ON THE RIEMANN HYPOTHESIS - PART 3

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October 2009

1. INTRODUCTION

In the report "On the Riemann Hypothesis - Part 1" (denoted by **P1**) we began to follow an approach to the resolution of the Riemann Hypothesis (RH) based on determining the signs of the determinants of a certain infinite set of matrices

$$\det\left[K(n,\lambda)\right], \quad n = 0,1,\ldots;\ \lambda = 1,2,\ldots\ . \tag{1.1}$$

The RH is true if and only if all the signs are positive. The objective is to approximate each determinant accurately enough to produce an unambiguous estimate for its sign. This task is complicated by the property, indicated by numerical calculations, that the extent of cancellations in the evaluation of a determinant increases as n, λ become larger.

The key result of **P1** is the decomposition of a scaled version of the determinant given in **P1**(2.26)

$$\det\left[K(n,\lambda)\right]_S \approx \sum_{i=0} y^i \sum_{m=0} \sum_{j=1}^{np(m)} I(m,j)\ \underline{w}(i)\cdot\underline{T}(m,j), \tag{1.2}$$

where the quantities involved are defined on **P1** page 6. The importance of this formula lies in the property **P1**(2.27), which states that

$$\underline{w}(i)\cdot\underline{T}(m,j) = 0, \quad j = 1,2\ldots,np(m) \quad \text{if} \quad m+i < \lambda(\lambda-1)/2, \tag{1.3}$$

which algebraic calculations have shown to be true for $\lambda \le 6$, and which we think is likely to be true for all λ (see the Cross Product tableaux in **P1** file 'data2') . For larger values of n (λ fixed), it is terms in (1.2) containing the largest relative values of $\left|I(m,j)\right|$ that will be omitted due to (1.3). It appears that this feature accounts for almost all the cancellation observed in numerical calculations.

In this report we give a modified version of (1.2), which differs in several respects from its form in **P1**. Thus

- The formula is now exact;

- The sums in (1.2) are now finite rather than infinite;
- The values of $I(m, j)$ may be found in terms of the moments (both even and odd) of $\Phi(t)$ (see **P1**(2.2)), and the parameter τ of **P1**(2.18), no matter how τ is chosen;
- We now retain all types (see **P1** page 6), whether the powers involved are even or odd;
- Formula (1.3) still holds (in general, we believe), with a modified condition on m, i.

The formalism is valid for any value of the order λ, but in this report we illustrate the most of the main ideas by applying it to the case $\lambda = 2$.

2. ALTERNATIVE DECOMPOSITION FORMULA

1. We begin by repeating the relevant basic formulas of **P1**, specialized as appropriate to the case $\lambda = 2$.

An important entire function $F(z)$, related to the Riemann zeta-function, defined by Csordas et al (denoted by **CNV**) is

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

To determine the coefficients of $F(z)$ we use the following definitions from **CNV**.

$$\phi_m(t) = \left[2m^4\pi^2 e^{9t} - 3m^2\pi e^{5t}\right]\exp(-m^2\pi e^{4t}), \quad m = 1,2,\ldots \tag{2.2}$$

$$\Phi(t) = \sum_{m=1}^{\infty} \phi_m(t) \tag{2.3}$$

$$b_n = \int_0^{\infty} t^{2n}\Phi(t)dt, \quad n = 0,1,2,\cdots \tag{2.4}$$

$$a_n = b_n / \Gamma(2n+1) \tag{2.5}$$

We call $\{b_n\}$ the (even) moments of $\Phi(t)$.

The semi-infinite matrix A is defined as

$$A_{m,n} = a_{n-m}, n \ge m; \quad = 0, n < m, \quad m, n = 0,1,2,\ldots, \tag{2.6}$$

with the matrix $K(n,\lambda)$ of order λ as

$$K(n,\lambda)_{i,j} = A_{i,j+n}, \quad i,j = 1,2,\ldots,\lambda\,, \tag{2.7}$$

Henceforth we shall use only $K(n) = K(n,2)$ given by

$$K(n) = \begin{bmatrix} a_n & a_{n+1} \\ a_{n-1} & a_n \end{bmatrix}, \quad n = 1,2,\ldots \tag{2.8}$$

The case of $K(0)$ is trivial.

2. The components of $\det[M(n)]$ defined in **P1**(2.12) are

$$B(1) = b_n^2\,, \quad B(2) = b_{n-1}b_{n+1} \qquad \text{and } \varepsilon(1) = 1, \quad \varepsilon(2) = -1\,, \tag{2.9}$$

and, with (2.5), we have

$$\det[K(n)] = \frac{B(1)}{\Gamma(2n+1)^2} - \frac{B(2)}{\Gamma(2n-1)\Gamma(2n+3)}\,. \tag{2.10}$$

With $y = 1/2n$ we multiply (2.10) by $\Gamma(2n+1)\Gamma(2n+3)y^2$ to produce a scaled version of $\det[K(n)]$ (not the same as the one used in P1)

$$\det[K(n)]_S = (1+3y+2y^2)B(1) - (1-y)B(2)\,. \tag{2.11}$$

Alternatively we may write

$$\det[K(n)]_S = \sum_{i=0}^{2} y^i C(i,n) \tag{2.12}$$

where

$$C(0,n) = B(1) - B(2), \quad C(1,n) = 3B(1) + B(2), \quad C(2,n) = 2B(1)\,. \tag{2.13}$$

3. Next we follow the procedure similar to that described in **P1**(2.16) et seq. We write

$$B(k) = \int_0^\infty\!\!\int_0^\infty dt_1 dt_2 \Phi(t_1)\Phi(t_2) t_1^{2n-2} t_2^{2n-2} R(k,t_1,t_2) \tag{2.14}$$

where

$$R(1,t_1,t_2) = t_1^2 t_2^2, \quad R(2,t_1,t_2) = (t_1^4 + t_2^4)/2 \tag{2.15}$$

For a quantity τ to be defined later we set

$$t_i = \tau(1 + x_i), \quad i = 1,2, \tag{2.16}$$

and substitute in (2.14). The result expresses $B(k)$ in the form

$$B(k) = \sum_{m=0}^{4} \sum_{j=1}^{np(m)} T(m,j,k) I(m,j), \quad k = 1,2, \tag{2.17}$$

where

$$I(m,j) = \tau^4 \int_0^\infty \int_0^\infty dt_1 dt_2 \Phi(t_1)\Phi(t_2) t_1^{2n-2} t_2^{2n-2} x_1^{\nu(1,m,j)} x_2^{\nu(2,m,j)} . \tag{2.18}$$

The values of the exponents $\nu(i,m,j)$ are given in Table 2.1. The values of the coefficients $T(m,j,k)$ appear in Table 2.2.

4. Note that the quantity $I(m,j)$ in (2.18) is a product of two integrals of the form $\int_0^\infty dt\Phi(t)t^{2n-2}x^\nu$, and each may be written as a linear combination of several moments of $\Phi(t)$, both even and odd. For example, suppose that we denote the moments by

$$\beta_n = \int_0^\infty dt\Phi(t)t^n . \tag{2.19}$$

Then, since $x = \tau^{-1}t - 1$,

$$\int_0^\infty dt\Phi(t)t^{2n-2}x = \tau^{-1}\beta_{2n-1} - \beta_{2n-2}, \tag{2.20}$$

where $\beta_{2n-2} = b_{n-1}$, etc.

5. The equivalent of the decomposition formula **P1**(2.26) follows by using the above relations. Firstly, if we define $w(i,k), \quad i = 0,1,2; k = 1,2$ by

$$\underline{w}(0) = (1,1); \quad \underline{w}(1) = (3,-1); \quad \underline{w}(2) = (2,0), \tag{2.21}$$

and also define (as in **P1**(2.23))

$$\underline{W}(n) = \sum_{i=0}^{2} y^i \, \underline{w}(i) \,, \tag{2.22}$$

then (2.12) may be written (as in **P1**(2.25))

$$\det[K(n)]_S = \underline{B} \cdot \underline{W}(n) = \sum_{k=1}^{2} B(k)\varepsilon(k)W(n,k) \,. \tag{2.23}$$

Substituting for $\underline{B}$ according to (2.17) leads to the new version of **P1**(2.26)

$$\det[K(n)]_S = \sum_{i=0}^{2} \sum_{m=0}^{4} \sum_{j=1}^{np(m)} I(m,j) \; \underline{w}(i) \cdot \underline{T}(m,j) \,. \tag{2.24}$$

The values of the cross product coefficients $\underline{w}(i) \cdot \underline{T}(m,j)$ are given in Table 2.3. The crucial fact that these coefficients are zero if $i = 0$ and $m = 0,1$ is the equivalent of **P1**(2.27).

3. COMPUTATIONS

1. The numerical calculation of the quantities introduced above sheds considerable light on the features of the decomposition of $\det[K(n)]$, which can guide attempts to analytically demonstrate its positivity. We note that, in 1986, **CNV** proved an inequality that contains the relation

$$a_n^2 - a_{n-1}a_{n+1} > 0, \quad n = 0,1,2,\ldots, \tag{3.1}$$

which implies $\det[K(n)] > 0$. Even earlier, Grosswald (**G**), using a different method, had shown that (3.1) holds for all *n* greater than a certain unknown value. However it is not likely that either method can be generalized to the case $\lambda > 2$. Our purpose in studying a different approach to the proof of a known result is the belief that our method will be applicable to at least some cases with $\lambda > 2$. The proof for all such cases, i.e. the proof of the RH, will most likely require some additional techniques, but it may turn out that a step-by-step procedure is the best way to solve this difficult problem.

2. We may regard the decomposition process described in Sec. 2 as consisting of two steps.

- The expansion (2.12) in powers of *y*, leading to the coefficients $C(i,n), \quad i = 0,1,2$. These coefficients are given in terms of the even moments $\{b_n\}$, which **P1** explained may be calculated with great precision up to very large values of *n*. For example we have calculated that, for $n = 500{,}000$, $b_n = 10^{370247} \times 5.260869$, which could be given to 500 decimal places or more.

- The substitution of the formula (2.17) for the components of $\det[K(n)]$ into the expression (2.13) for $C(i,n)$, with the resulting terms being classified by type. The integrals required to determine these terms involve both odd and even moments. While Gauss-Jacobi integration provides adequate results for odd moments for moderately large values of *n*, we are at present unable to deal with very large values such as in the example above.

Below we describe computations for both steps of the process.

3. File 'decompeven' contains three sets of results on calculations of $y^i C(i,n)$ for values of *n* in the range $1-500000$. The first three columns are the values of $y^{i-1}C(i,n)/C(1,n), \quad i = 0,1,2$. The evidence suggests that $yC(1,n)$ increasingly dominates the sum, the $i = 2$ term rapidly becoming insignificant as *n* increases. The relative contribution of the $i = 0$ term also decreases steadily at a slower rate.

The fifth column of the file, $\det[K(n)]/b_n^2$, lists a measure of the extent of cancellations in the calculation of $\det[K(n)]$ - the smaller its value the more cancellation. It is seen that, especially for larger values of *n*, there has been much more cancellation than occurs between the three terms in (2.12). This characteristic must be attributed to the cancellation provided by the first step in the decomposition process.

4. For the second step we calculate and list in file 'decompodd' the values of the individual terms in (2.24) for $n = 2-49$. Each entry in the file begins with the value of τ used, followed by the ratio $\det[K(n)]/B(1)$, repeated from file 'decompeven'. Next, under the heading "total(i)", are the values of the quantities $y^i C(i,n)$ in the sum (2.12). There follows in sets of 8 for each value of $i = 0,1,2$ the individual terms corresponding to 8 possible types for the integrand of $I(m, j)$. As will be explained below we expect, for larger values of *n*, that the largest term in absolute magnitude for a given *i* will be as listed in Table 3.1.

The data in file 'decompodd' confirm the expectation for all values of *n* listed. The second number in the line below each set of 8 is [1 - (largest term)/total(i)]. The observed decreasing trend of these numbers supports the view that terms corresponding to Table 3.1 are indeed dominant as *n* becomes large. Moreover, the results from Step 1 in file 'decmpeven' strongly suggest that, for very large *n*, the term with $i = 1,\ m = 0,\ j = 1$ is dominant.

4. APPLICATION OF THE LAPLACE METHOD

1. The results of **P1** suggest that the Laplace method can provide useful approximations to terms in (2.24), particularly the dominant terms. As remarked in Sec. 3.4, progress towards a detailed proof of our conjectures for the case $\lambda = 2$ depends on a study of the integrals

$$Q(i) = \int_0^\infty dt \varphi_1(t) t^{2n-2} y^i, \quad i = 0-4, \qquad \text{with} \quad y = t - \tau, \tag{4.1}$$

where we have used the fact, not difficult to demonstrate, that, for the current purposes, it is permissible to approximate $\Phi(t)$ by $\varphi_1(t)$ as given by (2.2).

Our approach is a slight modification of the exposition of **P1** Sec. 2.3. We set

$$\phi_1(t) t^{2n} = f(t) = 2\pi^2 \exp[p(t)], \tag{4.2}$$

so that

$$p(t) = -\pi e^{4t} + 2n \log(t) + 9t + \log(g(t)), \tag{4.3}$$

where

$$g(t) = 1 - \frac{3}{2\pi} e^{-4t}. \tag{4.4}$$

We see that

$$p^{(1)}(t) = \frac{dp}{dt} = -4\pi e^{4t} + 2n/t + 9 + g^{(1)}/g, \quad \text{with } g^{(1)}(t) = \frac{6}{\pi} e^{-4t}. \tag{4.5}$$

Also

$$p^{(2)}(t) = \frac{d^2 p}{dt^2} = -16\pi e^{4t} - 2n/t^2 + g^{(2)}/g - \left[g^{(1)}/g\right]^2, \tag{4.6}$$

where

$$g^{(2)}(t) = -\frac{24}{\pi} e^{-4t}. \tag{4.7}$$

Since every term of (4.6) is negative, we see that $p^{(2)}(t) < 0, \quad 0 \le t < \infty$, so that $p^{(1)}(t)$ is a decreasing function of t. The function $p^{(1)}(t)$ is positive for small *t*, negative for large *t*, so that there is a unique point τ, with $0 < \tau < \infty$, where $p^{(1)}(\tau) = 0$.

Since $p^{(1)}(t)$ increases for fixed *t* as *n* increases, and because $p^{(1)}(t)$ has a negative slope, it follows that $\tau = \tau(n)$ is an increasing function of *n*, $n > 0$.

The function $p(t)$ has a maximum at the point $t = \tau$, and near there we can approximate $p(t)$ by

$$p(t) \approx p(\tau) - \alpha(t-\tau)^2 , \tag{4.8}$$

where

$$\alpha = -\frac{1}{2} p^{(2)}(\tau). \tag{4.9}$$

If α is sufficiently large it may be possible to obtain a useful approximation to $\int_0^\infty dt \exp[p(t)](t-\tau)^i$ by replacing $p(t)$ by (4.8) , extending the lower limit to $-\infty$, and evaluating the integral analytically. Then for application to (4.1) we also need to derive error bounds, and perhaps include corrections that produce tighter estimates. These points are discussed below.

4. From (4.5) and $p^{(1)}(\tau) = 0$ we can find an expression for $2n$ in terms of τ. Subsituting into (4.6) and (4.9) leads to

$$\alpha = +8\pi e^{4t} + \frac{2\pi e^{4\tau}}{\tau} + \frac{9}{2\tau} - \frac{g^{(2)}(\tau)}{2g(\tau)} + \frac{1}{2}\left[\frac{g^{(1)}(\tau)}{g(\tau)}\right]^2 - \frac{g^{(1)}(\tau)}{2g(\tau)} . \tag{4.10}$$

We see that $\alpha \to \infty$ as $\tau \to \infty$, and also that $n \to \infty$ as $\tau \to \infty$, so that $\alpha \to \infty$ as $n \to \infty$. We therefore expect that the Laplace approximation will be accurate for large *n*, and numerical calculations have confirmed this.

REFERENCES

CNV G. Csordas, T.S. Norfolk and R. S.Varga, "The Riemann Hypothesis and the Turan Inequalities," Trans. Amer.Math. Soc., **296**, 521 (1986)

G E.Grosswald, "Generalization of a formula of Hayman, and its applications to the study of Riemann's zeta function," Illinois J. Math. **10**, (1966).

P1 J. Nuttall, "On the Riemann Hypothesis - Part 1" Unpublished (2009) . Available at http://publish.uwo.ca/~jnuttall/RHPart3.pdf

TABLES

i=1

m	1	2	3
0	0		
1	1		
2	2	1	
3	3	2	
4	4	3	2

i=2

m	1	2	3
0	0		
1	0		
2	0	1	
3	0	1	
4	0	1	2

Table 2.1. The values of the exponents appearing in (2.18).The column headings refer to the value of *j*. The two exponents for a given *m*,*j* are known as the type of the corresponding monomial.

k=1

m	1	2	3
0	1		
1	4		
2	2	4	
3	0	4	
4	0	0	1

k=2

m	1	2	3
0	1		
1	4		
2	6	0	
3	4	0	
4	1	0	0

Table 2.2. The values of the coefficients $T(m, j, k)$ appearing in (2.17). The column headings refer to the value of j.

i=0

m	1	2	3
0	0		
1	0		
2	-4	4	
3	-4	4	
4	-1	0	1

i=1

m	1	2	3
0	4		
1	16		
2	12	12	
3	4	12	
4	1	0	3

i=2

m	1	2	3
0	2		
1	8		
2	4	8	
3	0	8	
4	0	0	2

Table 2.3. The values of the cross product coefficients $\underline{w}(i) \cdot \underline{T}(m, j)$. The column headings refer to the value of j.

i	m	j	type
0	2	1	x^2
1	0	1	1
2	0	1	1

Table 3.1. The (m, j) values and type of the expected most important terms for a given value of i in (2.24).

DATA FILES Available on request from jnuttall@uwo.ca