
so that \( \vDash_S x \triangleleft A U \mapsto x \triangleleft A \mathcal{X}^{-1}[Tr] \) and by property (iii) of universal precovers, \( \vDash_S x \triangleleft A \mathcal{X}^{-1}[Tr] \mapsto \mathcal{X}(x) \triangleleft q_1 Tr \). Putting these equivalences together gives the required result. \( \Box \)

### 6 Equivalence between modalities and universal precovers

Our next task is to show that a modality is equivalent to a universal precover. In fact we will show that a universal precover induces a modality which, in its turn, gives rise to a universal precover which turns out to be the same as the original one. We will prove the same thing starting with a modality.

First, we show how to obtain a modality from a universal precover.

**Proposition 6.1.** Let \( \alpha \) be a universal precover on a local set theory \( S \) and \( \mu : \Omega \rightarrow \Omega \) the \( S \)-function defined by \( \mu(\omega) \equiv \omega \triangleleft q_1 Tr \). Then \( \mu \) is a modality.

**Proof.** We have to prove that \( \mu \) satisfies the three properties of a modality.

(i) We need to show that \( \alpha \vDash S \alpha \triangleleft q_1 Tr \). Since \( \vDash_S \omega = (\omega = \text{true}) \), it follows that \( \vDash_S \omega \triangleleft \omega \in Tr \) so that \( \omega \vDash_S \omega \in Tr \). From the first property of a universal precover we have \( \omega \in Tr \vDash_S \omega \triangleleft q_1 Tr \), whence \( \omega \vDash_S \omega \triangleleft q_1 Tr \), as required.

(ii) We have to show that from \( \alpha \vDash S \beta \) we can infer \( \alpha \triangleleft q_1 Tr \vDash_S \beta \triangleleft q_1 Tr \). From the hypothesis \( \alpha \vDash S \beta \) we infer \( \vDash_S \alpha \rightarrow \beta \) and hence, recalling the definition of \( \alpha \rightarrow \beta \), the equality axiom yields \( \alpha \triangleleft q_1 Tr \vDash_S \alpha \land \beta \triangleleft q_1 Tr \). So our task reduces to showing that \( \alpha \triangleleft \beta \triangleleft q_1 Tr \vDash_S \beta \triangleleft q_1 Tr \), and to this end, we consider the \( S \)-function
& : Ω × Ω → Ω defined by &((ω, ω')) = ω ∧ ω'. By the third property of universal precovers, we have \( \vdash_S (ω ∧ ω') ≡ ω ∧ ω' \). By the third property of universal precovers, we have \( \vdash_S (ω ∧ ω') \Rightarrow (ω, ω') ∈ &^{-1}[Tr] \). Now, if we can show that

\[
(4) \quad \vdash_S &^{-1}[Tr] = Tr × Tr,
\]

we may infer \( \vdash_S (ω ∧ ω') ≡ (ω, ω') \Rightarrow Tr × Tr \), so that

\[
(ω ∧ ω') ≡ ωₐ S Tr → (ω, ω') \Rightarrow Tr × Tr.
\]

Proposition 5.5.(ii) then implies \( (ω ∧ ω') ≡ ωₐ S Tr ∧ ω' ≡ ωₐ S Tr \) from which follows \( (ω ∧ ω') ≡ ωₐ S Tr ∧ ω' ≡ ωₐ S Tr \), so that, substituting \( α \) for \( ω \) and \( β \) for \( ω' \), we get what we want.

So we turn our attention to proving (4). To do this, we observe that, by the comprehension axiom, \( \vdash_S (ω, ω') ∈ &^{-1}[Tr] \Leftrightarrow (ω, ω') ∈ Ω × Ω ∧ (ω ∧ ω') ∈ Tr \), so that \( \vdash_S (ω, ω') ∈ &^{-1}[Tr] \Rightarrow (ω, ω') ∈ Ω × Ω ∧ (ω ∧ ω') = true \). Therefore, we have \( \vdash_S (ω', ω') ∈ &^{-1}[Tr] \Rightarrow (ω, ω') ∈ Ω × Ω ∧ (ω ∧ ω') \) from \( \vdash_S (ω ∧ ω') = (true, true) \), whence \( \vdash_S (ω ∧ ω') → ω ∈ Tr ∧ ω' ∈ Tr \). So we have \( \vdash_S (ω, ω') ∈ &^{-1}[Tr] \Rightarrow (ω, ω') ∈ Ω × Ω ∧ (ω ∈ Tr ∧ ω' ∈ Tr) \), whence

\[
\vdash_S (ω, ω') ∈ &^{-1}[Tr] \Rightarrow (ω, ω') ∈ Tr × Tr,
\]

from which we infer \( \vdash_S &^{-1}[Tr] = Tr × Tr \) by extensionality.

(iii) Finally, we must prove \( μ(μ(α)) \equiv μ(α) \), i.e., \( (α ≡ ωₐ S Tr) → (α ∈ Tr^Δ) \). It follows that

\[
(α ≡ ωₐ S Tr) → (α ∈ Tr^Δ) \Rightarrow ωₐ S Tr.
\]

By proposition 5.6, we have \( \vdash_S (α ∈ Tr^Δ) ≡ Tr → α ≡ ωₐ S Tr^Δ \). Hence

\[
(5) \quad \vdash_S (α ≡ ωₐ S Tr^Δ) → α ≡ ωₐ S Tr^Δ.
\]

From the second property of universal precovers, we have

\[
α ≡ ωₐ S Tr^Δ, Tr^Δ \Rightarrow ωₐ S Tr \Rightarrow α \equiv ωₐ S Tr.
\]

Since clearly \( \vdash_S Tr^Δ \Rightarrow Tr \), we obtain \( α \equiv ωₐ S Tr^Δ \Rightarrow α \equiv ωₐ S Tr \), which, together with (5) gives the required conclusion. ⊓⊔

Now we establish the equivalence between modalities and universal precovers.

Theorem 6.2. Every modality in a local set theory S is equivalent to a universal precover in S, and further, every universal precover is equivalent to a modality.

Proof. Let \( μ \) be a modality in S. As we saw in Section 4, we can define a universal precover, \( α \equiv μ(α ∈ U) \), which, in its turn, induces a modality \( μₐ \) as follows:

\[
μₐ(α) ≡ α \equiv μ(α ∈ U).
\]

We show that \( \vdash_S μₐ(α) \equiv μ(α) \). Since \( \vdash_S α → α \equiv μₐ(α) \), we can apply the second property of modalities and obtain

\[
\vdash_S μ(α) \equiv μ(α ∈ U),
\]

as claimed.
Conversely, given a universal precover \( \mathpmb{\wedge} \), we can define a modality \( \mu \) as in the previous proposition and then a universal precover \( \mathpmb{\wedge}_\mu \) by

\[
x \mathpmb{\wedge}_\mu U \equiv \mu(x \in U) \equiv (x \in U) \mathpmb{\wedge}_\Omega Tr.
\]

By Proposition 5.6 we have \( \vdash_S x \mathpmb{\wedge}_X U \leftrightarrow (x \in U) \mathpmb{\wedge}_\Omega Tr \) which we can rewrite as \( \vdash_S x \mathpmb{\wedge}_X U \leftrightarrow x \mathpmb{\wedge}_\mu U \) so that \( \mathpmb{\wedge}_\mu \) is the same as \( \mathpmb{\wedge} \).

\[\square\]

7 Stability

We have defined a universal precover on a local set theory by assigning to each \( S \)-set \( X \) a relation \( \mathpmb{\wedge}_X \) satisfying certain properties. When the \( S \)-set \( X \) is a monoid, we have noted that the first two properties correspond to the reflexive and transitive properties of a precover, respectively. Here we will show that, when \( X \) is a monoid, \( \mathpmb{\wedge}_X \) also satisfies stability so that \( \mathpmb{\wedge}_X \) is a precover on \( X \).

Let \( L \) be a local language in which we can write the monoid axioms and let \( M \) be a type of a monoid. We write \( M \) for the \( S \)-set \( \{x_M : true\} \) of type \( P(M) \), and if \( U \) and \( V \) are \( S \)-sets of type \( P(M) \), we write \( U \cdot V \) for the term \( \{x \cdot y : x \in U \land y \in V\} \).

We will say that a universal precover satisfies stability if

\[
a \mathpmb{\wedge}_M U, b \mathpmb{\wedge}_M V \vdash_S a \cdot b \mathpmb{\wedge}_M U \cdot V.
\]

**Proposition 7.1.** If \( \mathpmb{\wedge} \) is a universal precover on a local set theory \( S \), then \( \mathpmb{\wedge}_M \) satisfies stability.

**Proof.** Let \( f : M \times M \rightarrow M \) be the \( S \)-function defined by \( f((x, y)) \equiv x \cdot y \). From 5.5.(i) we infer \( (x, y) \in M \times M \), \( (x, y) \mathpmb{\wedge}_X U \times V \vdash_S f((x, y)) \mathpmb{\wedge}_M f(U \times V) \) and so, since \( \vdash_S (x, y) \mathpmb{\wedge}_X U \times V \leftrightarrow x \mathpmb{\wedge}_M U \land y \mathpmb{\wedge}_M V \) by Proposition 5.5.(ii), we obtain

\[
(x, y) \in M \times M, x \mathpmb{\wedge}_M U \land y \mathpmb{\wedge}_M V \vdash_S x \cdot y \mathpmb{\wedge}_M U \cdot V.
\]

Since clearly \( \vdash_S (x, y) \in M \times M \), the conclusion follows. \[\square\]

8 Precovers on \( \Omega \)

We have seen in Theorem 6.2 that specifying a modality on a local set theory \( S \) is equivalent to specifying a universal precover, but to specify a universal precover we have to define a relation for every \( S \)-set \( X \) whereas, to specify a modality we have to define an \( S \)-function only on the \( S \)-set \( \Omega \). This fact leads one to suspect that a universal precover may be obtained just by defining a relation on the \( S \)-set \( \Omega \), which is what we finally show.

**Definition 8.1.** A precover-on-\( \Omega \) in a local set theory \( S \) is an \( S \)-set \( \mathpmb{\wedge}_\Omega \) such that \( \mathpmb{\wedge}_\Omega \subseteq \Omega \times P(\Omega) \), and

(i) if \( \vdash_S U \subseteq \Omega \), then \( x \in U \vdash_S x \mathpmb{\wedge}_\Omega U \);

(ii) if \( \vdash_S U \subseteq \Omega \) and \( \vdash_S V \subseteq \Omega \), then \( x \mathpmb{\wedge}_\Omega U, U \mathpmb{\wedge}_\Omega V \vdash_S x \mathpmb{\wedge}_\Omega V \);

(iii) \( \vdash_S (\omega \land \omega') \mathpmb{\wedge}_\Omega Tr \rightarrow \omega \mathpmb{\wedge}_\Omega Tr \land \omega' \mathpmb{\wedge}_\Omega Tr \);  

(iv) if \( f : \Omega \rightarrow \Omega \) is an \( S \)-function and \( \vdash_S U \subseteq \Omega \), then \( \vdash_S \omega \mathpmb{\wedge}_\Omega f^{-1}[U] \leftrightarrow f(\omega) \mathpmb{\wedge}_\Omega U \).
We now show that this relation determines a universal precover.

Theorem 8.2. Let $S$ be a local set theory and $\precover$ a precover-on-$\Omega$ in $S$. If for each $S$-set $X$ we define the relation $\precover_X$, $\precover_X \subseteq X \times P(X)$, by

$$x \precover_X U \equiv (x \in U) \precover \Omega \Tr,$$

then we obtain a universal precover on $S$.

Proof.

(i) Since $\omega \models_S \omega \in \Tr$, we obtain $x \in U \models_S (x \in U) \in \Tr$ from which follows $x \in U \models_S (x \in U) \precover (x \in V) \precover \Omega \Tr$ by the first property of a precover-on-$\Omega$, and if we rewrite the conclusion using the definition of $\precover_X$, we get $x \in U \models_S x \precover_X U$.

(ii) We must show

$$(y \in U) \precover \Omega \Tr, \forall x ((x \in U) \rightarrow (x \in V) \precover \Omega \Tr) \models_S (y \in V) \precover \Omega \Tr.$$

Now, from the definition of $\alpha \rightarrow \beta$ we infer

$$(x \in U) \rightarrow (x \in V) \precover \Omega \Tr, (x \in U) \models_S ((x \in U) \land (x \in V) \precover \Omega \Tr) \precover \Omega \Tr$$

and the third property of $\precover$ implies

$$(x \in U) \rightarrow (x \in V) \precover \Omega \Tr, (x \in U) \models_S (x \in U) \precover \Omega \Tr \land ((x \in V) \precover \Omega \Tr) \precover \Omega \Tr.$$

Hence $(x \in U) \rightarrow (x \in V) \precover \Omega \Tr, (x \in U) \models_S ((x \in U) \precover \Omega \Tr) \precover \Omega \Tr$ so that

$$\forall x ((x \in U) \rightarrow (x \in V) \precover \Omega \Tr), (y \in U) \precover \Omega \Tr \models_S ((y \in V) \precover \Omega \Tr) \precover \Omega \Tr.$$

Now, from Proposition 5.6 we infer $((y \in V) \precover \Omega \Tr) \precover \Omega \Tr \models_S (y \in V) \precover \Omega \Omega \Tr$ and so, since $\models_S \Omega \Omega \Tr \precover \Omega \Tr$, we deduce

$$\forall x ((x \in U) \rightarrow (x \in V) \precover \Omega \Tr), (y \in U) \precover \Omega \Tr \models_S (y \in V) \precover \Omega \Tr,$$

as required.

(iii) Let $f : X \rightarrow Y$ be an $S$-function and $\models_S U \subseteq Y$. The equality axiom gives us $(f(x) \in U) = (x \in f^{-1}[U]), (f(x) \in U) \precover \Omega \Tr \models_S (x \in f^{-1}[U]) \precover \Omega \Tr$, and then, observing that $X \models_S f(x) \in U \rightarrow x \in f^{-1}[U]$ follows from the definition of $f^{-1}[U]$, we obtain $x \in X, (f(x) \in U) \precover \Omega \Tr \models_S (x \in f^{-1}[U]) \precover \Omega \Tr$. Similarly we infer $x \in X, (x \in f^{-1}[U]) \precover \Omega \Tr \models_S (f(x) \in U) \precover \Omega \Tr$ from which follows

$$x \in X \models_S (x \in f^{-1}[U]) \precover \Omega \Tr \rightarrow (f(x) \in U) \precover \Omega \Tr.$$

If we rewrite the conclusion using $\precover_X$, we get $x \in X \models_S x \precover f^{-1}[U] \rightarrow f(x) \precover U$. □

References


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