ON THE RIEMANN HYPOTHESIS - PART 1

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1. INTRODUCTION

In their recent book Borwein et al [Bo08] tell us that the Riemann Hypothesis (RH) is perhaps the greatest unsolved problem in mathematics. Many eminent mathematicians in the last 150 years have tried to solve it, without success. No doubt in part because of all these failures, the problem is regarded as being very difficult. Even brilliant students are advised to avoid the problem for fear that their careers will be wasted. Since I do not expect to have any sort of career in the future, I have now nothing to lose but my time, so I have been investigating the RH.

In his book Edwards [Ed74], p.178 provides a diagram of a section of the function Z. Plotted for real values this function oscillates about zero, and has zeros corresponding to the zeros of the zeta function on the line $\mathrm{Re}(s) = 1/2$. It is known that the RH would be true if and only if it could be shown that the value of the second derivative of Z at every turning point had sign opposite to that of Z at that point. Numerical calculations have previously demonstrated the existence of the Lehmer phenomenon where a turning point occurs at a very low value of $|Z|$, so that the RH is almost contradicted, but no example has been found where the turning point sign rule is violated. This behavior suggests that there is some hidden mechanism in the structure of the functions involved that ensures the truth of the RH.

In the present studies I have made extensive use of numerical calculations. I believe that the computer provides us with a valuable tool that could give us an advantage over the mathematicians of the past. Below I describe some products of this work, which show how helpful the computer might be in pointing the way towards rigorous mathematical results.

My approach has been to study the entire function $F(z)$ as defined by Csordas et al [Cs86] (denoted by **CNV**), which is closely related to Z. The RH is known to be equivalent to the statement that all the zeros of $F(z)$ are real and negative. Karlin [Ka68] tells us that this condition is ensured by a requirement on the coefficients $\{a_n, n = 0,1,\ldots\}$ of the power series $F(z)$, from which is formed the semi-infinite matrix A

$$A_{m,n} = a_{n-m}, n \geq m; \quad = 0, n < m, \quad m,n = 0,1,2,\ldots. \tag{1.1}$$

Thus, if we define the matrix $K(n,\lambda)$ of order λ as

$$K(n,\lambda)_{i,j} = A_{i,j+n}, \quad i,j = 1,2,\ldots,\lambda , \tag{1.2}$$

the RH is equivalent to the condition that

$$\det[K(n,\lambda)] > 0, \quad n = 0,1,\ldots; \lambda = 1,2,\ldots \tag{1.3}$$

In this case the matrix A is said to be 'totally positive' or TP, and the sequence $\{a_n\}$ is called a one-sided sequence that is a Polya frequency sequence of infinite order (see Karlin [Ka68], p. 393).

With the help of high precision software (kindly made available by David Bailey of UCal Berkeley) it is possible on a laptop computer to evaluate the determinants of $K(n,\lambda)$ for values of n,λ into the hundreds or more. In this context the difficulty of the RH appears in the fact that, as n,λ become larger, the value of the determinant (always positive of course) gets increasingly smaller compared to the product of the elements on the main diagonal. To understand the structure of the problem, with the hope of eventually proving the RH, we have to understand the mechanism for this cancellation in the evaluation of the determinants.

In the next section we describe what we believe to be a compelling explanation (supported in Section 3 by accurate numerical predictions of determinant values, etc.) for the cancellation in the situation n large, λ fixed. Given a proof of some algebraic results verified in special cases by the numerical approach, I believe that the explanation could be turned into a rigorous proof of (1.3) in this case for any fixed λ. For small values of λ, where computation has proved these results for a range of values of *n*, an application of standard techniques for finding error bounds, such as may be found in the book by Olver [Ol74], should lead to a proof in this part of the (n,λ)-space.

I regard this development as a sign that the RH problem <u>may</u> not be as difficult as is commonly believed, although we may have just picked the 'low-hanging fruit'. It is plausible to speculate that there will be analogous properties of the determinants in other parts of (n,λ)-space. Progress may now be a matter of discovering the appropriate structures and the reasons for their existence, and this is where I am currently devoting my efforts.

2. DETERMINANTS FOR LARGE *n*

1. We first note that (1.3) is trivially satisfied if either $n = 0$ or $\lambda = 1$.

Over 20 years ago **CNV** proved a result that implies that (1.3) holds for $\lambda = 2, \quad n = 0,1,\ldots$, but an important property used in the proof cannot be generalized beyond $\lambda = 2$. The present work discusses higher fixed values of λ.

2. To determine the coefficients of $F(z)$ we use the following definitions from **CNV**.

$$\phi_m(t) = \left[2m^4\pi^2 e^{9t} - 3m^2\pi e^{5t}\right]\exp(-m^2\pi e^{4t}), \quad m = 1,2,\ldots \tag{2.1}$$

$$\Phi(t) = \sum_{m=1}^{\infty}\phi_m(t) \tag{2.2}$$

$$b_n = \int_0^{\infty} t^{2n}\Phi(t)dt, \quad n = 0,1,2,\cdots \tag{2.3}$$

$$a_n = b_n / \Gamma(2n+1) \tag{2.4}$$

We call $\{b_n\}$ the moments of $\Phi(t)$, and $\{a_n\}$ the normalized moments.

3. Our approach is to observe that, for suitably large values of n, we can find approximations to the moments $\{b_n\}$ that are accurate enough to show that (1.3) holds for the set of values of (n,λ) referred to above, i.e. $n > N(\lambda)$ for some suitable choice of $N(\lambda)$.

It will be seen that, as t increases, $|\Phi(t) - \phi_1(t)| / \Phi(t) \to 0$ at a high rate compared to n, so that there are regions of (n,λ) - space where it is adequate in evaluating (2.3) to replace $\Phi(t)$ by $\phi_1(t)$. For large n it is relatively larger values of t that are required, and we restrict attention to that case. Therefore, let us use a familiar technique (called the Laplace method by Olver [Ol74]) and write

$$\phi_1(t)t^{2n} = f(t,n) = q(t)\exp[p(t,n)], \tag{2.5}$$

where

$$\begin{aligned} p(t,n) &= -\pi e^{4t} + 2n\log(t) \\ q(t) &= 2\pi^2 e^{9t} - 3\pi e^{5t} \end{aligned} \tag{2.6}$$

The function $p(t,n)$ has a maximum at a point $t = \tau(n)$, where

$$\frac{dp(\tau,n)}{dt} = -4\pi e^{4\tau} + 2n/\tau = 0. \tag{2.7}$$

This is equivalent to

$$n = 2\pi\tau \exp[4\tau] \quad . \tag{2.8}$$

The function $\tau(n)$ increases towards infinity monotonically with n. Near the maximum point, $p(t,n)$ may be approximated by

$$p(t,n) \approx p(\tau,n) - \alpha(n)(t-\tau)^2 \tag{2.9}$$

where

$$\alpha(n) = -\frac{1}{2}\frac{d^2 p(\tau,n)}{dt^2} = 8\pi\, e^{4\tau} + n/\tau^2 = 4n/\tau + n/\tau^2 = 2\pi \exp[4\tau]\left\{4 + \frac{1}{\tau}\right\}, \tag{2.10}$$

which also increases towards infinity with n.

Assuming the validity of the above approximations as $n \to \infty$, we conclude that the function $p(t,n)$ has an increasingly sharp peak at $t = \tau$, so that the integrand in (2.3) has a much sharper peak. The integral for the coefficient b_n is dominated for large n by values of t near $\tau(n)$. In file 'data1' we have plotted some values of $\Phi(t)t^{4000}$ that illustrate these remarks.

4. Consider the matrix $K(n,\lambda)$ with a_n replaced by b_n, i.e.

$$M(n,\lambda)_{i,j} = b_{j+n-i}, \quad i, j = 1,2,\ldots,\lambda\,. \tag{2.11}$$

We write

$$\det[M(n,\lambda)] = \sum_{k=1}^{\lambda!} B(k)\varepsilon(k)\,, \tag{2.12}$$

where $B(k)$ is a product of λ matrix elements, each from a different row and column, so that $\lambda!$ is the number of permutations of the numbers $1,2,\ldots,\lambda$, with $\varepsilon(k) = \pm 1$ corresponding to the sign of the permutation. We call $B(k)$ a 'component', and choose $k = 1$ to correspond to the component $B(1) = (b_n)^\lambda$, which arises from the product of the elements on the principal diagonal. Throughout this discussion we shall assume that $n > \lambda$.

5. The first salient feature to be observed is that, as $n \to \infty$ for given λ, numerical evidence suggests that

$$B(k)/B(1) \to 1, \quad k = 1,2,\ldots p(\lambda)\,. \tag{2.13}$$

This property shows that $\det[M(n,\lambda)]/B(1) \to 0$ as $n \to \infty$.

Some of the determinants $\det M(n,\lambda)$ are negative (including all those with $\lambda = 2$), so that, if the RH is to hold, it must often be due to the effect of the normalizing factor in (2.4), i.e. $\Gamma(2n+1)^{-1}$. We can argue that, if all the moments $\{b_n\}$ were equal to unity, then $B(k) = 1, \quad k = 1,2,\ldots\lambda!$. In that case we would have $K(n,\lambda) = G(n,\lambda)$, where the matrix $G(n,\lambda)$ is given by

$$G(n,\lambda)_{i,j} = \Gamma(2(j+n-i)+1)^{-1}, \ j > i; \quad = 0, \ j < i. \tag{2.14}$$

It is therefore favorable for the validity of the RH that Lemma A of the Appendix proves that

$$\det G(n,\lambda) > 0, \quad n = 0,1,\ldots; \ \lambda = 1,2,\ldots \tag{2.15}$$

6. A given component $B(k)$ may be written as an integral over λ variables. For example, when $\lambda = 3$, $B(k)$ could be written as

$$B(k) = \int_0^\infty\int_0^\infty\int_0^\infty dt_1 dt_2 dt_3 \Phi(t_1)\Phi(t_2)\Phi(t_3) t_1^{2n} t_2^{2n} t_3^{2n} t_1^{2\nu_1} t_2^{2\nu_2} t_3^{2\nu_3}\,, \tag{2.16}$$

where (ν_1, ν_2, ν_3) are sets of integers chosen, in the case $\lambda = 3$, from the 6 possibilities $(0,0,0)$, $(0,1,-1)$, $(1,-1,0)$, $(1,1,-2)$, $(2,-1,-1)$, $(2,0,-2)$ according to the value of k. Note that in all cases

$$\sum_{i=1}^{\lambda} \nu_i = 0 \ . \tag{2.17}$$

The above discussion indicates that, for large *n*, the integral (2.16) is dominated by points near $t_1 = t_2 = t_3 = \tau(n)$, so we write

$$t_i = \tau(1 + x_i), \quad i = 1,\ldots,\lambda\,, \tag{2.18}$$

with the expectation that each x_i may be regarded as a small quantity in the important part of the domain of integration. We then expand the factor $t_1^{2\nu_1} t_2^{2\nu_2} t_3^{2\nu_3}$ to produce an approximating polynomial in x_1, x_2, x_3. We repeat the process for the different permutations of $(1,2,3)$ in (2.16) and average the results to obtain a formal expansion

$$B(k) \approx \sum_{m=0} \sum_{j=1}^{np(m)} T(m,j,k) I(m,j)\,. \tag{2.19}$$

Here $T(m,j,k)$ is a numerical coefficient relating to a term $Q = Av\prod_{i=1} x_i^{\sigma_i}$, where the average is over permutations of x_i, $i = 1,\dots,\lambda$. The degree $m = \sum_{i=1}\sigma_i$, and j labels the different types of power sets that occur for that degree, which are designated by $[\sigma_1,\sigma_2,\sigma_3,\dots]$, with positive integers σ_i that satisfy $\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots$. Zero powers are omitted in the type designation.

Since, in the leading term of the approximation to $\Phi(t)$ using (2.9) we have

$$\Phi(t) \approx \text{const}\exp(-\alpha\tau^2 x^2), \tag{2.20}$$

we omit any types that contain one or more odd powers σ because the corresponding integral would be zero. We have shown computationally that the procedure is legitimate for $\lambda = 2-5$. As a consequence of this step, we replace the type designation by $[\sigma_1/2,\sigma_2/2,\sigma_3/2,\dots]$.

The quantity $I(m,j)$ is the integral

$$I(m,j) = \int\int\int dt_1 dt_2 dt_3 \Phi(t_1)\Phi(t_2)\Phi(t_3) t_1^{2n} t_2^{2n} t_3^{2n} Q \tag{2.21}$$

with the domain of integration restricted to an appropriate neighborhood of (τ,τ,τ).

7. To complete the structure needed for our remarks, we derive an expansion of the normalizing factors from (2.4) valid for large *n*. We note that multiplication of all elements of the matrix $K(n,\lambda)$ by a common positive factor does not affect the sign of $\det[K(n,\lambda)]$, and define

$$H(i,j,y) = \Gamma(2n+1)/\Gamma(2(j+n-i)+1), \quad i,j = 1,2,\dots,\lambda, \quad \text{with } y = (2n)^{-1}. \tag{2.22}$$

We assert that the components $W(n,k), \quad k = 1,2,\dots,\lambda!$ of the scaled normalization matrix *H* have an expansion valid for $n \to \infty$ of the form

$$W(n,k) = \sum_{i=0} w(i,k) y^i, \quad k = 1,2,\dots,\lambda!. \tag{2.23}$$

It is convenient to regard *B(k)*, *W(n,k)*, etc. as the components of vectors in a space of dimension $\lambda!$, with scalar product

$$\underline{B}\cdot\underline{W}(n) = \sum_{k=1}^{\lambda!} B(k)\varepsilon(k)W(n,k) \tag{2.24}$$

With this notation we observe that a scaled version of $\det[K(n,\lambda)]$ may be written as

$$\det\left[K(n,\lambda)\right]_S = \underline{B}\cdot\underline{W}(n)\,, \tag{2.25}$$

which may be approximated by

$$\det\left[K(n,\lambda)\right]_S \approx \sum_{i=0} y^i \sum_{m=0} \sum_{j=1}^{np(m)} I(m,j)\ \underline{w}(i)\cdot\underline{T}(m,j)\,. \tag{2.26}$$

8. The properties of the scalar products in (2.26) , although not yet proven in general, are essential to the proof. We speculate, on the basis of analytical and numerical evidence, that

for any value of $j = 1,2,\ldots,np(m)$ the scalar product in (2.26) satisfies the relation

$$\underline{w}(i)\cdot\underline{T}(m,j) = 0, \qquad j = 1,2\ldots,np(m) \quad \text{if} \quad m+i < \lambda(\lambda-1)/2\,. \tag{2.27}$$

We believe that this property is the key to the cancellations required to prove (1.3) for fixed λ and large *n*.

3. CALCULATIONS

1. The first task is to calculate with sufficient precision the moments $\{b_n\}$ of $\Phi(t)$. This involves numerical evaluation of the integrals (2.3), in the present case for up to $n = 4000$. In 1986 **CNV** reported using the Romberg method for up to $n = 20$, and they provided a table of moments $\{b_n\}$ useful for checking purposes.

The Romberg method involves the use of certain corrections to the simple trapezoidal rule, where the integrand must be evaluated at a set of equally spaced points. We have discovered by experiment that, for larger numbers of points, the trapezoidal method is in this case far more accurate than the Romberg method. I believe the reason may be understood in the context of the Euler-Maclaurin summation formula [Ed74], p.98, which also involves corrections to the trapezoidal rule. The corrections relating to the ends of the intervals vanish if the odd derivatives of the integrand are zero at the ends. As **CNV** point out, this is the case for the function $\Phi(t)$ at the end $t = 0$ and the same applies to the integrand. As the table in file 'data1' demonstrates, the same effectively applies at the upper end if we approximate the limit ∞ by say $t = 5.0$, since $\Phi(t)$ decreases very rapidly as *t* increases. It must be that the remaining Euler-Maclaurin correction gets small very quickly. For numbers of points in the vicinity of 2^{14} or higher, the increase in the precision of the trapezoidal approximation for a doubling of the number of points is in the hundreds of decimal places. This allows us to be confident in the accuracy of calculations of even high order determinants where a lot of cancellation is present.

2. An example may help to clarify the manipulations described in Sec. 2. Consider the case of $\lambda = 3$, where $\det M(n,\lambda)$ has 6 components $B(k)$, with corresponding signs $\varepsilon(k)$ as follows.

$$\begin{array}{ll}
B(1) = b_n b_n b_n & \varepsilon(1) = 1 \\
B(2) = b_n b_{n+1} b_{n-1} & \varepsilon(2) = -1 \\
B(3) = b_{n+1} b_{n-1} b_n & \varepsilon(3) = -1 \\
B(4) = b_{n+1} b_{n+1} b_{n-2} & \varepsilon(4) = 1 \\
B(5) = b_{n+2} b_{n-1} b_{n-1} & \varepsilon(5) = 1 \\
B(6) = b_{n+2} b_n b_{n-2} & \varepsilon(6) = -1
\end{array} \tag{3.1}$$

Inside the integral (2.16) corresponding to $B(2)$, for example, we write , with $z_i = x_i^2$,

$$\begin{aligned}
t_1^{2\nu_1} t_2^{2\nu_2} t_3^{2\nu_3} &= t_1^0 t_2^2 t_3^{-2} = (1+x_2)^2 (1+x_3)^{-2} \approx (1+z_2)(1+3z_3+5z_3^2+\ldots) \\
&\approx 1 + z_2 + 3z_3 + 5z_3^2 + 3z_2 z_3 + O(z^3)
\end{aligned} \tag{3.2}$$

As explained above, we have retained only even powers of *x*.

Next we permute and average the *z* variables to obtain the formal expression $1+4z+5z^2+3zz$. This means that with $\lambda = 3,\ k = 2$ we have for the quantity $T(m, j, k)$ appearing in (2.19) the values

$$T(0,1,2)=1; \quad T(1,1,2)=4; \quad T(2,1,2)=5; \quad T(2,2,2)=3; \tag{3.3}$$

For $\lambda = 3,\ k = 2$ and degree *m* = 0 there is one type [0] or 1.
For degree *m* = 1 there is one type [1] or z.
For degree *m* = 2 there are two types, [2] or z^2 for j = 1 and [1 1] or *zz* for j = 2.
For degree *m* = 3 there are three types, [3] for j = 1, [2 1] for j = 2 and [1 1 1] for j = 3.

k	ε	type=0	type=1	type=2	type=1 1	type=3	type=2 1	type=1 1 1
1	1	1	0	0	0	0	0	0
2	-1	1	4	5	3	7	5	0
3	-1	1	4	5	3	7	5	0
4	1	1	12	35	21	84	70	10
5	1	1	12	11	45	14	96	54
6	-1	1	16	36	60	84	220	0

Table 3.1 . Some values of the coefficients $T(m, j, k)$ for $\lambda = 3$ and degrees *m* = 0 - 3.

k	0	1	2	3	4
1	1	0	0	0	0
2	1	-4	10	-22	46
3	1	-4	10	-22	46
4	1	-12	70	-282	922
5	1	-12	86	-490	2466
6	1	-16	136	-856	4576

Table 3.2. Some values of the coefficients $w(i)$ for $\lambda = 3$ and i = 0 - 4.

Given these values in these tables, it may be shown that the relations (2.27) hold for all types corresponding to $m + i < 3$. For example take degree $m = 2$, type = (1 1) and $i = 0$, where we see that $1\times 0 - 1\times(-4) - 1\times(-4) + 1\times(-12) + 1\times(-12) - 1\times(-16) = 0$

It is also important for the later development to note that $\underline{w}(3)\cdot\underline{T}(0,1) = 128$, a positive number. Calculations up to $\lambda = 5$ have verified (2.27), and in each case $\underline{w}(i)\cdot\underline{T}(0,1)$ is positive for $i = \lambda 2 = \lambda(\lambda - 1)/2$.

Using data such as appears in the above two tables we have calculated the cross products $\underline{w}(i)\cdot\underline{T}(m, j)$ for a range of values of λ, i and m. The results are shown in file 'data2'.

At the beginning of each λ section we have displayed an array with rows m = 1,2,.. and columns i = 0,1,.., in which 0 means that all cross products are zero, 1 means that at least one cross product is not zero. The striking pattern corresponds to (2.27), and I have little doubt that it continues for higher values of λ than those in file 'data2'.

3. Next we use (2.26) to approximate a normalized scaled value $\det[K(n,\lambda)]_S / B(1)$ of $\det[K(n,\lambda)]$ that we define as

$$\det[K(n,\lambda)]_{SN} \approx \left\{\sum_{i=0} y^i \sum_{m=0} \sum_{j=1}^{np(m)} I(m, j)\ \underline{w}(i)\cdot\underline{T}(m, j)\right\} \Big/ \left\{\int dt\Phi(t)t^{2n}\right\}^{\lambda} \qquad (3.4)$$

In each of the integrals over t that occur in (3.4) we take the leading term of the Laplace expansion. Pairing individual integrals from numerator and denominator, we see that some common factors cancel and we are left with terms containing the form

$$\int_{-\infty}^{\infty} dx \exp(-\beta x^2) x^{2\eta} \Big/ \int_{-\infty}^{\infty} dx \exp(-\beta x^2) = c(\eta)\beta^{-\eta} , \tag{3.5}$$

where

$$\beta = \alpha(n)\tau(n)^2 . \tag{3.6}$$

Here, $c(\eta)$ is defined by $c(1) = 1/2$, $c(m) = c(m-1)\times\left[(2m-1)/2\right]$, m = 2, 3, .

Now consider the contribution to the numerator of (3.4) from the terms with given values of m and i. Take for example the case λ = 3, i = 0 and m =3, where there are three types corresponding to z^3, $z^2 z$, zzz . Using the values of the cross products from file 'data2' , the contribution to the numerator becomes

$$0\times\frac{15}{8\beta^3} - 64\times\frac{3}{4\beta^2}\times\frac{1}{2\beta} + 64\times\left[\frac{1}{2\beta}\right]^3 = -16\beta^{-3} . \tag{3.7}$$

This quantity may be compared with the exact value of $\underline{B}\cdot\underline{w}(0)/B(1)$, which is done in file 'data3', Section lam = 3, heading ' Approximations to cross product factors'. In the column headed 0, we have shown, for various values of n from 200 to 4000, the quantity $\underline{B}\cdot\underline{w}(0)\beta^3 / B(1)$. It is seen that this quantity appears to be converging to the -16 of (3.7) as n increases.

This behavior may be explained on the basis that corrections to the approximation to $\underline{B}\cdot\underline{w}(0)/B(1)$ would involve other terms from the numerator of (3.4) with i = 0 and higher values of m, and these would contain additional factors of β^{-1}, which $\to 0$ as $n \to \infty$.

Generalizing this explanation, we propose that a useful approximation for large n to the normalized scaled determinant (which may be written as $\sum_{i=0} \underline{B}.\underline{w}(i) y^i$) is given by

$$\det\left[K(n,\lambda)\right]_{SN} \approx \sum_{i=0}^{\lambda 2} P(i) y^i \beta^{i-\lambda 2} , \tag{3.8}$$

where $P(i)$ is analogous to the -16 in (3.7) and depends only on i and λ . For $\lambda = 2-5$ the values of $P(i)$ are listed in the 'Theory' line at the foot of the first table for each λ .

The other columns under the heading ' Approximations to cross product factors' indicate the quality of these approximations for each λ and i. Note that the columns for $i > \lambda 2$ relate to an approximation of the form $P(i)y^i$.

The second table in each λ section of file 'data3' is headed 'Relative contributions to exact determinants'. The entries denote the quantities $\underline{B}.\underline{w}(i)y^i / \underline{B}.\underline{w}(\lambda 2)y^{\lambda 2}$. It is seen that there is a trend as *n* increases for the contribution to $\det[K(n,\lambda)]_{SN}$ from $i = \lambda 2$ to become dominant, although the value of *n* for which dominance becomes apparent rapidly increases with λ.

The third table in each λ section is headed 'Determinants'. The first column headed 'approx/exact' gives the ratio of the approximation (3.8) to the exact value of $\det[K(n,\lambda)]_{SN}$. The second column headed 'exact/diag' gives the ratio of the exact determinant to the corresponding product of the elements on the main diagonal, a measure of the amount of cancellation occurring in the calculation of the determinant. It is seen that the amount of cancellation increases rapidly with *n* and λ. These results appear to be a compelling demonstration of the power of the theoretical analysis and the accuracy of the computations.

4. The approximate determinant formula (3.8) may be written in the form

$$\det[K(n,\lambda)]_{SN} \approx (2n)^{-\lambda 2} \sum_{i=0}^{\lambda 2} P(i) \left[\frac{2n}{\beta}\right]^{\lambda 2 - i}, \qquad (3.9)$$

where from (2.10 and (3.6)

$$\frac{2n}{\beta} = \frac{1}{2\tau(n)[1 + 1/(4\tau(n))]}. \qquad (3.10)$$

As (2.8) shows, $\tau(n) \to \infty$ as $n \to \infty$, so that $2n/\beta \to 0$ as $n \to \infty$, but slowly. For all λ the term in (3.9) corresponding to $i = \lambda 2$ will eventually dominate, but our calculations show that the other terms are significant for values of *n* in the 1000s and beyond, the more so the higher λ.

4. THE QUESTION OF PROOF

1. Based on the above development a simple but weak proposed theorem states

PROPOSED THEOREM *For given order λ the scaled normalized determinant of the matrix $K(n,\lambda)$ has the property*

$$n^{\lambda 2} \det[K(n,\lambda)]_{SN} \to P \quad as \quad n \to \infty, \qquad (4.1)$$

where $\lambda 2 = \lambda(\lambda - 1)/2$ and P is a positive constant dependent on λ.

To construct a proof of this theorem the most important required missing inputs are

1. A proof of the algebraic relations (2.27);
2. A proof that the constant P is positive;
3. The analysis that, for large enough values of n, all the corrections to the approximation (3.9) are negligible.

In connection with these remarks we observe that

$$\det[K(n,\lambda)]_S = \int_0^\infty dt_1 \Phi(t_1) t_1^{2n} \dots \int_0^\infty dt_\lambda \Phi(t_\lambda) t_\lambda^{2n} J \tag{4.2}$$

where

$$J(\underline{x}, y) = \det\left[H(i,j,y)(1+x_i)^{2(j-i)}\right]_{i,j=1,\lambda} \tag{4.3}$$

The cross products $\underline{w}(i) \cdot \underline{T}(m,j)$ used in (2.26), (2.27) may be obtained from the results of symmetrizing $J(\underline{x}, y)$, differentiating several times with respect to the arguments, and setting $\underline{x}$ and y to zero. This method might lead to a general proof of (2.27) and information about the coefficients $P(i)$ for general λ.

An approach based on the structure of Olver's discussion [Ol74], p.82, might lead to the analysis required in Point 3 above.

It should also be possible to extend the proposed theorem to more values of n by using the approximation of (3.9).

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APPENDIX

LEMMA A. *Define an infinite sequence* $[g_n]$ *by*

$$g_n = [\Gamma(1+2n)]^{-1}, \qquad n \in X, \quad X = \{0,1,2,\cdots\}$$
$$g_n = 0, \qquad n < 0.$$

Then the sequence $[g_n]$ *gives rise to a kernel* $G(m,n)$ *on* $X \times X$ *, with* $G(m,n) = g_{n-m}$*, which is a one-sided Polya frequency sequence of infinite order, i.e. it is* PF_∞ *.*

Proof. First we prove the above result with g_n replaced by $h_n = [\Gamma(1+n)]^{-1}$. We note that the generating function $H(z) = \sum_{m=0}^{\infty} h_m z^m = e^z$. This function belongs to the class of generating functions to which the fundamental representation theorem for one-sided PF sequences applies. This proves that the sequence $[h_n]$ is PF_∞ , so that the kernel $H(m,n) = h_{n-m}$ is TP. That is, for all $k \geq 1$, the determinant of every finite square matrix formed by choosing k rows and columns from $H(m,n)$ is non-negative.

If we restrict this choice to rows and columns with even indices, we obtain all those determinants needed to prove that the kernel $G(m,n)$ is TP, which proves the lemma.