Determinants, Sign Regularity, and the Riemann Hypothesis

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Outline

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Pólya Question

Define

$$\Phi(u) = \sum_{m=1}^{\infty} (2m^4\pi^2 e^{9u} - 3m^2\pi e^{5u}) \exp(-m^2\pi e^{4u}), \tag{1}$$

with

$$\xi(z) = \int_0^\infty \Phi(u) \cos(uz)\, du. \tag{2}$$

Riemann Hypothesis: Function $\xi(z)$ has only real zeros.

Pólya Question (1925): *What properties of* $\Phi(u)$ *are sufficient to secure that* $\xi(z)$ *has only real zeros?*

Pólya: "*I have found some criteria answering the proposed question, but I will not give them here, because they are rather unsystematic and tentative in character.*"

Determinantal Method

Define the normalized (double) moments of $\Phi(u)$ as

$$\beta_n = \frac{1}{\Gamma(2n+1)} \int_0^\infty \mathrm{d}u\, \Phi(u) u^{2n}, \quad n = 0,\ 1,\ \ldots. \tag{3}$$

Define the matrix B as

$$B(i,j) = \begin{cases} \beta_{j-i}, & j \geq i, \\ 0, & j < i, \end{cases} \qquad i,\ j = 0,\ 1,\ 2,\ \ldots, \tag{4}$$

For $n \geq 0$ and $r \geq 1$, set

$$D(n,r) = \det\left[B(i, j+n)\right]_{i,\,j=1,\,\ldots,\,r}. \tag{5}$$

Then the Riemann Hypothesis holds if

$$D(n,r) > 0, \qquad n = 0,\ 1,\ \ldots, \quad r = 1,\ 2,\ \ldots. \tag{6}$$

In 1986 Csordas, Norfolk and Varga (CNV)[1] proved the Turán inequalities

$$(\beta_m)^2 > \left(\frac{m+1}{m}\right) \beta_{m-1}\beta_{m+1}, \quad m = 1,\ 2,\ \ldots, \tag{7}$$

that must hold if the RH is true.

Note that, if $r = 2$, then (6) leads to

$$(\beta_m)^2 > \beta_{m-1}\beta_{m+1}, \quad m = 1,\ 2,\ \ldots, \tag{8}$$

a true but weaker condition than (7).

Our aim is to prove (6) for $r > 2$.

[1] G. Csordas, T. Norfolk and R. Varga, *Trans. Am. Math. Soc.* **296**(2):521 (1986).

Outline

Sign-Regularity

Following Karlin[2] (K) take a kernel $Q(x, y)$. Let X be a linearly ordered set. For a given positive integer p the open simplex $\Delta_p(X)$ is

$$\Delta_p\,(X) = \{\boldsymbol{x} = (x_1, x_2,\, \ldots,\, x_p)\,| x_1 < x_2 < \ldots < x_p : \, x_i \in X\}. \qquad (9)$$

The compound kernel $Q_{[p]}(\boldsymbol{x}, \boldsymbol{y})$ is defined by

$$Q_{[p]}\,(\boldsymbol{x}, \boldsymbol{y}) = \begin{vmatrix} Q\,(x_1, y_1) & Q\,(x_1, y_2) & \ldots & Q\,(x_1, y_p) \\ Q\,(x_2, y_1) & Q\,(x_2, y_2) & \ldots & Q\,(x_2, y_p) \\ \vdots & \vdots & & \vdots \\ Q\,(x_p, y_1) & Q\,(x_p, y_2) & \ldots & Q\,(x_p, y_p) \end{vmatrix}. \qquad (10)$$

Define $\epsilon_p = (-1)^{p(p-1)/2}$, $p = 1,\, \ldots,\, r$. Then $Q(\boldsymbol{x},\, \boldsymbol{y})$ is sign-reverse regular of order r (*i.e.* RR_r) if $\epsilon_p Q_{[p]}\,(\boldsymbol{x}, \boldsymbol{y})$ is a non-negative function on $\Delta_p\,(X) \times \Delta_p\,(Y)$ for each $p = 1,\, 2,\, \ldots,\, r$. Similarly, if $\epsilon_p = 1$, $p = 1,\, \ldots,\, r$, then $Q(\boldsymbol{x}, \boldsymbol{y})$ is totally positive (*i.e.* TP_r).

[2] S. Karlin, Total Positivity (Stanford Univ. Press, 1968).

Kernels

Define the kernel

$$K(u,v) = \Phi(u+v), \quad u,\ v \geq 0. \tag{11}$$

Suppose that

$$\lambda(t) = \int_0^\infty \mathrm{d}v\, \phi(v,t)\Phi(v),$$

with the kernel $\phi(v,t)$ being

$$\phi(v,t) = \frac{v^{t-1}}{\Gamma(t)}, \quad t > 0. \tag{12}$$

Note from (3) that

$$\beta_n = \lambda(2n+1), \quad n \geq 0. \tag{13}$$

Define the kernel

$$\Lambda(s,t) = \lambda(s+t), \quad s,\ t > 0. \tag{14}$$

Double BCF Representation

The argument underlying Karlin's development shows that, for appropriate values of s and t,

$$\int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v\, \phi(u,s)\Phi(u+v)\phi(v,t) = \int_0^\infty \mathrm{d}v\, \phi(v,s+t)\Phi(v). \tag{15}$$

Thus we obtain

$$\Lambda(s,t) = \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v\, \phi(u,s)K(u,v)\phi(v,t). \tag{16}$$

From two applications of the basic composition formula (BCF) to equation (16) we find that for any $p > 0$

$$\Lambda_{[p]}(\boldsymbol{s},\boldsymbol{t}) = \int_0^\infty \mathrm{d}\boldsymbol{u} \int_0^\infty \mathrm{d}\boldsymbol{v}\, \phi_{[p]}(\boldsymbol{u},\boldsymbol{s})K_{[p]}(\boldsymbol{u},\boldsymbol{v})\phi_{[p]}(\boldsymbol{v},\boldsymbol{t}), \tag{17}$$

where $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{s}$, $\boldsymbol{t}$ are defined as in (9).

$D(n, 2)$ Formula

The relevance of the above relations becomes clear when we combine the information in (17), (14) and (13). Suppose in (17) we choose $p = 2$, and $\boldsymbol{s}$, $\boldsymbol{t}$ such that

$$s_1 = t_1 = n - \frac{1}{2}, \quad s_2 = t_2 = n + \frac{3}{2}, \quad n = 1,\ 2,\ \ldots. \tag{18}$$

The elements of the determinant $\Omega(n) = \Lambda_{[2]}(\boldsymbol{s}, \boldsymbol{t})$ are then

$$\Omega(n) = \begin{bmatrix} \lambda(s_1 + t_1) & \lambda(s_1 + t_2) \\ \lambda(s_2 + t_1) & \lambda(s_2 + t_2) \end{bmatrix}, \quad n = 1,\ 2,\ \ldots.$$

Thus, using (13), we have

$$\Omega(n) = \begin{vmatrix} \beta_{n-1} & \beta_n \\ \beta_n & \beta_{n+1} \end{vmatrix} = -D(n, 2), \quad n = 1,\ 2,\ \ldots \tag{19}$$

since $D(n, 2)$ is the determinant in (19) with columns reversed.

Consequences of Sign-Regularity

The kernel $\phi(v, t)$ is TP or totally positive, *i.e.* the determinant $\phi_{[p]}(\boldsymbol{u}, \boldsymbol{s})$ is positive for $(\boldsymbol{u},\ \boldsymbol{s})$ in the appropriate domain [see (9)], $p = 1$, 2, $\ldots$.

Now suppose that $K_{[2]}(\boldsymbol{u}, \boldsymbol{v}) < 0$ for all values of the arguments (*i.e.* $K(u, v)$ is RR_2), then it follows from (17) that $\Lambda(u, v)$ is RR_2, so that $\Omega(n) < 0$, $n = 1$, 2, $\ldots$, and hence $D(n, 2) > 0$, as required by the RH. The case of $n = 0$ is trivial.

Similarly, provided that the kernel $K(u, v)$ is RR_r, $r > 2$, it follows that $D(n, r) > 0$, $n = r$, $r + 1$, $\ldots$.

There follows a discussion of a method for evaluating the nature of the sign-regularity of $K(u, v)$. There is strong evidence that this kernel is RR_r, $r = 3$, 4, but the same is definitely not true for $r > 4$. A method for dealing with cases with $r > 4$ appears later.

Wronskian Sign-Regularity Test

Karlin Test: Suppose that $\psi(x)$ is analytic in a neighborhood of $X = (0, \infty)$, and that the kernel $Q(x, y) = \psi(x + y)$ with x, $y \in (0, \infty)$. Define $w_p(u) = \det \left|\psi^{(i+j-2)}(u)\right|_{i,j=1}^{p}$. If $\epsilon_p w_p(u) > 0$, $u \geq 0$, $p = 1$, 2, $\ldots$, r, then Karlin shows that $Q(x, y)$ is RR_r.

To apply this test to the kernel $K(u, v)$ of (11) set $\psi(x) = \Phi(x)$ and choose $r = 2$. For $p = 1$ the above condition requires that $\psi(u) > 0$, $u \geq 0$, since $\epsilon_1 = 1$. This is true since $\Phi(u) > 0$, $u \geq 0$.

For $p = 2$ the condition becomes, since $\epsilon_2 = -1$,

$$-\begin{vmatrix} \Phi(u) & \Phi^{(1)}(u) \\ \Phi^{(1)}(u) & \Phi^{(2)}(u) \end{vmatrix} = \Phi^{(1)}(u)^2 - \Phi(u)\Phi^{(2)}(u) > 0, \quad u \geq 0. \tag{20}$$

This follows from the work of CNV while proving (7).

Outline

Sign-Regularity for Order 3 – Part 1

Using Csordas and Varga (CV)[3] gives, with $y = \pi e^{4u}$,

$$\Phi^{(k)}(u) = \pi e^{5u-y}\left\{p_{k+1}(y) + 4e^{-3y}p_{k+1}(4y) + \Psi_{2,k}(u)\right\}, \tag{21}$$

where

$$\Psi_{2,k}(u) = e^{y}\sum_{m=3}^{\infty} m^2 p_{k+1}(m^2 y)e^{-m^2 y}. \tag{22}$$

Polynomials $p_k(y)$, $k = 1, \ldots, 5$ are defined as

$$\begin{aligned}
p_1(y) &= 2y - 3\\
p_2(y) &= -8y^2 + 30y - 15\\
p_3(y) &= 32y^3 - 224y^2 + 330y - 75\\
p_4(y) &= -128y^4 + 1440y^3 - 4232y^2 + 3270y - 375\\
p_5(y) &= 512y^5 - 8448y^4 + 41408y^3 - 68096y^2 + 30930y - 1875.
\end{aligned} \tag{23}$$

[3] G. Csordas and R. Varga, *Constr. Approx.* **4**(2):175 (1988).

Sign-Regularity for Order 3 – Part 2

For $r = 3$ the Karlin test requires

$$w_3(u) = \begin{vmatrix} \Phi(u) & \Phi^{(1)}(u) & \Phi^{(2)}(u) \\ \Phi^{(1)}(u) & \Phi^{(2)}(u) & \Phi^{(3)}(u) \\ \Phi^{(2)}(u) & \Phi^{(3)}(u) & \Phi^{(4)}(u) \end{vmatrix} \tag{24}$$

Now substitute (21) into (24). Each column of (24) may be thought of as a sum of three columns corresponding to the three terms in (21), so that $w_3(u)$ may be written as the sum of $3 \times 3 \times 3 = 27$ determinants formed by choosing one from each of the three columns comprising a column of $w_3(u)$. The result is

$$w_3(u) = \left(\pi e^{5u-y}\right)^3 W_3(u).$$

where

$$\begin{aligned} W_3(u) =& F_0(y) + e^{-3y} F_{1,1}(y) + e^{-6y} F_{1,2}(y) + e^{-9y} F_{1,3}(y) \\ &+ F_{2,1}(y) + e^{-3y} F_{2,2}(y) + e^{-6y} F_{2,3}(y) + F_3(y) + F_4(y). \end{aligned} \tag{25}$$

Sign-Regularity for Order 3 – Part 3

The first term $F_0(y)$ in $W_3(u)$ is obtained by choosing the first term $p_{k+1}(y)$ in each element of $w_3(t)$, so that

$$F_0(y) = \begin{vmatrix} p_1(y) & p_2(y) & p_3(y) \\ p_2(y) & p_3(y) & p_4(y) \\ p_3(y) & p_4(y) & p_5(y) \end{vmatrix} = \begin{matrix} 860160y^3 - 737280y^4 \\ +294912y^5 - 65536y^6 \end{matrix} \tag{26}$$

$F_0(y)$ has three zeros, apart from $y = 0$,

$$F_0(y) = -65536y^3[(y-a)^2 + b^2](y-c),$$
$$a = 1.192509\ldots,\ b = 2.187155\ldots,\ c = 2.114980\ldots.$$

Now $u \geq 0$, *i.e.* $y \geq \pi$, so that, since $\pi > 2.11498...$, it follows that

$$F_0(y) < 0, \quad y \geq \pi, \tag{27}$$

so that $-F_0(y)$ increases steadily as y increases from π. Thus the least upper bound of $F_0(y)$, $y > \pi$, is $F_0(\pi) \approx -1.7904\times 10^7$.

Sign-Regularity for Order 3 – Part 4

The next term in $W_3(u)$, *i.e.* $e^{-3y}F_{1,1}(y)$, uses two columns of type $p_{k+1}(y)$ and one of type $4e^{-3y}p_{k+1}(4y)$. Thus

$$\begin{aligned} F_{1,1}(y) = & \begin{vmatrix} 4p_1(4y) & p_2(y) & p_3(y) \\ 4p_2(4y) & p_3(y) & p_4(y) \\ 4p_3(4y) & p_4(y) & p_5(y) \end{vmatrix} + \begin{vmatrix} p_1(y) & 4p_2(4y) & p_3(y) \\ p_2(y) & 4p_3(4y) & p_4(y) \\ p_3(y) & 4p_4(4y) & p_5(y) \end{vmatrix} \\ & + \begin{vmatrix} p_1(y) & p_2(y) & 4p_3(4y) \\ p_2(y) & p_3(y) & 4p_4(4y) \\ p_3(y) & p_4(y) & 4p_5(4y) \end{vmatrix} \\ = & 72253440y^3 - 482181120y^4 + 792281088y^5 \\ & -579403776y^6 + 228261888y^7 - 42467328y^8. \end{aligned} \tag{28}$$

As above it may be shown that the least upper bound to $e^{-3y}F_{1,1}(y)$ is $e^{-3\pi}F_{1,1}(\pi) \approx -5.8783 \times 10^6$.

Sign-Regularity for Order 3 – Part 5

Using the above techniques, and a bound on $\Psi_{2,k}(u)$ provided by CV, it is found that, for $y \geq \pi$, the nine contributions to $W_3(u)$ in (25) have upper bounds as follows

1	$F_0(y)$	-1.7904×10^{7}
2	$e^{-3y}F_{1,1}(y)$	-5.8783×10^{6}
3	$e^{-6y}F_{1,2}(y)$	-5.6349×10^{4}
4	$e^{-9y}F_{1,3}(y)$	-6.1328×10^{0}
5	$F_{2,1}(y)$	3.083×10^{3}
6	$e^{-3y}F_{2,2}(y)$	2.422×10^{2}
7	$e^{-6y}F_{2,3}(y)$	1.379×10^{1}
8	$F_3(y)$	3.497×10^{-2}
9	$F_4(y)$	3.036×10^{-12}

For all $t \geq 0$, the function $W_3(u) < 0$, so that the Karlin test shows that $K(u,v)$ is RR_3, which means that $D(n,3) > 0$, $n = 2, 3, \ldots$.

Outline

Cumulant – Part 1

A calculation shows that there is no doubt that $K(u, v)$ is not RR_r for $r \geq 5$. To make progress for general values of r introduce the cumulants $\{\Psi_m(u)\}$, where

$$\Psi_m(u) = \int_u^\infty \mathrm{d}t\, \Psi_{m-1}(t), \quad m = 1,\ 2,\ \ldots; \quad \Psi_0(u) = \Phi(u). \tag{29}$$

There is a corresponding set of kernels

$$K(u, v; m) = \Psi_m\,(u + v), \qquad u,\ v \geq 0, \quad m = 0,\ 1,\ \ldots. \tag{30}$$

As in (15) it may be shown that, for appropriate values of m, s, t,

$$\int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v\, \phi(u, s)\Psi_m(u + v)\phi(v, t) = \int_0^\infty \mathrm{d}v\, \phi(v, s + t)\Psi_m(v).$$

Integrating by parts m times leads to

$$\int_0^\infty \mathrm{d}v\, \phi(v, s + t)\Psi_m(v) = \int_0^\infty \mathrm{d}v\, \phi(v, s + t + m)\Phi(v) = \lambda(s + t + m),$$

where t, $s > 0$ and $m = 0$, 1, 2, $\ldots$.

Thus

$$\Lambda(s+m/2, t+m/2) = \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v\, \phi(u,s) K(u,v;m) \phi(v,t). \tag{31}$$

With the compound kernel $K_{[p]}(\boldsymbol{u},\boldsymbol{v};m)$ corresponding to $K(u,v;m)$,

$$\Lambda_{[p]}(\boldsymbol{s}+m\boldsymbol{w}/2, \boldsymbol{t}+m\boldsymbol{w}/2) = \int_0^\infty \mathrm{d}\boldsymbol{u} \int_0^\infty \mathrm{d}\boldsymbol{v}\, \phi_{[p]}(\boldsymbol{u},\boldsymbol{s}) K_{[p]}(\boldsymbol{u},\boldsymbol{v};m) \phi_{[p]}(\boldsymbol{v},\boldsymbol{t}), \tag{32}$$

where $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{s}$, $\boldsymbol{t}$ are defined as in (9), and $\boldsymbol{w}$ is a vector with all components equal to unity.

Now we adapt the procedure of (17), (18) to relate the compound kernel appearing in the LHS of (32) to some of the determinants needed in (6). In (32) we choose $p = r > 1$, and $\boldsymbol{s}$, $\boldsymbol{t}$ such that

$$s_j = t_j = \mu + 2j, \quad j = 1, \ \ldots, \ r.$$

Cumulant – Part 3

The elements of the determinant $\Lambda_{[r]}(\boldsymbol{s} + m\boldsymbol{w}/2, \boldsymbol{t} + m\boldsymbol{w}/2)$ are then

$$\lambda(2\mu + 2i + 2j + m), \quad i, j = 1, \ldots, r. \tag{33}$$

If the argument of λ appearing in (33) is odd and positive for all entries, then we can use (14) to express all elements of $\Lambda_{[r]}(\boldsymbol{s} + m\boldsymbol{w}/2, \boldsymbol{t} + m\boldsymbol{w}/2)$ in terms of the coefficients β_n given by (13). For use in (32) the inequality (12) requires that $\mu + 2 > 0$. It follows that the determinant with elements given by (33) is the same as $D(n, r)$ with the order of the rows reversed, and an appropriate choice of n. As before, the reversal of the rows changes the determinant by a factor of ϵ_r.

It follows that all values of k are possible so long as $k \geq k_L$. It is found that

$$k_L = n_L + r - 1,$$

where $n_L = m/2$ for even m, and $n_L = (m+1)/2$ for odd m.

Sign-Regularity of Cumulants – Part 1

Numerical evidence based on the Karlin test provides evidence to support

CONJECTURE

For any given order $r > 1$, there is a lowest integer $m(r)$ such that $K(u, v; m(r))$ is RR_r.

Karlin shows that, because of the relation (29), if $K(u, v; m_1)$ is RR_r, then so is $K(u, v; m_2)$ for all $m_2 > m_1$. Thus the conjecture implies that, if $m \geq m(r)$, then $K(u, v; m)$ is RR_r. It also follows that, if $r_2 > r_1$, then $m(r_2) \geq m(r_1)$.

The conjecture means that the compound kernel $K_{[r]}(\boldsymbol{u}, \boldsymbol{v}; m(r))$ has sign ϵ_r, so that, as before, the same applies to $\Lambda_{[r]}(\boldsymbol{s} + m(r)\boldsymbol{w}/2, \boldsymbol{t} + m(r)\boldsymbol{w}/2)$ from (32), since the other factors in the integrand are non-negative.

Sign-Regularity of Cumulants – Part 2

The conclusion is that, given the conjecture, we can prove the positivity of $D(n, r)$ needed in (6) for all values of n and r such that $n \geq k_L(m(r))$. Thus, if the conjecture is correct, the proof of the RH reduces to showing that

$$D(n, r) > 0, \qquad n = 0, \ldots, \quad \eta(r), \; r = 1, 2, \ldots . \tag{34}$$

To examine in a non-rigorous numerical way whether the conjecture might be valid we have applied the Karlin test to the kernels $K(x, y; m)$, $m = 0$, 1, 2, $\ldots$; $r = 2$, 3, $\ldots$. The results, subject to further checking of the accuracy of the numerical methods, are consistent with the conjecture, some values of $m(r)$ being listed below.

Note the sets of three consecutive equal values of $m(r)$, colored red. There is one set of two consecutive equal values colored green. Between these sets are sequences of values with an increment of 2. The increments on either end of a triplet are one and two.

Sign-Regularity of Cumulants – Part 3

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(r)$	0	0	0	1	1	1	2	4	6	7	7	7	9	11	13
r	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$m(r)$	15	15	15	16	18	20	22	24	24	24	25	27	29	31	33
r	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
$m(r)$	34	34	34	36	38	40	42	44	45	45	45	47	49	51	53
r	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
$m(r)$	55	57	57	57	58	60	62	64	66	68	69	69	69	71	73
r	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
$m(r)$	75	77	79	81	81	81	82	84	86	88	90	92	94	94	94
r	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
$m(r)$	95	97	99	101	103	105	107	108	108	109	111	113	115	117	119
r	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106
$m(r)$	121	122	122	122	124	126	128	130	132	134	136	137	137	137	139
r	107	108	109	110	111										
$m(r)$	141	143	145	147	149										

Outline

Single Moment Method – Part 1

With the definition of β_n in (3) the determinant $D(n, r)$ of order r is given by

$$D(n,r) = \begin{vmatrix} \beta_n & \beta_{n+1} & \cdots & \beta_{n+r-1} \\ \beta_{n-1} & \beta_n & \cdots & \beta_{n+r-2} \\ \vdots & \vdots & & \vdots \\ \beta_{n-r+1} & \beta_{n-r+2} & \cdots & \beta_n \end{vmatrix}, \tag{35}$$

where $\beta_n = 0$, $n < 0$.

Single Moment Method – Part 2

Another set of normalized moments, both even and odd, is defined by

$$b_n = \frac{1}{\Gamma(n+1)} \int_0^\infty \mathrm{d}u \, \Phi(u) u^n, \quad n = 0, 1, \ldots . \tag{36}$$

In this case we define the "moments" for negative n by

$$b_n \; = (-1)^{n+1} \Phi^{(-n-1)}(0), \quad n = -1, \; -2, \; \ldots . \tag{37}$$

A corresponding set of determinants is

$$\Delta(n,r) = \begin{vmatrix} b_n & b_{n+1} & \ldots & b_{n+r-1} \\ b_{n-1} & b_n & \ldots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-r+1} & b_{n-r+2} & \ldots & b_n \end{vmatrix}. \tag{38}$$

Note that, if $n = 2j$, $j = \ldots, -2, -1, 0, 1, 2, \ldots$, then $\beta_j = b_n$. This relation applies even for negative j, since then $\beta_j = 0$.

Single Moment Method – Part 3

Karlin discussed cumulants (he called them modified kernels), and he showed that

$$\Psi_m(u) = \frac{1}{\Gamma(m)} \int_u^\infty \mathrm{d}t \, \Phi(t)(t-u)^{m-1}, \quad m \geq 1. \tag{39}$$

This formula may be verified by differentiating with respect to u, which shows that

$$\Psi_m^{(1)}(u) = -\Psi_{m-1}(u), \quad m \geq 1. \tag{40}$$

consistent with (29).

The Karlin test shows that the requirement for the kernel $K(u, v; m) = \Psi_m(u + v)$ to be RR_r is that

$$\epsilon_p w(p, m; u) > 0, \qquad u \geq 0, \quad p = 1, \ 2, \ \ldots, \ r,$$

where

$$w(p, m; u) = \det \left| \Psi_m^{(i+j-2)}(u) \right|_{i,j=1}^{p}. \tag{41}$$

Single Moment Method – Part 4

Suppose that m, r are chosen so that $K(u, v; m)$ is RR_r according to the conjecture above. We note that from (39)

$$\Psi_m(0) = \frac{1}{\Gamma(m)} \int_0^\infty \mathrm{d}t\, \Phi(t) t^{m-1} = b_{m-1}, \quad m \geq 1, \tag{42}$$

so that from (40)

$$\Psi_m^{(n)}(0) = (-1)^n \Psi_{m-n}(0) = (-1)^n b_{m-n-1}, \quad n = 0,\ 1,\ \ldots. \tag{43}$$

It follows that

$$\begin{aligned}\epsilon_r w(r, m; 0) =& \epsilon_r \begin{vmatrix} b_{m-1} & -b_{m-2} & \ldots & (-1)^{r-1} b_{m-r} \\ -b_{m-2} & b_{m-3} & \ldots & (-1)^{r-2} b_{m-r-1} \\ \vdots & \vdots & & \vdots \\ (-1)^{r-1} b_{m-r} & (-1)^{r-2} b_{m-r-1} & \ldots & b_{m-2r+1} \end{vmatrix} \\ =& \Delta(n, r) \end{aligned} \tag{44}$$

where $n = m - r$.

Single Moment Method – Part 5

We begin by considering the case $r = 2$, and note that in (20) we proved that $\epsilon_2 w(2, 0; 0) > 0$, which means that $\Delta(-2, 2) > 0$. Since $K(u, v; 0)$ being RR_2 implies that $K(u, v; m)$, $m > 0$, is also RR_2, it immediately follows that $\Delta(n, 2) > 0$, $n \geq -2$. The coefficient $b_{-2} = 0$, but $b_n > 0$, $n \geq -1$, so that we can apply [K Thm. 3.2, p. 59], with rows and columns interchanged, to the matrix

$$\begin{bmatrix} b_{-1} & b_0 & b_1 & b_2 & \dots \\ b_{-2} & b_{-1} & b_0 & b_1 & \dots \end{bmatrix}. \tag{45}$$

The theorem states that, since all 2×2 minors of (45) with consecutive columns are positive, and also $b_n > 0$, $n \geq -1$, then all 2×2 minors are also positive. In the applications below we shall need the values of a few other 2×2 minors that are easily determined by inspection.

Next turn to matrices of the form

$$\begin{bmatrix} b_n & b_{n+1} & b_{n+2} \\ b_{n-1} & b_n & b_{n+1} \\ b_{n-2} & b_{n-1} & b_n \end{bmatrix}. \tag{46}$$

Assume that $n = 2k$, $k \geq 1$ so that the sole 2×2 minor of (46) that can be formed by taking only odd rows and columns is

$$D(k,2) = \begin{vmatrix} \beta_k & \beta_{k+1} \\ \beta_{k-1} & \beta_k \end{vmatrix}. \tag{47}$$

Applying [K Thm. 3.2, p. 59] to the first two columns of (46) and again to the last two columns, leads to the result that all the 2×2 minors with consecutive columns of the following matrix are > 0:

$$\begin{bmatrix} b_n & b_{n+1} & b_{n+2} \\ b_{n-2} & b_{n-1} & b_n \end{bmatrix}, \quad n > 0. \tag{48}$$

Apply the same theorem to (48) to show that $D(k,2)$ is > 0.

Single Moment Method – Part 7

In the case $n = k = 0$ the theorem does not apply, since the element $b_{-2} = 0$. To deal with this case we apply the technique used in the proof of Karlin [K Thm. 3.2, p. 59]. Consider the first two columns of (46). The formula of Karlin shows that

$$b_{-1} \begin{vmatrix} b_0 & b_1 \\ b_{-2} & b_{-1} \end{vmatrix} = b_{-2} \begin{vmatrix} b_0 & b_1 \\ b_{-1} & b_0 \end{vmatrix} - b_0 \begin{vmatrix} b_{-2} & b_{-1} \\ b_{-1} & b_0 \end{vmatrix}. \tag{49}$$

Since $b_{-2} = 0$ and $b_{-1} > 0$, (48) shows that

$$\begin{vmatrix} b_0 & b_1 \\ b_{-2} & b_{-1} \end{vmatrix} > 0. \tag{50}$$

From this point the discussion used in the case of $k > 0$ applies, and we find that $D(0, 2) > 0$. Of course we knew that was the case, but the point is to illustrate how [K Thm. 3.2, p. 59] helps us prove that $D(k, 2) > 0$, $k = 0, 1, \ldots$, given some information about the 2×2 minors of $\Delta(n, 3)$, $n = 0, 1, \ldots$.

Single Moment Method – Part 8

The single moment procedure can be extended to $r \geq 3$. However, the kernel $K(u, v; m)$ is definitely not RR_r unless $m \geq m(r)$, where $m(r)$ resulting from our calculations is given in a table up to $r = 111$. Taking account of this constraint, we repeatedly apply the theorem in [K Thm. 3.2, p. 59]. First we observe that the RR_r property of the kernel $K(u, v; m)$ may be used to show that

$$\Delta(n, r) > 0, \quad n \geq m(r) - r. \tag{51}$$

Next consider the determinant $\Delta(N, 2r - 1)$, where N is even. All the $r \times r$ minors with consecutive rows and columns will have a top left corner element of the form b_k. The minor for which k is least occurs in the bottom right corner, and it may be seen that

$$k = N - r + 1.$$

To apply the Karlin theorem, (51) requires that $k \geq m(r) - r$, which implies that

$$N \geq m(r) - 1. \tag{52}$$

In addition the Karlin theorem also requires that $(r-1) \times (r-1)$ minors of $\Delta(N, 2r-1)$ with consecutive rows and columns should be positive. An argument similar to the above leads to the requirement

$$N \geq m(r-1) + 1. \tag{53}$$

Thus, if both (52) and (53) hold, all $r \times r$ minors are positive. In particular, (with $[\,]$ meaning integer part), if we set

$$\mu = [(N+1)/2] = N/2,$$

then it follows that $D(\mu, r) > 0$.
In the above discussion we have not taken account of the fact that the method described for the case $r = 3$ uses a bottom left 2×2 matrix that is not positive but zero. We have shown that the method of (49) applies up to $r = 7$, so that we may set $\mu(r) = 0$, $r \leq 7$. We have obtained a significantly stronger result than the Karlin BCF method produces, but there remain cases to be treated.

Outline

Using the Structure of $\Phi(u)$ – Part 1

Define the theta-function $\theta(x)$ by

$$\theta(x) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi x}, \tag{54}$$

If we set

$$\chi(u) = \theta(e^{4u}), \tag{55}$$

then it will be seen that

$$\Phi(u) = \frac{e^u}{16}\left[\frac{\mathrm{d}^2\chi}{\mathrm{d}u^2} + 2\frac{\mathrm{d}\chi}{\mathrm{d}u}\right]. \tag{56}$$

The functional equation[4] for $\theta(x)$ states that

$$\theta(x) = x^{-1/2}\theta(1/x), \tag{57}$$

from which it follows that

$$\Phi(-u) = \Phi(u). \tag{58}$$

[4] E. Titchmarsh, The Theory of the Riemann Zeta-Function (Clarendon Press, Oxford 1986).

Using the Structure of $\Phi(u)$ – Part 2

Writing $Y_m = m^2 y$, $y = \pi e^u$, leads to

$$\Phi(u) = e^u \sum_{m=1}^{\infty} Y_m e^{-Y_m} p_1(Y_m), \qquad (59)$$

so that it becomes clear that the form of $p_1(y) = 2y - 3$ is responsible for the even nature of $\Phi(u)$ in (58).
In the analysis related to $r = 3$ Wronskian it was found that the critical factor in the dominant term was (26) $F_{0,3}$ — similarly for $r = 2$

$$F_{0,3}(y) = \begin{vmatrix} p_1(y) & p_2(y) & p_3(y) \\ p_2(y) & p_3(y) & p_4(y) \\ p_3(y) & p_4(y) & p_5(y) \end{vmatrix} = \begin{array}{l} 2^{13} \cdot y^3 \left(3{\cdot}5{\cdot}7 - 2{\cdot}3^2{\cdot}5{\cdot}y \right. \\ \left. +2^2{\cdot}3^2{\cdot}y^2 - 2^3{\cdot}y^3\right) \end{array}$$

$$F_{0,2}(y) = \begin{vmatrix} p_1(y) & p_2(y) \\ p_2(y) & p_3(y) \end{vmatrix} = -2^4{\cdot}y\left(3{\cdot}5 - 2^2{\cdot}3{\cdot}y + 2^3{\cdot}y^2\right).$$

Using the Structure of $\Phi(u)$ – Part 3

It is known by explicit calculation that $F_{0,k}(y) < -C_k$, $k = 2, 3$, where C_k is positive and relatively large, as needed to avoid contradicting the RH. The discussion on the previous slide suggests that it should be possible to prove these inequalities purely by using the properties of $p_1(y)$.

One important property of the polynomial $F_{0,k}(y)$ is that the coefficients of the monomials y^j, $j = 0$, $k(k-1)/2 - 1$ are zero. We proved[5] this property for any value of k solely on the knowledge of the form of the polynomials $p_j(y)$. These polynomials may be defined by a recursion relation starting with $p_1(y)$. In this way we have demonstrated part of the connection between the basic structure of $\Phi(u)$ and the positivity of the determinants $D(n, r)$, $r = 2, 3$ required in the determinantal proof of the RH.

[5] J. Nuttall, Determinantal Inequalities and the Riemann Hypothesis, *Nuttall web site* Report 6 (2011).

Outline

Concluding Remarks

To prove the RH using the methods described above there are two problems to be solved:

1. Prove that the conjecture about cumulants is correct and find the form of $m(r)$ for general r.
2. Extend the single moment method so that it will treat $D(n, r)$ for all cases not currently covered.

Remark 1. Perhaps a precise knowledge of $m(r)$ is not necessary, *e.g.* a proof that $m(r+1) - m(r) \leq 2$ may be adequate.

Remark 2. Prove that $D(n, r) > 0$ for $r \geq 7$, all r. Study the mechanisms behind results.

Remark 3. Continue to use numerical calculations (experimental mathematics) that have been very helpful so far.

Remark 4. As suggested by Brian Conrey, apply the methods used above to a set of 8 cases postulated by Conrey and Ghosh[6] to resemble the Riemann case.

[6] J. B. Conrey and A. Ghosh, *Trans. Am. Math. Soc.* **342**(1):407–419 (1994).

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