DETERMINANTAL INEQUALITIES AND THE RIEMANN HYPOTHESIS

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ABSTRACT. It is known that the Riemann Hypothesis is correct if a certain set of determinantal inequalities $D(n,r) > 0,\ n = 0, 1, \ldots$ hold for all positive integer values of the order r of the determinants. The condition has been previously verified for orders $1, 2$. Here the result is proved for order $r = 3$.

1. INTRODUCTION

1.1. About 25 years ago Csordas, Norfolk and Varga [1] reported the first progress in what might be called the determinantal method for proving the Riemann hypothesis (RH). This method involves the study of the coefficients in the Taylor expansion of the Riemann ζ-function, which in the notation of [1] (slightly modified) is written

$$F(z) = \sum_{n=0}^{\infty} \beta_n z^n \tag{1.1}$$

where the series coefficients are

$$\beta_n = \frac{1}{\Gamma(2n+1)} \int_0^{\infty} dt \Phi(t)\, t^{2n}, \tag{1.2}$$

with the function $\Phi(t)$ given by

$$\Phi(t) = \sum_{m=1}^{\infty} \left[2m^4\pi^2 e^{9t} - 3m^2\pi e^{5t}\right] \exp\left(-m^2\pi e^{4t}\right). \tag{1.3}$$

The RH is equivalent to the statement that all the zeros of the entire function $F(z)$ are real and negative.

1.2. Karlin [3, Theorem 5.3, p. 412; p. 393; Lemma 9.3, p. 89] tells us that this condition is ensured by a requirement on the coefficients $(\beta_n, n = 0, 1, \ldots)$ of the series $F(z)$, from which is formed a semi-infinite matrix B,

$$B_{i,j} = \begin{cases} \beta_{j-i}, & j \geq i; \\ 0, & j < i; \end{cases} \quad i, j = 0,\ 1,\ 2,\ \ldots. \tag{1.4}$$

Thus, if we define the minors $D(n,r)$ of order r by

$$D(n,r) = \det[B_{i,j+n}]_{i,j=1,\ldots,r}, \tag{1.5}$$

the RH is equivalent to the condition that

$$D(n,r) > 0, \quad n = 0, 1, \ldots; \ r = 1, 2... \tag{1.6}$$

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In this case the matrix B is said to be totally positive or TP (see [3]). The ultimate objective is to prove (1.6), which we call the determinantal inequalities, and so demonstrate the truth of the RH.

1.3. In [1] attention was restricted to the case of order $r = 2$ ($r = 1$ is trivial). In that case the condition (1.6) reduces to

$$D(n,2) = \begin{vmatrix} \beta_n & \beta_{n+1} \\ \beta_{n-1} & \beta_n \end{vmatrix} > 0, \quad n = 1, 2, \ldots, \tag{1.7}$$

while for the case $n = 0$ we require

$$D(0,2) = \begin{vmatrix} \beta_0 & \beta_1 \\ 0 & \beta_0 \end{vmatrix} > 0, \tag{1.8}$$

which is trivially true.

In the process of deriving some stronger results Csordas et al. [1] (see also Csordas and Varga [2]) proved (1.7). Their main technique was originally introduced by Karlin, Proschan and Barlow [4] and later described by Karlin [3] . In a simplified version suitable for our purposes we define the kernel $K(x, y,),\ 0 \leq x, y < \infty$, by

$$K(x, y) = \Phi(x + y). \tag{1.9}$$

The corresponding compound kernel of order 2 is given by

$$K_{[2]}\left(\underline{x}, \underline{y}\right) = \det \begin{bmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{bmatrix} \tag{1.10}$$

where

$$\underline{x} = (x_1, x_2), \quad 0 \leq x_1 < x_2, \tag{1.11}$$

and similarly for $\underline{y}$.

In [2] (see (2.7) for details) it is shown that $\log[\Phi(x)]$ is strictly concave on $(0, \infty)$, which means that

$$\det \begin{bmatrix} \Phi(x) & \Phi^{(1)}(x) \\ \Phi^{(1)}(x) & \Phi^{(2)}(x) \end{bmatrix} < 0, \ \ x \geq 0. \tag{1.12}$$

It is shown in [1] that consequently $K_{[2]}\left(\underline{x}, \underline{y}\right) < 0$ where (1.11) holds. See Lemma 2.1 below for a more general proof of this result.

In the terminology of Karlin [3] we say that $K(x, y)$ is sign-reverse regular of order 2, i.e. RR_2. As explained by Csordas et al. [1, p. 526] , the RR_2 condition leads to a proof of (1.7).

1.4. It is natural to ask whether the above techniques can be generalized to prove (1.6) for values of order $r > 2$. Our response here is to show that the corresponding compound kernel of order 3, namely

$$K_{[3]}\left(\underline{x}, \underline{y}\right) = \det \begin{bmatrix} K(x_1, y_1) & K(x_1, y_2) & K(x_1, y_3) \\ K(x_2, y_1) & K(x_2, y_2) & K(x_2, y_3) \\ K(x_3, y_1) & K(x_3, y_2) & K(x_3, y_3) \end{bmatrix} \tag{1.13}$$

is RR_3. This is what is done below in Sect.3.

As we describe in Section 2 the method of [4] then shows, just as it did for $r = 2$, that (1.6) holds for $r = 3,\ n = 2, 3, \ldots$.

1.5. In Section 2 we show how to prove (1.7) and (2.3) given that the respective compound kernels are $RR_r,\ r = 2, 3$.

Section 3 uses a theorem of [3] to prove that $K_{[3]}\left(\underline{x}, \underline{y}\right) < 0$ is RR_3. This involves a number of elementary estimates, for which the techniques and information in [2] Sect. 3 are very helpful.

Section 4 contains a brief discussion of the prospects for further progress.

2. SIGN-REGULARITY AND POSITIVE RIEMANN DETERMINANTS

2.1. We first review the precise general definition of sign-reverse regularity when applied to the kernel $K(x, y)$ of (1.9), as stated by [3, p. 12]. The set X is the non-negative real line. The open simplex $\Delta_p(X)$ is

$$\Delta_r(X) = \{\underline{x} = (x_1, x_2, \ldots, x_p)\, | x_1 < x_2 < \ldots < x_p : x_i \in X\} \tag{2.1}$$

The compound kernel $K_{[p]}\left(\underline{x}, \underline{y}\right)$ is defined by

$$K_{[p]}\left(\underline{x}, \underline{y}\right) = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \ldots & K(x_1, y_p) \\ K(x_2, y_1) & K(x_2, y_2) & \ldots & K(x_2, y_p) \\ \vdots & \vdots & & \vdots \\ K(x_p, y_1) & K(x_p, y_2) & \ldots & K(x_p, y_p) \end{vmatrix} \tag{2.2}$$

where

$$\underline{x} = (x_1, x_2, \ldots, x_p) \in X; \quad \underline{y} = (y_1, y_2, \ldots, y_p) \in X \tag{2.3}$$

With $\epsilon_p = (-1)^{p(p-1)/2}$, we say that $K(x, y)$ is RR_r if $\epsilon_p K_{[p]}\left(\underline{x}, \underline{y}\right)$ is a non-negative function on $\Delta_p(X) \times \Delta_p(X)$ for each $p = 1, 2, \ldots, r$.

2.2. The following lemma is important in the later development.

Lemma 2.1. *Suppose that* $\psi(x)$ *is analytic in a neighborhood of* $(0, \infty)$*, and that the kernel* $k(x, y) = \psi(x + y),\ x, y \in (0, \infty)$*. Define* $w_p(t) = \det\left|\psi^{(i+j-2)}(t)\right|_{i,j=1}^{p}$*. If* $\epsilon_p w_p(t) > 0,\ t \geq 0,\ \ p = 1, 2, \ldots, r$ *then* $k(x, y)$ *is* RR_r*.*

Proof. This result is a special case of Theorem 2.6 of [3, p. 55]. The analyticity of $\psi(x, y)$ ensures that the differentiability requirements of the theorem are satisfied. The relation

$$\det\left|\frac{\partial^{i+j-2}}{\partial x^{i-1}\partial y^{j-1}} k(x, y)\right|_{i,j=1}^{p} = \det\left|\psi^{(i+j-2)}(t)\right|_{i,j=1}^{p}, \quad t = x + y, \tag{2.4}$$

together with [3, (1.3), p. 48] demonstrates that the requirements on the compound kernel appearing in the statement of the theorem hold. □

Now define $W_p(t)$ as

$$W_p(t) = \det\left|\Phi^{(i+j-2)}(t)\right|_{i,j=1}^{p}. \tag{2.5}$$

Theorem 2.2. *The kernel* $K(x, y)$ *defined by (1.9) is sign-regular of type* RR_3*.*

Proof. From Lemma 2.1 we must show, for $t \geq 0$, that $W_1(t) > 0$, while for $p = 2, 3$, that $W_p(t) < 0$.

Since $\Phi(t) > 0,\ t \geq 0$ the relation $W_1(t) > 0,\ t \geq 0$ is true.

Next, from [1, p. 523] we have $\Phi^{(1)}(t) \le 0$, $t \ge 0$, from [2, (3.18)] that $\Phi^{(2)}(0) < 0$, and from [2, p. 197] we have $g(t) > 0,\ t > 0$, where

$$g(t) = -t\left(\Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2\right) + \Phi(t)\Phi^{(1)}(t) \tag{2.6}$$

It follows that

$$W_2(t) = \Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2 < 0,\quad t \ge 0. \tag{2.7}$$

The proof that $W_3(t) < 0,\ t \ge 0$ is given in Theorem 3.9. □

2.3. In the remainder of this section we use the fact that $K(x, y)$ is RR_3, and demonstrate how to prove the determinantal inequalities (1.6) for order $r = 3$, using the techniques of [4] and [1]. We define the function $\lambda(s),\ s > -1$, as

$$\lambda(s) = \frac{1}{\Gamma(s+1)}\int_0^\infty dx\Phi(x)\,x^s. \tag{2.8}$$

There is a corresponding kernel

$$\Lambda(s,t) = \lambda(s+t),\quad s, t > -1/2. \tag{2.9}$$

A useful third type of kernel is

$$G(s,u) = \frac{u^{s-1/2}}{\Gamma(s+1/2)},\quad u > 0;\ s > -1/2. \tag{2.10}$$

[1, p. 526], and previously [4], show that

$$\begin{aligned}\Lambda(s,t) &= \frac{1}{\Gamma(s+t+1)}\int_0^\infty dx\Phi(x)\,x^{s+t}\\ &= \int_0^\infty du\int_0^\infty dv\,G(s,u)\,K(u,v)\,G(t,v),\quad s,t > -1/2.\end{aligned} \tag{2.11}$$

2.4. The last component of the machinery of the proof is the application of what Karlin [3, p. 17] calls the Basic Composition Formula (BCF). In the present case it may be used to relate the compound kernels corresponding to the kernels appearing in (2.11), so that

$$\Lambda_{[3]}(\underline{s},\underline{t}) = \int_0^\infty d\underline{u}\int_0^\infty d\underline{v}\,G_{[3]}(\underline{s},\underline{u})\,K_{[3]}(\underline{u},\underline{v})\,G_{[3]}(\underline{t},\underline{v}). \tag{2.12}$$

In (2.12) we have

$$\underline{u} = (u_1,u_2,u_3),\ \ 0 \le u_1 < u_2 < u_3\ \ d\underline{u} = du_1du_2du_3, \tag{2.13}$$

and similarly for $\underline{v}$. Also we have

$$\underline{s} = (s_1,s_2,s_3),\ \ -1/2 \le s_1 < s_2 < s_3, \tag{2.14}$$

and similarly for $\underline{t}$.

The compound kernel $G_{[3]}(\underline{s},\underline{u})$ may be written as

$$G_{[3]}(\underline{s},\underline{u}) = \frac{{u_1}^{s_1-1/2}{u_2}^{s_1-1/2}{u_3}^{s_1-1/2}}{\Gamma(s_1+1/2)\Gamma(s_2+1/2)\Gamma(s_3+1/2)}\begin{vmatrix} 1 & {u_1}^{s_2-s_1} & {u_1}^{s_3-s_1}\\ 1 & {u_2}^{s_2-s_1} & {u_2}^{s_3-s_1}\\ 1 & {u_3}^{s_2-s_1} & {u_3}^{s_3-s_1}\end{vmatrix} \tag{2.15}$$

Also $\Lambda_{[3]}(\underline{s},\underline{t})$ may be written as

$$\Lambda_{[3]}(\underline{s},\underline{t}) = \begin{vmatrix} \lambda(s_1+t_1) & \lambda(s_1+t_2) & \lambda(s_1+t_3)\\ \lambda(s_2+t_1) & \lambda(s_2+t_2) & \lambda(s_2+t_3)\\ \lambda(s_3+t_1) & \lambda(s_3+t_2) & \lambda(s_3+t_3)\end{vmatrix}. \tag{2.16}$$

Relations (2.13) and (2.14) show that the two G compound kernels in (2.12) are positive for all arguments. Since the compound kernel $K_{[2]}$ is negative for all arguments, (2.12) demonstrates that $\Lambda_{[3]}\left(\underline{s}, \underline{t}\right) < 0$ for all valid $\underline{s}, \underline{t}$.

2.5. With this in mind we have

Theorem 2.3. *For all integer* $n \geq 2$

$$D(n,3) = \begin{vmatrix} \beta_n & \beta_{n+1} & \beta_{n+2} \\ \beta_{n-1} & \beta_n & \beta_{n+1} \\ \beta_{n-2} & \beta_{n-1} & \beta_n \end{vmatrix} > 0, \tag{2.17}$$

Proof. To prove these inequalities we choose

$$s_1 = t_1 = n-2, \quad s_2 = t_2 = n, \quad s_3 = t_3 = n+2, \quad n = 2, 3, \ldots \tag{2.18}$$

and use the relation (2.12), which gives

$$\begin{vmatrix} \beta_{n-2} & \beta_{n-1} & \beta_n \\ \beta_{n-1} & \beta_n & \beta_{n+1} \\ \beta_n & \beta_{n+1} & \beta_{n+2} \end{vmatrix} = \Lambda_{[3]}\left(\underline{s}, \underline{t}\right) < 0. \tag{2.19}$$

The restriction $n \geq 2$ arises from the need to satisfy the condition $s > -1/2$ in (2.10).

Interchanging the rows of (2.19) produces the determinants in (2.17) and also changes the sign of the original determinants, so that (2.17) is verified. ☐

Remark 2.4. Just as for order $r = 2$ the case $n = 0$ is trivial, but for $n = 1$ the relation

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_0 & \beta_1 & \beta_2 \\ 0 & \beta_0 & \beta_1 \end{vmatrix} > 0, \tag{2.20}$$

does not follow from the general method. Its validity may be checked by inserting the numerical values of β_j listed by [1, p. 540].

3. PROOF OF SIGN-REGULARITY OF ORDER 3

3.1. The aim of this section is to prove that

$$W_3(t) < 0, \ \ t \geq 0, \tag{3.1}$$

where $W_3(t)$ is defined by (2.5). We find it convenient to work in terms of $f(t) = \log(\Phi(t))$. It may be shown that

$$\begin{aligned} W_3\left(t\right) &= \begin{vmatrix} \Phi(t) & \Phi^{(1)}(t) & \Phi^{(2)}(t) \\ \Phi^{(1)}(t) & \Phi^{(2)}(t) & \Phi^{(3)}(t) \\ \Phi^{(2)}(t) & \Phi^{(3)}(t) & \Phi^{(4)}(t) \end{vmatrix} \\ &= \left(f^{(2)}(t)\left[2f^{(2)}(t)^2 + f^{(4)}(t)\right] - f^{(3)}(t)^2\right)\exp\left[3\Phi(t)\right] \end{aligned} \tag{3.2}$$

After presenting a number of subsidiary results, we end this section with a proof of (3.1) in Theorem 3.9.

3.2. To prove (3.1) we make frequent use of the notation, results and techniques in [2, Sec. 3]. Thus we write

$$a_n(t) = \pi n^2 \left(2\pi n^2 e^{4t} - 3\right) \exp\left(5t - \pi n^2 e^{4t}\right) \tag{3.3}$$

$$\Phi_j(t) = \sum_{n=j+1}^{\infty} a_n(t) \tag{3.4}$$

The derivatives of $a_n(t)$ are written as

$$a_n^{(j)}(t) = \pi n^2 p_{j+1}\left(\pi n^2 e^{4t}\right) \exp\left(5t - \pi n^2 e^{4t}\right), \quad j = 1, 2, \ldots, \tag{3.5}$$

where the relevant polynomials p_k are

$$\begin{aligned} p_1(y) &= 2y - 3, \\ p_2(y) &= -8y^2 + 30y - 15, \\ p_3(y) &= 32y^3 - 224y^2 + 330y - 75, \\ p_4(y) &= -128y^4 + 1440y^3 - 4232y^2 + 3270y - 375, \\ p_5(y) &= 512y^5 - 8448y^4 + 41408y^3 - 68096y^2 + 30930y - 1875. \end{aligned} \tag{3.6}$$

We set

$$f_0(t) = \log(a_1(t)); \quad h(t) = 1 + \Phi_1/a_1; \quad f_e(t) = \log(h(t)) \tag{3.7}$$

so that

$$f(t) = f_0(t) + f_e(t) \tag{3.8}$$

With $\alpha_j(t) = a_1^{(j)}(t)/a_1(t)$ we find that

$${f_0}^{(2)}(t) = \alpha_2(t) - \alpha_1(t)^2 \tag{3.9}$$

$${f_0}^{(4)}(t) = \alpha_4(t) - 4\alpha_1(t)\alpha_3(t) - 3\alpha_2(t)^2 + 12\alpha_1(t)^2\alpha_2(t) - 6\alpha_1(t)^4. \tag{3.10}$$

Further, we have

$${f_e}^{(2)}(t) = \frac{h^{(2)}(t)}{h(t)} - \frac{h^{(1)}(t)^2}{h(t)^2} \tag{3.11}$$

$$\begin{aligned} {f_e}^{(4)}(t) = {} & \frac{h^{(4)}(t)}{h(t)} - 4\frac{h^{(1)}(t)h^{(3)}(t)}{h(t)^2} - 3\frac{h^{(2)}(t)^2}{h(t)^2} \\ & +12\frac{h^{(1)}(t)^2 h^{(2)}(t)}{h(t)^3} - 6\frac{h^{(1)}(t)^4}{h(t)^4}. \end{aligned} \tag{3.12}$$

If we set

$$\begin{aligned} \sigma_0(t) &= 1, \\ \sigma_1(t) &= -\alpha_1(t), \\ \sigma_2(t) &= -\alpha_2(t) + 2\alpha_1^2, \\ \sigma_3(t) &= -\alpha_3(t) + 6\alpha_1(t)\alpha_2(t) - 6\alpha_1(t)^3, \\ \sigma_4(t) &= -\alpha_4(t) + 8\alpha_1(t)\alpha_3(t) + 6\alpha_2(t)^2 - 36\alpha_1(t)^2\alpha_2(t) + 24\alpha_1(t)^4, \end{aligned} \tag{3.13}$$

then we find that

$$h^{(k)}(t) = \sum_{i=0}^{k} \frac{\Phi_1^{(k-i)}(t)}{a_1(t)} \sigma_i(t) C_i^k, \tag{3.14}$$

where C_i^k is the binomial coefficient. Finally we define $h_0^{(k)}$ and $h_e^{(k)}$ as

$$h_0^{(k)}(t) \quad = \quad \sum_{i=0}^{k} \frac{a_1^{(k-i)}(t)}{a_1(t)} \sigma_i(t) C_i^k \tag{3.15}$$

$$h_e^{(k)}(t) \quad = \quad \sum_{i=0}^{k} \frac{\Phi_2^{(k-i)}(t)}{a_1(t)} \sigma_i(t) C_i^k. \tag{3.16}$$

Since $\Phi_1(t) = \Phi_2(t) + a_2(t)$ it follows that

$$h^{(k)}(t) = h_0^{(k)}(t) + h_e^{(k)}(t), \quad k = 1, 4 \tag{3.17}$$

3.3. Next we derive some useful information about the functions defined above. Throughout Section 3 we use $u = \pi e^{4t}$. We begin with

Lemma 3.1. *For all* $t \geq 0$ *and* $k = 0, 1, \ldots, 4$ *we have* $|\sigma_k(t)| < \sigma_{b,k}(t)$, *where*

$$\sigma_{b,k}(t) = \frac{8^k u^{2k}}{(2u-3)^k} \tag{3.18}$$

Proof. Substituting the definitions of p_k and α_j into the formulas for σ_k leads to

$$\sigma_k(t) = \frac{\eta_k(u)}{(2u-3)^k}, \quad k = 0, 1, \ldots, 4 \tag{3.19}$$

where

$$\begin{aligned}
\eta_0(u) &= 1 \\
\eta_1(u) &= 15 - 30u + 8u^2, \\
\eta_2(u) &= 225 - 660u + 948u^2 - 416u^3 + 64u^4, \\
\eta_3(u) &= 3375 - 12330u + 18348u^2 - 24984u^3 + 14624u^4 - 4224u^5 \\
&\quad + 512u^6, \\
\eta_4(u) &= 50625 - 219240u + 429048u^2 - 271200u^3 + 413712u^4 \\
&\quad - 331520u^5 + 146944u^6 - 36864u^7 + 4096u^8,
\end{aligned} \tag{3.20}$$

We note that the coefficient of the highest power of u in each of these polynomials is 8^k. Following the technique of [2], for each $k = 1, 2, \ldots, 4$, we calculate the location of the zeros of $8^k u^{2k} - \eta_k(u)$. It is found that in no case is there a real zero of value $\geq \pi$. Since for large u the expression is positive, it follows that $\eta_k(u) < 8^k u^{2k}$. Similarly, there are no real zeros $\geq \pi$ of $8^k u^{2k} + \eta_k(u)$, so that $\eta_k(u) > -8^k u^{2k}$. The conclusion is that $|\eta_k(u)| < 8^k u^{2k}$ and(3.18) follows. ☐

Lemma 3.2. *For all* $t \geq 0$ *and* $k = 1, \ldots, 4$ *we have* $|h_0^{(k)}(t)| < h_{0b,k}(t)$, *where*

$$h_{0b,k}(t) = \frac{12^k 2^{k+5} u^{2k+1} e^{-3u}}{(2u-3)^{(k+1)}} \tag{3.21}$$

Proof. Substituting the definitions of a_1^k and σ_k into (3.15) leads to

$$h_0^{(k)}(t) = \frac{u\rho_k(u)e^{-3u}}{(2u-3)^{k+1}}, \quad k = 1, 2, \ldots, 4 \tag{3.22}$$

where

$$\begin{aligned}
\rho_1(u) &= -720 + 1440u - 768u^2, \\
\rho_2(u) &= 8640 - 54720u + 96768u^2 - 68352u^3 + 18432u^4 \\
\rho_3(u) &= -103680 + 1486080u - 5223168u^2 + 8077824u^3 \\
&\quad - 6435840u^4 + 2598912u^5 - 442368u^6 \\
\rho_4(u) &= 1244160 - 36910080u + 220243968u^2 - 569650176u^3 \\
&\quad + 802308096u^4 - 664166400u^5 + 326762496u^6 \\
&\quad - 88915968u^7 + 10616832u^8
\end{aligned} \tag{3.23}$$

An argument analogous to that used in Lemma 3.1 produces the desired result. □

Lemma 3.3. *For all* $t \geq 0$ *and* $k = 0, 1, \ldots, 4$ *we have* $|\Phi_2^{(k)}(t)| < \Phi_{2b,k}(t)$*, where*

$$\Phi_{2b,k}^{(k)}(t) = 1.03\pi^{k+2}2^{2k+1}3^{2k+4}e^{(4k+9)t-9u} \tag{3.24}$$

Proof. The relation (3.24) has been proved in the case $k = 6$ by [2, (3.32)]. Their method of proof applies equally well to the cases listed in Lemma 3.3. □

Lemma 3.4. *For all* $t \geq 0$ *and* $k = 1, 2, \ldots, 4$ *we have* $|h_e^{(k)}(t)| < h_{eb,k}(t)$*, where*

$$h_{eb,k}(t) = \frac{334\pi^2 u^{k+1}(80u-108)^k e^{-8u}}{(2u-3)^{(k+1)}} \tag{3.25}$$

Proof. If we insert the bounds from Lemma 3.1 and Lemma 3.3 into (3.16), we obtain

$$|h_e^{(k)}(t)| < \frac{334\pi^2}{(2u-3)} \sum_{i=0}^{k} (36u)^{k-i} \Big(\frac{8u^2}{(2u-3)}\Big)^i C_i^k. \tag{3.26}$$

or

$$|h_e^{(k)}(t)| < 1.03\frac{324\pi^2}{(2u-3)}\Big[36u + \frac{8u^2}{(2u-3)}\Big]^k \tag{3.27}$$

which leads to (3.25). □

Lemma 3.5. *For all* $t \geq 0$ *and* $k = 1, 2, \ldots, 4$*, we have* $|h^{(k)}(t)| < h_{b,k}(t)$*, where*

$$h_{b,k}(t) = 1.01\frac{12^k 2^{k+5} u^{2k+1} e^{-3u}}{(2u-3)^{(k+1)}} \tag{3.28}$$

Proof. If we use the bounds from Lemma 3.2 and Lemma 3.4, we find that

$$\frac{|h_{eb,k}^{(k)}(t)|}{|h_{0b,k}^{(k)}(t)|} = Q(u) = \frac{334\pi^2(80u-108)^k e^{-5u}}{12^k 2^{k+5} u^k} \tag{3.29}$$

We see that for $u \geq \pi$, equivalent to $t \geq 0$,

$$\frac{d}{du}\log Q(u) = k\Big[\frac{80}{80u-108} - \frac{1}{u}\Big] - 5 = k\frac{108}{80u-108} - 5 < 0. \tag{3.30}$$

Thus $\frac{|h^{(k)}_{eb,k}(t)|}{|h^{(k)}_{0b,k}(t)|}$ is a strictly decreasing function of t, and therefore no greater than its value at $t = 0$, i.e. $u = \pi$. For the range of values of k specified, the largest value of $\frac{|h^{(k)}_{eb,k}(0)|}{|h^{(k)}_{0b,k}(0)|}$ occurs for $k = 4$, when it is $0.2027\ldots 10^{-3}$. Since $|h^{(k)}(t)| < |h_{0b,k}(t)| + |h_{eb,k}(t)|$ it follows from (3.21) that (3.28) holds. □

Lemma 3.6. *For all* $t \geq 0$ *we have* $2{f_0^{(2)}}^2 + {f_0}^{(4)} = H_1(t)$, *where* $H_1(t)$ *is a strictly increasing function of* t *defined below.*

Proof. Substituting the expressions (3.5) for $a_n^{(j)}(t)$ into the formulas (3.9), (3.10), we find that $H_1(t)$ may be written as as

$$H_1(t) = \frac{uF(u)}{(2u-3)^4} \tag{3.31}$$

where

$$F(u) = -34560 + 133632u - 245760u^2 + 159744u^3 - 53248u^4 + 8192u^5. \tag{3.32}$$

The derivative of $H_1(t))$ with respect to u has the form $J(u)/(2u-3)^5$, where $J(u)$ is polynomial in u of degree 6, with highest term $32768u^6$, and no real zeros. It follows that $H_1(t)$ is an increasing function of u, and therefore of t, for all $t \geq 0$. □

Lemma 3.7. *For all* $t \geq 0$ *we have* $4|f_0^{(2)}(t)f_e^{(2)}(t)| < H_2(t)$, *where*

$$H_2(t) = 4|f_0^{(2)}(t)|[h_{b,2}(t) + h_{b,1}(t)^2] \tag{3.33}$$

is a strictly decreasing function of t.

Proof. Since $|1/h(t)| < 1$, it follows from (3.11) and Lemma 3.5 that

$$|f_e^{(2)}(t)| < h^{(2)}(t) + {h^{(1)}(t)}^2 < h_{b,2}(t) + h_{b,1}(t)^2, \quad t \geq 0 \tag{3.34}$$

Now substitute (3.5) and (3.20) into (3.9) to obtain

$$f_0^{(2)}(t) = \frac{u\mu_2(u)}{(2u-3)^2} \tag{3.35}$$

where

$$\mu_2(u) = -240 + 192u - 64u^2. \tag{3.36}$$

Therefore from (3.28) we have

$$H_2(t) = c_1\frac{u^6\mu_2(u)e^{-3u}}{2u-3)^5} + c_2\frac{u^7\mu_2(u)e^{-6u}}{(2u-3)^6}, \quad t \geq 0. \tag{3.37}$$

where c_1, c_2 are positive constants independent of t. For the first term in (3.37) we copy the procedure used in Lemma 3.5, defining $Q(u)$ as

$$Q(u) = \frac{u^6\mu_2(u)e^{-3u}}{(2u-3)^5}. \tag{3.38}$$

Again, by examining the zeros of the numerator, it is found that $\frac{d}{du}\log Q(u)$ is a negative rational function of u for $t \geq 0$. The same procedure and result applies equally to the second term in (3.37), and thus $H_2(t)$ is indeed a strictly decreasing function of t as Lemma 3.5 states. □

Lemma 3.8. *For all* $t \geq 0$ *we have* $f_e^{(4)}(t) > -H_3(t)$, *where*

$$H_3(t) = h_{eb,4}(t) + 4h_{b,1}(t)h_{b,3}(t) + 3h_{b,2}(t)^2 + 12h_{b,1}(t)^2h_{b,2}(t) + 6h_{b,1}(t)^4 \tag{3.39}$$

is a strictly decreasing function of t.

Proof. It follows from (3.11) and Lemma 3.5 that

$$f_e{}^{(4)}(t) = \frac{h_0^{(4)}}{h(t)} + S(t) \tag{3.40}$$

where

$$S(t) = \frac{h_e^{(4)}(t)}{h(t)} - 4\frac{h^{(1)}(t)h^{(3)}(t)}{h(t)^2} - 3\frac{h^{(2)}(t)^2}{h(t)^2} + 12\frac{h^{(1)}(t)^2h^{(2)}(t)}{h(t)^3} - 6\frac{h^{(1)}(t)^4}{h(t)^4} \tag{3.41}$$

From (3.22) we have

$$h_0^{(4)}(t) = \frac{u\rho_4(u)e^{-3u}}{(2u-3)^5} \tag{3.42}$$

where $\rho_4(u)$ is given by (3.23). Using familiar arguments it may be checked that the polynomial $\rho_4(u) > 0,\ t \geq 0$, so that $\frac{h_0^{(4)}(t)}{h(t)} > 0,\ t \geq 0$.

Lemmas 3.2, 3.4 show that $H_3(t)$ is an upper bound for $|S(t)|$, so that $f_0^{(4)}(t) > -H_3(t),\ t \geq 0$ as stated. Arguments applied to the terms of $H_3(t)$ similar to those described above demonstrate the strictly decreasing nature of the function. □

3.4. We now come to the main result of Section 3.

Theorem 3.9. *For all* $t \geq 0$, *the function* $W_3(t) < 0$, *where* $W_3(t)$ *is defined in (2.5).*

Proof. Since $f^{(3)^2} \geq 0$ we see from (3.2)that the Theorem will hold if we can prove that

$$f^{(2)}(t)\left[2f^{(2)}(t)^2 + f^{(4)}(t)\right] < 0, \quad t \geq 0 \tag{3.43}$$

Now from (2.7)

$$\Phi(t)^2 f^{(2)}(t) = \Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2 = W_2(t) < 0 \quad t \geq 0, \tag{3.44}$$

so that (3.43) reduces to

$$2f^{(2)}(t)^2 + f^{(4)}(t) > 0, \quad t \geq 0. \tag{3.45}$$

Substituting for $f(t)$ according to (3.8) leads to

$$2f^{(2)}(t)^2 + f^{(4)}(t) = [2f_0{}^{(2)^2} + f_0{}^{(4)}] + [4f_0{}^{(2)}f_e{}^{(2)}] + [f_e{}^{(4)}] + [2f_e{}^{(2)^2}] \tag{3.46}$$

From Lemmas 3.6, 3.7 and 3.8 we have

$$\begin{aligned} f_0{}^{(2)^2} + f_0{}^{(4)} &> H_1(0) - 1 \\ |4f_0{}^{(2)}f_e{}^{(2)}| &< H_2(0) \\ f_e{}^{(4)} &> -H_3(0) \end{aligned} \tag{3.47}$$

Therefore it follows on calculating the values of $H_j(0)$, $j = 1, 2, 3$, that

$$\begin{aligned} 2{f^{(2)}(t)}^2 + f^{(4)}(t) &> H_1(0) - 1 - H_2(0) - H_3(0) \\ &> 6295.544\ldots - 1 - 4075.900\ldots - 1199.914\ldots \\ &> 1018.0\ldots \end{aligned}$$

Thus, in view of (3.45), the theorem is proved. □

4. DISCUSSION

Numerical study suggests that the kernel $K(x, y)$ is RR_4, in which case the determinantal inequalities (1.6) could be proved for $r = 4$, $n \geq 3$. Calculation of $W_5(0)$ makes it certain that $K(x, y)$ is not RR_5. We discuss possible further steps towards the RH in an unpublished report [5].

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