THE SINGLE MOMENT METHOD IN THE DETERMINANTAL APPROACH TO THE RIEMANN HYPOTHESIS

JOHN NUTTALL

ABSTRACT. On the assumption that our previous conjecture relating to the properties of cumulants is correct, we describe a method that increases the set of values of n for which it may be proved that the Riemann determinants $D(n,r) > 0$.

1. INTRODUCTION

1.1. This note relates to the problem of proving the positivity of a set of determinants $D(n,r)$, $n = 0, 1, \ldots;\ r = 1, 2, \ldots$. If all these inequalities hold, then the Riemann hypothesis is true.

We shall assume that Conjecture 1 of [4, Sec. 4, p. 10] is correct. This conjecture relates to the kernel $K(u, v; m)$, which is constructed from the cumulant $\Psi_m(u)$ defined in terms of the function $\Phi(u)$ of [4, (2.1), p. 3]. The conjecture asserts that there is a non-decreasing integer function $m(r)$ such that, if $m \geq m(r)$, the kernel has sign-regularity of type RR_r (see [2, p. 12]) for any given order r, and that $m(r)$ is the smallest such integer for that r for which the property holds.

In [4, Sec. 3] we pointed out that the method using a kernel based on $\Phi(u)$ (i.e. $K(u, v; 0)$) described by [1] for order 2, and recently applied to order 3 in [5], to study the determinants $D(n,r)$ of [4, (2.6), p. 3], did not apply to any order $r > 4$. We explained that, given the validity of the conjecture, the method using cumulants could probably be extended to arbitrarily large values of r, with the exception of some lower values of n. In [4, Sec. 4.4, p. 12] we presented a table showing $\eta(r)$, the first value of n above which the method applies.

We suggested an alternative method [4, Sec. 5, p. 12] that might be able to treat the easiest non-trivial exceptional case $D(1, 3)$, but the prospect is that this approach would become increasingly complicated for further cases, if it could be applied at all.

Here we propose a simpler scheme, still based on cumulants, for dealing with some of the exceptional cases. As explained in [4, Sec. 2.1] the original method, which goes back to [3], involves the use of the semi-group property [2, (5.13), p. 129] and the double application of the basic composition formula (BCF) [2, (2.5), p. 17]. The reason for this procedure, instead of the simpler single BCF and no semi-group, is that the normalized moments [4, (2.4), p. 3] required are only the even ones. It would seem that the simpler method along the lines of [2, p. 200] could not directly produce the determinants required.

We believe that this difficulty can be overcome in some cases by the procedure described in Sec. 3. Some notation and previous results are first described in Sec. 2.

2. NOTATION AND PREVIOUS RESULTS

2.1. We begin by introducing some notation used in [4]. The even normalized moments of $\Phi(u)$ are defined by

$$\beta_n = \frac{1}{\Gamma(2n+1)} \int_0^\infty du \Phi(u) u^{2n}, \quad n = 0, 1, \ldots, \tag{2.1}$$

with the convention that $\beta_n = 0$ if $n < 0$. The determinant $D(n, r)$ of order r is given by

$$D(n,r) = \begin{vmatrix} \beta_n & \beta_{n+1} & \cdots & \beta_{n+r-1} \\ \beta_{n-1} & \beta_n & \cdots & \beta_{n+r-2} \\ \vdots & \vdots & & \vdots \\ \beta_{n-r+1} & \beta_{n-r+2} & \cdots & \beta_n \end{vmatrix} \tag{2.2}$$

Another set of normalized moments, both even and odd, is defined by

$$b_n = \frac{1}{\Gamma(n+1)} \int_0^\infty du \Phi(u) u^n, \quad n = 0, 1, \ldots. \tag{2.3}$$

In this case we define the 'moments' for negative n by

$$b_n = (-1)^{n+1} \Phi^{(-n-1)}(0), \quad n = -1, -2, \ldots. \tag{2.4}$$

A corresponding set of determinants is

$$\Delta(n,r) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+r-1} \\ b_{n-1} & b_n & \cdots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-r+1} & b_{n-r+2} & \cdots & b_n \end{vmatrix} \tag{2.5}$$

Note that, if $n = 2j,\ j = \ldots, -2, -1, 0, 1, 2, \ldots$, then $\beta_j = b_n$. This relation applies even for negative j, since the function $\Phi(u)$ is even and $\beta_j = 0$. We defined the cumulants $\{\Psi_m(u)\}$ by the relations [4, Sec. 4, p. 10]

$$\Psi_m(u) = \int_u^\infty du \Psi_{m-1}(u), \quad m = 1, 2, \ldots \tag{2.6}$$

and

$$\Psi_0(u) = \Phi(u). \tag{2.7}$$

There is a corresponding set of kernels

$$K(u, v; m) = \Psi_m(u + v), \quad 0 \le u, v; \quad m = 0, 1, \ldots. \tag{2.8}$$

Karlin [2, p. 193] discussed cumulants (he called them modified kernels), and he showed that

$$\Psi_m(u) = \frac{1}{\Gamma(m)} \int_u^\infty dt \Phi(t)(t-u)^{m-1}, \quad m \ge 1. \tag{2.9}$$

This formula may be verified by differentiating with respect to u, which shows that

$$\Psi_m^{(1)}(u) = -\Psi_{m-1}(u), \quad m \ge 1, \tag{2.10}$$

consistent with (2.6) and (2.7).

Karlin [2, p. 128] also showed that, if $K(u, v; m)$ is RR_r, then so is $K(u, v; m+1)$.

2.2. In [4, Sec. 2.7, p. 3], following Karlin, we described a useful technique for proving that a kernel is sign-regular. Applied to the kernel $K(u, v; m) = \Psi_m(u+v)$ the requirement for being RR_r is that

$$\epsilon_p w(p, m; u) > 0, \ \ u \geq 0, \quad p = 1, 2, \ldots, \tag{2.11}$$

where

$$w(p, m; u) = \det \left| \Psi_m^{(i+j-2)}(u) \right|_{i,j=1}^{p}. \tag{2.12}$$

The quantity $\epsilon_p = (-1)^{p(p-1)/2}$.

3. THE SINGLE MOMENT METHOD

3.1. Let us suppose that m, r are chosen so that $K(u, v : m)$ is RR_r according to the conjecture above. We note that from (2.9)

$$\Psi_n(0) = \frac{1}{\Gamma(n)} \int_u^\infty dt \Phi(t) t^{n-1} = b_{n-1}, \quad n \geq 1, \tag{3.1}$$

so that from (2.10)

$$\Psi_m^{(n)}(0) = (-1)^n \Psi_{m-n}(0) = (-1)^n b_{m-n-1}, \quad n = 0, 1, \ldots. \tag{3.2}$$

It follows that

$$w(r, m; 0) = \begin{vmatrix} b_{m-1} & -b_{m-2} & \cdots & (-1)^{r-1} b_{m-r} \\ -b_{m-2} & b_{m-3} & \cdots & (-1)^{r-2} b_{m-r-1} \\ \vdots & \vdots & & \vdots \\ (-1)^{r-1} b_{m-r} & (-1)^{r-2} b_{m-r-1} & \cdots & b_{m-2r+1} \end{vmatrix} \tag{3.3}$$

Suppose that we multiply the even rows of (3.3) by -1. The columns will then have alternating signs $+, -, +, -, \ldots$, so that if we now multiply the even columns by -1, all the elements will have a positive sign. The net effect of the multiplications will create no change in the value of the determinant. In addition, reversing the order of the columns in (3.3) will multiply $w(r, m; 0)$ by ϵ_r. Thus we have effectively proved

Lemma 3.1. *The determinant $w(r, m; 0)$ of (3.3) satisfies*

$$\epsilon_r w(r, m; 0) = \Delta(n, r) \tag{3.4}$$

where $n = m - r$.

3.2. We begin by considering the case $r = 2$, and note that in [5] we proved that $\epsilon_2 w(2, 0; 0) > 0$, which means from Lemma 3. 1 that $\Delta(-2, 2) > 0$. Since $K(u, v; 0)$ being RR_2 implies that $K(u, v; m), \ m > 0$ is also RR_2, it immediately follows that $\Delta(n, 2) > 0, \ n \geq -2$. The coefficient $b_{-2} = 0$, but $b_n > 0, \ n \geq -1$, so that we can apply [2, Thm. 3.2, p. 59], with rows and columns interchanged, to the matrix

$$\begin{bmatrix} b_{-1} & b_0 & b_1 & b_2 & \cdots \\ b_{-2} & b_{-1} & b_0 & b_1 & \cdots \end{bmatrix}. \tag{3.5}$$

Since all 2×2 minors of (3.5) with consecutive columns are positive, it follows that all 2×2 minors are also positive. In the applications below we shall need the values of a few other 2×2 minors that are easily determined by inspection.

Next we turn to matrices of the form

$$\begin{bmatrix} b_n & b_{n+1} & b_{n+2} \\ b_{n-1} & b_n & b_{n+1} \\ b_{n-2} & b_{n-1} & b_n \end{bmatrix} \tag{3.6}$$

Note that, for even $n = 2k,\ k \geq 1$ the sole 2×2 minor of (3.6) that can be formed by taking only odd rows and columns is

$$D(k,2) = \begin{vmatrix} \beta_k & \beta_{k+1} \\ \beta_{k-1} & \beta_k \end{vmatrix}. \tag{3.7}$$

Applying [2, Thm. 3.2, p. 59] to the first two columns of (3.6) and again to the last two columns, leads to the result that all the 2×2 minors of the matrix

$$\begin{bmatrix} b_n & b_{n+1} & b_{n+2} \\ b_{n-2} & b_{n-1} & b_n \end{bmatrix},\ n > 0 \tag{3.8}$$

are positive. Finally apply the same theorem once again to (3.8) to show that $D(k,2)$ is positive.

In the case $n = k = 0$ the theorem does not apply, since the element $b_{-2} = 0$. To deal with this case we apply the technique used in the proof of Karlin [2, Thm. 3.2, p. 59]. Consider the first two columns of (3.6). The formula of Karlin shows that

$$b_{-1} \begin{vmatrix} b_0 & b_1 \\ b_{-2} & b_{-1} \end{vmatrix} = b_{-2} \begin{vmatrix} b_0 & b_1 \\ b_{-1} & b_0 \end{vmatrix} - b_0 \begin{vmatrix} b_{-2} & b_{-1} \\ b_{-1} & b_0 \end{vmatrix} \tag{3.9}$$

Since $b_{-2} = 0$ and $b_{-1} > 0$, (3.8) shows that

$$\begin{vmatrix} b_0 & b_1 \\ b_{-2} & b_{-1} \end{vmatrix} > 0. \tag{3.10}$$

From this point the discussion used in the case of $k > 0$ applies, and we find that $D(0,2) > 0$. Of course we knew that was the case, but the point is to illustrate how [2, Thm. 3.2, p. 59] helps us prove that $D(k,2) > 0,\ k = 0.1. \ldots$, given some information about the 2×2 minors of $\Delta(n,3),\ n = 0,1,\ldots$. Note that we have not used the fact that $\Delta(n,3) > 0$ in this argument.

3.3. The method can be extended to $r \geq 3$. In the case of $r = 3$, the procedure described in [4, Sec. 3, p. 9] was unable to prove that $D(1,3) > 0$, but the single moment method achieves this gaol, as well as handling the trivial case $D(0,3)$.

Previously in [5] we used the choice $m = 0$ to prove that $\epsilon_3 w_3(0) > 0$, or from Lemma 3.1, $\Delta(-3,3) > 0$. It immediately follows that $\Delta(n,3) > 0,\ n \geq -3$. Now for $n \geq 0$ consider the 5×5 matrix

$$\begin{bmatrix} b_n & b_{n+1} & b_{n+2} & b_{n+3} & b_{n+4} \\ b_{n-1} & b_n & b_{n+1} & b_{n+2} & b_{n+3} \\ b_{n-2} & b_{n-1} & b_n & b_{n+1} & b_{n+2} \\ b_{n-3} & b_{n-2} & b_{n-1} & b_n & b_{n+1} \\ b_{n-4} & b_{n-3} & b_{n-2} & b_{n-1} & b_n \end{bmatrix} \tag{3.11}$$

Extending the technique that we applied to (3.6), we have

$$D(k,3) = \begin{vmatrix} \beta_k & \beta_{k+1} & \beta_{k+2} \\ \beta_{k-1} & \beta_k & \beta_{k+1} \\ \beta_{k-2} & \beta_{k-1} & \beta_k \end{vmatrix}. \tag{3.12}$$

Since $b_{2k} = \beta_k$, we see that $D(k,3)$ is the only 3×3 minor of (3.11) that can be formed by using only odd rows and columns. Again the idea is to apply [2, Thm. 3.2, p. 59] to each of the three sets of three consecutive columns of (3.11). For each set, the three 3×3 minors with consecutive rows have positive determinants. Except for the case $n = 0$, all the 2×2 minors present in the first two columns of the set (not necessarily with consecutive rows) have positive determinants. Thus we can use [2, Thm. 3.2, p. 59] to prove that the 3×3 minor in a set with rows $(1,3,5)$ will have a positive determinant. The determinants so formed are the three 3×3 minors with consecutive columns of

$$\begin{bmatrix} b_n & b_{n+1} & b_{n+2} & b_{n+3} & b_{n+4} \\ b_{n-2} & b_{n-1} & b_n & b_{n+1} & b_{n+2} \\ b_{n-4} & b_{n-3} & b_{n-2} & b_{n-1} & b_n \end{bmatrix}, \tag{3.13}$$

where $n = 2k$, $k \geq 1$. Another application of [2, Thm. 3.2, p. 59] to columns $(1,3,5)$ of (3.13) shows that $D(k,3) > 0$.

In the case $n = k = 0$ the Karlin theorem once again does not apply to the first three columns of (3.11), so that we must use the Karlin formula relating various minors. In this case we begin with

$$\begin{aligned} &\begin{vmatrix} b_{-2} & b_{-1} \\ b_{-3} & b_{-2} \end{vmatrix} \begin{vmatrix} b_{-1} & b_0 & b_1 \\ b_{-2} & b_{-1} & b_0 \\ b_{-4} & b_{-3} & b_{-2} \end{vmatrix} = \\ &\begin{vmatrix} b_{-2} & b_{-1} \\ b_{-4} & b_{-3} \end{vmatrix} \begin{vmatrix} b_{-1} & b_0 & b_1 \\ b_{-2} & b_{-1} & b_0 \\ b_{-3} & b_{-2} & b_{-1} \end{vmatrix} - \begin{vmatrix} b_{-1} & b_0 \\ b_{-2} & b_{-1} \end{vmatrix} \begin{vmatrix} b_{-2} & b_{-1} & b_0 \\ b_{-4} & b_{-3} & b_{-2} \\ b_{-3} & b_{-2} & b_{-1} \end{vmatrix} \end{aligned} \tag{3.14}$$

In this expression the first 2×2 minor is positive, the second is zero, and the third is positive. On the right hand side, the first 3×3 minor is positive, while the second is negative as it is $\Delta(-2,3)$ with the last two rows interchanged. Consequently we find that the first 3×3 minor, with rows $(2,3,5)$, is positive.

Our second application of the Karlin formula is

$$\begin{aligned} &\begin{vmatrix} b_{-2} & b_{-1} \\ b_{-1} & b_0 \end{vmatrix} \begin{vmatrix} b_0 & b_1 & b_2 \\ b_{-2} & b_{-1} & b_0 \\ b_{-4} & b_{-3} & b_{-2} \end{vmatrix} = \\ &\begin{vmatrix} b_{-2} & b_{-1} \\ b_{-4} & b_{-3} \end{vmatrix} \begin{vmatrix} b_0 & b_1 & b_2 \\ b_{-2} & b_{-1} & b_0 \\ b_{-1} & b_0 & b_1 \end{vmatrix} - \begin{vmatrix} b_0 & b_1 \\ b_{-2} & b_{-1} \end{vmatrix} \begin{vmatrix} b_{-2} & b_{-1} & b_0 \\ b_{-4} & b_{-3} & b_{-2} \\ b_{-1} & b_0 & b_1 \end{vmatrix} \end{aligned} \tag{3.15}$$

Again the first term on the right is zero, and the second term is negative. Since the 2×2 minor on the left is negative, we deduce that the corresponding 3×3 minor

$$\begin{vmatrix} b_0 & b_1 & b_2 \\ b_{-2} & b_{-1} & b_0 \\ b_{-4} & b_{-3} & b_{-2} \end{vmatrix} > 0. \tag{3.16}$$

We repeat the process for columns $(2,3,4)$ and $(3,4,5)$, leading to the conclusion that

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ b_{-1} & b_0 & b_1 \\ b_{-3} & b_{-2} & b_{-1} \end{vmatrix} > 0 \qquad \begin{vmatrix} b_2 & b_3 & b_4 \\ b_0 & b_1 & b_2 \\ b_{-2} & b_{-1} & b_0 \end{vmatrix} > 0. \tag{3.17}$$

A final repeat of the process using the determinants (3.16) and (3.17) proves that

$$D(0,3) = \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_{-1} & \beta_0 & \beta_1 \\ \beta_{-2} & \beta_{-1} & \beta_0 \end{vmatrix} = \begin{vmatrix} b_0 & b_2 & b_4 \\ b_{-2} & b_0 & b_2 \\ b_{-4} & b_{-2} & b_0 \end{vmatrix} > 0. \tag{3.18}$$

The conclusion is that, using the single moment method, we can prove that $D(n,3) > 0,\ \ n = 0,1,\dots$ without using the BCF double moment technique employed by [1] for $r = 2$ and [5] for $r = 3$. There are no exceptional cases and in particular the method applies to $D(1,3)$ where the previous method failed.

3.4. The single moment procedure described above can be extended in an obvious manner to $r > 3$. However, as we explained in [4, Sec. 4.4, p. 12], the kernel $K(u,v;m)$ is definitely not RR_r unless $m \geq m(r)$, where $m(r)$ resulting from our caclulations is given in the attached table up to $r = 20$.

Taking account of this constraint, we repeatedly apply the theorem in [2, Thm. 3.2, p. 59] in the same manner as for $r = 3$. First we observe that the RR_r property of the kernel $K(u,v;m)$ may be used to show that

$$\Delta(n,r) > 0,\ \ n \geq m(r) - r. \tag{3.19}$$

Next consider the determinant (or rather the corresponding matrix) $\Delta(N, 2r-1)$, where N is even. All the $r \times r$ minors with consecutive rows and columns will have a top left corner element of the form b_k. The minor for which k is least occurs in the bottom right corner, and it may be seen that

$$k = N - r + 1. \tag{3.20}$$

To apply the Karlin theorem, (3.19) requires that $k \geq m(r) - r$, which implies that

$$N \geq m(r) - 1. \tag{3.21}$$

In addition the Karlin theorem also requires that $(r-1) \times (r-1)$ minors of $\Delta(N, 2r-1)$ with consecutive rows and columns should be positive. An argument similar to the above leads to the requirement

$$N \geq m(r-1) + 1. \tag{3.22}$$

Thus, if both (3.21) and (3.22) hold, all $r \times r$ minors are positive. In particular, (with [ ] meaning integer part), if we set

$$\mu = [(N+1)/2] = N/2 \tag{3.23}$$

then it follows that $D(\mu, r) > 0$. In the table below we list the lowest allowed values of $\mu(r)$ and corresponding lowest values $\eta(r)$ using the previous BCF double moment method.

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mu(r)$	0	0	1	1	1	1	1	2	3	4	4	4	4	5	6	7	8	8	8
$\eta(r)$	1	2	3	5	6	7	8	10	12	14	15	16	18	20	22	24	25	26	27

In the above discussion we have not taken account of the fact that the method described for the case $r = 3$ uses a bottom left 2×2 matrix that is not positivebut zero. We have shown that the method of Sec. 3.3 applies up to $r = 7$, so that the table may be extended by substituting $\mu(r) = 0,\ \ r \leq 7$. Of course it is obvious that $D(0,r) = 0$ for any r, but here we have been concerne with methodology.

4. DISCUSSION

4.1. The single moment method appears to lead to a significant reduction in the number of exceptional cases, values of n where the earlier method is unable to prove that $D(n,r) > 0$. It does this despite the fact that less information is used in the argument. Thus the standard method is based on the the values of $w(r,m;u)$, $u \geq 0$, whereas the new approach requires only $w(r,m;0)$.

References

1. G. Csordas, T. S. Norfolk, and R. S. Varga, *The riemann hypothesis and the turan inequalities*, Trans. Amer. Math. Soc. **296** (1986), 521–541.
2. S. Karlin, *Total positivity*, Stanford University Press, Stanford, CA, 1968.
3. S. Karlin, F. Proschan, and R. E. Barlow, *Moment inequalities of polya frequency functions*, Pacific J. Math. **11** (1961), 1023–1033.
4. J. Nuttall, *Determinantal approach to a proof of the riemann hypothesis*, unpublished.
5. ______, *Riemann determinantal inequalities of order 3*, unpublished.

E-mail address: `jnuttall@uwo.ca`