STRUCTURE OF WRONSKIANS RELATED TO THE RIEMANN HYPOTHESIS

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Abstract. This report introduces the concept of 'splits' to analyze the two-way Wronskians that appear in the study of the Riemann Hypothesis.

1. INTRODUCTION

1.1. In previous reports we have emphasized the importance of the cumulants of the function $\Phi(t)$ (as defined by [1]) in analyzing the veracity of the Riemann Hypothesis (RH). A key function based on $\Phi(t)$ is the two-way Wronskian

$$w(r,m;t) = \det \left| \Phi(t)_m^{(i+j-2)}(t) \right|_{i,j=1}^{r}, \quad t \geq 0; \;\; r = 1,2,\ldots; \;\; m = 0,1,2,\ldots, \tag{1.1}$$

where the superscript implies differentiation with respect to t. Here, $\Phi_m(t)$ is the cumulant of level m defined by

$$\Phi_m(t) = \int_t^\infty du\, \Phi_{m-1}(u), \quad m \geq 1, \tag{1.2}$$

and

$$\Phi_0(t) = \Phi(t). \tag{1.3}$$

The definition of the cumulant shows that

$$\Phi_m^{(1)}(t) = -\Phi_{m-1}(t), \quad m \geq 1. \tag{1.4}$$

We have provided numerical evidence [JN Sec. 2.1.9] (still to be checked independently) that there exists a non-decreasing integer function $m(r)$, $r = 1,2,\ldots,142$, with $m(142) = 197$, such that, with $\epsilon_p = (-1)^{p(p-1)/2}$,

$$\epsilon_p w(p, m(r); t) > 0, \;\; t \geq 0; \;\; p = 1,2,\ldots,r; \;\; r = 1,2,\ldots,142. \tag{1.5}$$

It is said that (1.5) is a manifestation of the sign-regularity (SR) of $\Phi(t)$ and its cumulants. If this evidence could be confirmed, and (1.5) extended to arbitrary r, it is possible (but by no means certain) that further development of techniques described by Karlin [2] would lead to a proof of the RH.

This report describes a numerical investigation of the structure of the Wronskians (1.1), aimed at identifying the mechanism behind the results of (1.5). Understanding such a structure might lead to a proof of (1.1) for all r.

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2. WRONSKIAN SPLITS

2.1. The definition of $\Phi(t)$ by [1] is

$$\Phi(t) = \sum_{n=1}^{\infty} a_n(t), \tag{2.1}$$

where

$$a_n(t) = \pi n^2 \left[2n^2 \pi e^{4t} - 3\right] \exp(5t - n^2 \pi e^{4t}). \tag{2.2}$$

To reduce the problem of finding the cumulants $\Phi_m(t)$ to a single integration, we use an identity of [2, Remark 4.1, p.194] that effectively states

$$\Phi_m(t) = \frac{1}{\Gamma(m)} \int_t^{\infty} du\, \Phi(u)(u-t)^{m-1}, \quad m \geq 1. \tag{2.3}$$

Similarly we can define the m-level cumulant of $a_n(t)$ by

$$\alpha_m(n,t) = \frac{1}{\Gamma(m)} \int_t^{\infty} du\, a_n(u)(u-t)^{m-1}, \;\; n = 1, 2, \ldots, ; \;\; m \geq 1. \tag{2.4}$$

For each n we introduce a set of r column vectors $\overline{\alpha}_m(n,i,t),\ i = 1, 2, \ldots, r$, each of dimension r, with elements $\alpha_m^{(i+j-2)}(n,t),\ \ j = 1, 2, \ldots, r;\ \ i = 1, 2, \ldots, r$. Now suppose that the series (2.1) is truncated at $n = n_c$. This leads to an approximation to the Wronskian of (1.1) that takes the form

$$w_{n_c}(r, m; t) = \left|\ \overline{M}_1 \quad \overline{M}_2 \quad \ldots \quad \overline{M}_r\ \right| \tag{2.5}$$

where the column

$$\overline{M}_i = \sum_{n=1}^{n_c} \overline{\alpha}_m(n,i,t). \tag{2.6}$$

By choosing in turn one column $\overline{\alpha}_m(n,i,t)$ from each term of (2.5), the determinant in (2.5) may be written as the sum of $(n_c)^r$ separate determinants with columns chosen from the $\overline{\alpha}_m(n,i,t)$.

An example of this form, with $r = 3$, $m = 0$, $n_c = 2$ may be found in [JN Sec. 2.1.5](Sec. 3). The sum of the first four contributions to [JN Sec. 2.1.5](3.6) is equivalent to the $(n_c)^r = 2^3 = 8$ terms in the expanded (2.5). The four contributions correspond to the types (111), (112), (122), (222), which contain respectively the $1, 3, 3, 1$ entries described in [JN Sec. 2.1.5] (Secs. 3.2 - 3.5). We sum the determinants corresponding to a given type, and call the result a split $S(n_1, n_2, \ldots, n_r)$, where $n_1 \leq n_2 \ldots n_r \leq n_c$. In the example the four splits correspond to the functions

$$F_0(y), F_{1,1}(y), F_{1,2}(y), F_{1,3}(y), \tag{2.7}$$

where $y = \pi \exp(4t)$.

Note that, because of the exponential term in (2.2), $S(1,1,1)$ contains a factor $\exp(15t - 3y)$. In [JN Sec. 2.1.5](Secs. 3.2 - 3.5) this factor has been omitted from all four functions $F(y)$ listed in (2.7), with the resulting factor (i.e. relative to $S(1,1,1)$) being given explicitly.

Note also that the example relates to a case where the cumulant order is $m = 0$. For cases with non-zero m, the functions corresponding to (2.7) are no longer polynomials, and they cannot be expressed in such a simple form

3. CALCULATION OF WRONSKIAN SPLITS

3.1. We concentrate on using numerical methods to find the values of a relevant set of split determinant sums when r is small (in our case $r \leq 7$ - this could be increased), for several cumulant orders, and a range of values of t. The main obstacle is the evaluation of the integral in the formula (2.2) for $a_m(n,t)$. We have found that it is possible to obtain highly accurate values of $a_m(n,t)$ by using the analyticity described in [1, (1.14(ii), p.523] to approximate the required integral.

The calculations involve $\alpha_m(n,t)$ and its derivatives, with parameters in the range

$$m = 0, 1, \ldots, 5; \quad n = 1, 2, 3; \quad t = 0.0, 0.0025, \ (0.0025), \ 0.25, \tag{3.1}$$

so that $n_c = 3$. For each value of r we choose a set of types. For each type $(n_1, n_2, \ldots, n_r)$ we obtain the value of the splits $S(n_1, n_2, \ldots, n_r)$ for each value of t and m in the above list.

As in the example, for a given r, m, t, the sum of the splits is an approximation to the exact value of the coresponding Wronskian $w(r, m; t)$. We find that, for some of the possible splits, there is no value of t for which the corresponding $S(n_1, n_2, \ldots, n_r)$ has a significant contribution to the Wronskian. Those splits we omit from the table of results, since we are concerned only with determining the sign of the Wronskian.

The results are listed in the file 'splita'. There are $36 = 6 \times 6$ blocks corresponding to the values $r = 2-7$, $m = 0-5$. Each block has 101 lines, one for each value of t. The first column provides an 'exact' value of $\epsilon_r w(r, m; t)$. The second column gives the value of the same quantity obtained by summing all the chosen splits that are listed in columns ≥ 3. The numbers in the first two columns are close to each other, which suggests (but does not prove) that none of the omitted splits is significant.

The evidence of the tables suggests the following features.

- For fixed r, m the sign is positive for sufficiently large values of t.
- For fixed r and all t the sign is positive for sufficiently large values of m.
- The last factor in (2.2) decreases rapidly as n increases, which leads to the result that, if r is small, we need only a small value of n_c to obtain adequately accurate estimates for the Wronskians of (1.1).

References

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