Quantum Incompatibility and Noncommutativity

Here is a way of looking at quantum incompatibility of properties, suggested by Basil Hiley, which brings noncommutativity into full focus.

We are given a bunch of objects each of which is either a sphere or a cube, and each of which is either red or blue. Thus the objects are classified under two attributes SHAPE and COLOUR. SHAPE further breaks down into the two properties spherical and cubical, and COLOUR into the properties red and blue.

We suppose that each of the four properties spherical, cubical, red, blue comes with a test for determining whether an object has the property in question. Thus, for example, if an object passes the test for red, it is deemed to have the property red; if not, it is deemed to have the property blue; in other words, it would have passed the blue test. We assume that, if an object passes a test for a given property, it will also pass a test for that property applied immediately afterwards (i.e., without any intervening application of a test for a different property).

Now suppose that we test the objects for the property spherical, and assemble the objects passing the test into an ensemble $S$. Next, we subject the objects in $S$ to the test for red, and assemble the objects passing this subsequent test into an ensemble which we shall write as $S \star R$. 
Now perform this procedure in the opposite order, first testing for red and then for spherical. The resulting ensemble will then, naturally, be \( R \star S \).

If the attributes SHAPE and COLOUR behave \textit{classically}, we would expect the ensembles \( S \star R \) and \( R \star S \) to be identical, since then the operation \( \star \) would simply correspond to the logical conjunction \((\land)\) of properties (and of course \( \land \) is commutative). In other words, from a classical standpoint, \( \star \) would be a commutative operation.

But in the quantum world \( \star \) will not necessarily be a commutative operation, \textit{and so will not necessarily correspond to the logical operation of conjunction}. For in the quantum world attributes such as SHAPE and COLOUR may be \textit{incompatible}, that is, like position and momentum of electrons, not simultaneously \`testable`. In that case \( S \star R \) and \( R \star S \) may differ, so that the \( \star \) operation is \textit{noncommutative}.

What this means is that, if we take an arbitrary object from \( S \star R \) and subject it to the red test, it will always pass the test. But if we subject an arbitrary object from \( R \star S \) to the same, red, test, sometimes it will not pass the test and \textit{will then have to be deemed blue}, that is, it would have passed the blue test. In other words, \( S \star R \) consists entirely of red objects, but \( R \star S \) will contain some blue objects.

The startling nature of this conclusion can be made further evident by tracking the history of an object, \( a \), in \( R \star S \) which fails to pass the test for red and so is deemed blue. Now \( a \) initially passed the red test and so was, correctly, included in \( R \). At this point \( a \) was definitely red. But then, for \( a \) to be included in \( R \star S \), it must subsequently pass the spherical test. Once \( a \) passes this test, it is then tested for blue and passes that
test as well. In other words, applying the **spherical** test to the **red** object \(a\) has caused it to change its colour to **blue**!