NUMERICAL PATTERNS RELATED TO THE RIEMANN AND CONREY-GHOSH Φ FUNCTIONS

JOHN NUTTALL

Abstract. This report summarizes several aspects of a numerical investigation into the functions $\Phi(u)$ connected to the Riemann Hypothesis and the eight analogous functions of Conrey and Ghosh. Some surprising patterns have emerged, as described in Sec. 3.

1. INTRODUCTION

1.1. **Definition of the $\Phi(u)$ functions.** An important function in the study of the Riemann Hypothesis (RH) is given by[1] [JN 2.1.10]

$$\Phi_0(u) = \sum_{m=1}^{\infty} \left[2m^4\pi^2 e^{9u} - 3m^2\pi e^{5u}\right] \exp\left(-m^2\pi e^{4u}\right). \tag{1.1}$$

Conrey and Ghosh [1] have conjectured that the eight functions $\{\Phi_k(u)\}$ have properties analogous to the Riemann function $\Phi_0(u)$. The 8 CG cases are labeled by $k = 1, 2, 3, 4, 6, 8, 12, 24$, and are obtained from the Riemann case by replacing $\Phi_0(u)$ with $\Phi_k(u)$, given by

$$\Phi_k(u) = \left[\exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right] \prod_{m=1}^{\infty} (1 - e^{-2\pi m e^u})\right]^k. \tag{1.2}$$

We shall use the symbol $\Phi(u)$ to represent any one of the nine functions defined above.

1.2. **Even moments and the determinants $D(n, r)$.** In previous reports [JN 2.1.1 - 2.1.12] we have described an approach to the proof of the Generalized Riemann Hypothesis (GRH) that involves showing the positivity of a set of determinants, i.e.

$$D(n, r) > 0, \quad n = 0, 1, \dots; \; r = 1, 2.... \tag{1.3}$$

The normalized even moments of $\Phi(u)$ are defined by

$$\beta_n = \frac{1}{\Gamma(2n+1)} \int_0^{\infty} du \Phi(u) u^{2n}, \quad n = 0, 1, \dots, \tag{1.4}$$

Date: August 2013.

The author is most grateful for the suggestions of J. B. Conrey and the brothers D. and G. Chudnovsky. The assistance provided by the multiple precision arithmetic package written by David H. Bailey is much appreciated. The use of the Sharcnet consortium computing resources has been very helpful.

[1] Previous work by the author is available on the web site http://publish.uwo.ca/ jnuttall. The following refers to a section on that web site.

with the convention that $\beta_n = 0$ if $n < 0$. The determinant $D(n,r)$ of order r is given by the Toeplitz form

$$D(n,r) = \begin{vmatrix} \beta_n & \beta_{n+1} & \cdots & \beta_{n+r-1} \\ \beta_{n-1} & \beta_n & \cdots & \beta_{n+r-2} \\ \vdots & \vdots & & \vdots \\ \beta_{n-r+1} & \beta_{n-r+2} & \cdots & \beta_n \end{vmatrix}. \tag{1.5}$$

1.3. **Single moments and the determinants $\Delta(n,r)$.** Another important set of normalized moments, both even and odd, is defined by

$$b_n = \frac{1}{\Gamma(n+1)} \int_0^\infty du \Phi(u) u^n, \quad n = 0, 1, \ldots. \tag{1.6}$$

In this case we define the 'moments' for negative n by

$$b_n = (-1)^{n+1} \Phi^{(-n-1)}(0), \quad n = -1, -2, \ldots. \tag{1.7}$$

A corresponding set of determinants is

$$\Delta(n,r) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+r-1} \\ b_{n-1} & b_n & \cdots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-r+1} & b_{n-r+2} & \cdots & b_n \end{vmatrix}. \tag{1.8}$$

Note that, if $n = 2j$, $j = \ldots, -2, -1, 0, 1, 2, \ldots$, then $\beta_j = b_n$. This relation applies even for negative j, since the function $\Phi(u)$ is even and $\beta_j = 0$.

2. RELATIONS BETWEEN THE DETERMINANTS $D(n,r)$ AND $\Delta(n,r)$

2.1. **Definition of various minors.** Karlin [2, p. 59] has described a set of identities that may be used to establish a relation between the two types of determinant.

To investigate $D(n,r)$ for a given n, r, we consider the $(2r-1) \times (2r-1)$ matrix

$$B(n,r) = \begin{bmatrix} b_n & b_{n+1} & \cdots & b_{n+2r-2} \\ b_{n-1} & b_n & \cdots & b_{n+2r-3} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \cdots & b_n \end{bmatrix}. \tag{2.1}$$

Note, for even n, that the sole $r \times r$ minor of $B(n,r)$ with only odd rows and columns has determinant $D(n/2, r)$, and our aim is to determine the sign of this minor.

The process consists of two stages. In the first stage (ST1) we consider in turn the r sub-matrices of $B(n,r)$ with $2r-1$ rows and r consecutive columns, so that the first sub-matrix is

$$B_{1,1}(n,r) = \begin{bmatrix} b_n & b_{n+1} & \cdots & b_{n+r-1} \\ b_{n-1} & b_n & \cdots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \cdots & b_{n-r+1} \end{bmatrix}. \tag{2.2}$$

Moving from left to right in (2.2) let us label the columns of the first sub-matrix $B_{1,1}(n,r)$ by $1, 2, \ldots, r$. Label the rows $1, 2, \ldots, 2r-1$ from top to bottom.

We use the symbol $M(r, j, k)$ to denote a certain $r \times r$ minor determinant of the sub-matrix as follows.

- $M(r, 0, k)$ contains the consecutive rows $k - r + 1, k - r + 2, \ldots, k$, with $k = r, r + 1, \ldots, 2r - 1$.
- $M(r, j, k)$, $j \geq 1$, uses consecutive rows of the sub-matrix ending in row k, except that, at the end of the list, there are j gaps of one row each.

For example, with $r = 4$,

- $M(r, 0, 6)$ has rows $3, 4, 5, 6$.
- $M(r, 1, 7)$ has rows $3, 4, 5, 7$.
- $M(r, 2, 7)$ has rows $2, 3, 5, 7$.

Lemma 2.1. *With the minors $M(r, j, k)$ defined as above, and appropriate restrictions on the values of j, k, the following relations hold.*

(2.3)

$$\begin{aligned}
&M(r, 1, k)M(r - 1, 0, k - 1)\\
&= M(r, 0, k - 1)M(r - 1, 1, k) + M(r, 0, k)M(r - 1, 0, k - 2)\\[1em]
&M(r, j, k)M(r - 1, j - 2, k - 2)\\
&= M(r, j - 2, k - 2)M(r - 1, jj, k) + M(r, j - 1, k)M(r - 1, j - 1, k - 2)\\[1em]
&jj = j,\ \ j \leq r - 2; \quad jj = j - 1,\ \ j = r - 1.
\end{aligned}$$

Proof. With the notation for determinants used by Karlin [2, p. 59] we may write

$$M(r, j, k) = B_{1,1} \begin{pmatrix} i_1, i_2, \ldots, i_r \\ 1,\ 2, \ldots, r \end{pmatrix} \tag{2.4}$$

where the top set of indices refers to rows, the bottom to columns, and

$$\begin{aligned}
&i_r = k \leq 2r - 1\\
&i_1 < i_2 < i_3 \ldots < i_r\\
&i_{m+1} - i_m = 1,\ \ m = 1, k - j - 1\\
&i_{m+1} - i_m = 2,\ \ m = k - j, k - 1;\ \ j \geq 1
\end{aligned} \tag{2.5}$$

For a given choice of r, j, k we determine the values of $i_1, i_2, \ldots, i_r$ and substitute them into the Karlin identity [2, p. 59] . Taking account of the fact that some of the indices in the Karlin determinants are not in increasing order, it will be found that (2.3) is valid. □

For example consider the 4×4 minor $M(4, 2, 7)$, so that $i_1, i_2, i_3, i_4 = 2, 3, 5, 7$. The Karlin relation involves an integer i^0 between i_1 and i_4, not equal to $2, 3, 5, 7$. Choose $i^0 = 4$, in which case the relation becomes

$$M(4, 2, 7)M(3, 0, 5) = M(4, 0, 5)M(3, 1, 7) + M(4, 1, 7)M(3, 1, 5), \tag{2.6}$$

as (2.3) states. Note that the choice of $i^0 = 6$ would give another relation, not needed for our purposes, although such a formula could easily be obtained.

2.2. **Stage 1 minors.** Let us call the value of j in $M(r, j, k)$ the layer number. Using several applications of (2.3) we can produce a set of recursion relations that connect minors of layer 0 to a single top minor $M(r, r-1, 2r-1)$. In the example above we need the following minors.

$$\begin{array}{llll} M(4,0,4) & & & \\ M(4,0,5) & M(4,1,5) & & \\ M(4,0,6) & & & \\ M(4,0,7) & M(4,1,7) & M(4,2,7) & M(4,3,7) \end{array} \tag{2.7}$$

The ST1 minors of form $M(4,0,k)$ may be written in terms of various $\Delta(n,4)$ using the formula

$$M(r,0,k) = \Delta(n+r-k, r). \tag{2.8}$$

We think of the information as flowing from left to right in this table. Given those minors in layer 0, we may deduce the value of the minors in layer 1, twice using the first equation in Lemma 3.1. Thus $M(4,1,7)$ is related to $M(4,0,7)$ and $M(4,0,6)$, while $M(4,1,5)$ is related to $M(4,0,5)$ and $M(4,0,4)$. Next, using the second equation in Lemma 3.1, the layer 2 minor $M(4,2,7)$ is related to $M(4,1,7)$ in layer 1 and $M(4,0,5)$ in layer 0. Finally the minor $M(4,3,7)$ depends on the minor in layer 2 $M(4,2,7)$ and $M(4,1,5)$ in layer 1. All these relations also involve minor factors for $r=3$. This tree-like structure is may be generalized to all cases.

2.3. **Stage 2 minors.** Continuing with the $r=4$ example we note that the $r \times r$ minor $M(4,3,7)$ may be written

$$M(4,3,7) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+r-1} \\ b_{n-2} & b_{n-1} & \cdots & b_{n+r-3} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \cdots & b_{n-r+1} \end{vmatrix} \tag{2.9}$$

which uses only the odd rows from the matrix $B_{1,1}(n,4)$. This is the result of the first part of ST1. We repeat the above steps $r-1$ times , each time increasing the value of n by 1. The result will be 4 versions called $M(4,3,7)^{(1)}_\nu$, $\nu = n, n+1, n+2, n+3$, given by (2.9) with the appropriate value of ν, where the superscript stands for ST1. Here the minor $M(4,3,7)^{(1)}_n$ is what we previously denoted by $M(4,3,7)$. The determinants $\{M(4,3,7)^{(1)}_\nu\}$ may be thought of as the 4 minors with consecutive columns of the $r \times (2r-1)$ matrix (with $r=4$)

$$B_2(n,r) = \begin{bmatrix} b_n & b_{n+1} & \cdots & b_{n+2r-2} \\ b_{n-2} & b_{n-1} & \cdots & b_{n+2r-4} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \cdots & b_n \end{bmatrix} \tag{2.10}$$

To proceed to the second stage (ST2) of the process, we bring $B_2(n,r)$ into a form analgous to $B_{1,1}$ of (2.2) by transposing, reversing the order of the columns, and then reversing the order of the rows to produce the $(2r-1) \times r$ matrix

$$B_{2,1}(n,r) = \begin{bmatrix} b_n & b_{n+2} & \cdots & b_{n+2r-2} \\ b_{n-1} & b_{n+1} & \cdots & b_{n+2r-3} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+4} & \cdots & b_n \end{bmatrix} \tag{2.11}$$

As in (2.2) we label the columns of $B_{2,1}(n,r)$ left to right by $1,2,\dots,r$ and the rows top to bottom by $1,2,\dots 2r-1$. From this point we repeat the steps after (2.2) to construct $r\times r$ minors from $B_{2,1}(n,r)$, that we now call $M(r,j,k)^{(2)}_n$ for ST2. The key point to note is that

$$M(r,0,2r-1-i)^{(2)}_n = M(r,r-1,2r-1)^{(1)}_{n+i},\ \ i=0,1,r-1. \tag{2.12}$$

This serves as a starting point for ST2. We use the relations in the generalization of formula (2.3) for minors $M(r,j,k)^{(2)}_n$, which leads to an expression for $M(r,r-1,2r-1)^{(2)}_n = D(n/2,r)$ for even n. The combined process involves two stages, with $2r-1$ layers in all.

2.4. **General structure of the identities.** The general form of the tree of minors such as (2.7) is given below for the case when r is even. To save space we have used the notation $p_j = r+j;\ \ u_j = r-j;\ \ t_j = 2r-j$. The symbol (j,k) is short for $M(r,j,k)$.

$$\begin{array}{llllll}
(0,r) & & & & & \\
(0,p_1) & (1,p_1) & & & & \\
(0,p_2) & & & & & \\
(0,p_3) & (1,p_3) & (2,p_3) & (3,p_3) & & \\
\vdots & & & \vdots & & \\
(0,t_4) & & & & & \\
(0,t_3) & (1,t_3) & (2,t_3) & (3,t_3) & \dots & (u_3,t_3) \\
(0,t_2) & & & & & \\
(0,t_1) & (1,t_1) & (2,t_1) & (3,t_1) & \dots & (u_3,t_1)\ (u_2,t_1)\ (u_1,t_1).
\end{array} \tag{2.13}$$

In this case the rows are denoted by $r,r+1,\dots,2r-1$ and the columns (layers) by $0,1,2,\dots,(r-1)$. The formulas in Lemma 3.1 describe the other two $r\times r$ minors that are connected with $M(r,j,k)$. and similarly the three $(r-1)\times(r-1)$ minors.

When r is odd the corresponding table is

$$\begin{array}{llllll}
(0,r) & & & & & \\
(0,p_1) & & & & & \\
(0,p_2) & (1,p_2) & (1,p_3) & & & \\
(0,p_3) & & & & & \\
(0,p_4) & (1,p_4) & (2,p_4) & (3,p_4) & (3,p_5) & \\
\vdots & & & \vdots & & \\
(0,t_4) & & & & & \\
(0,t_3) & (1,t_3) & (2,t_3) & (3,t_3) & \dots & (u_3,t_3) \\
(0,t_2) & & & & & \\
(0,t_1) & (1,t_1) & (2,t_1) & (3,t_1) & \dots & (u_3,t_1)\ (u_2,t_1)\ (u_1,t_1).
\end{array} \tag{2.14}$$

2.5. **Example of a combined tree.** As an example of the combined tree, for $r = 4$ we have
(2.15)

$(0,4)^{(1)}_{n+3}$
$(0,5)^{(1)}_{n+3}$ $(1,5)^{(1)}_{n+3}$
$(0,6)^{(1)}_{n+3}$
$(0,7)^{(1)}_{n+3}$ $(1,7)^{(1)}_{n+3}$ $(2,7)^{(1)}_{n+3}$ $\left[(3,7)^{(1)}_{n+3} = (0,4)^{(2)}_{n}\right]$

$(0,4)^{(1)}_{n+2}$
$(0,5)^{(1)}_{n+2}$ $(1,5)^{(1)}_{n+2}$
$(0,6)^{(1)}_{n+2}$
$(0,7)^{(1)}_{n+2}$ $(1,7)^{(1)}_{n+2}$ $(2,7)^{(1)}_{n+2}$ $\left[(3,7)^{(1)}_{n+2} = (0,5)^{(2)}_{n}\right]$ $(1,5)^{(2)}_{n}$

$(0,4)^{(1)}_{n+1}$
$(0,5)^{(1)}_{n+1}$ $(1,5)^{(1)}_{n+1}$
$(0,6)^{(1)}_{n+1}$
$(0,7)^{(1)}_{n+1}$ $(1,7)^{(1)}_{n+1}$ $(2,7)^{(1)}_{n+1}$ $\left[(3,7)^{(1)}_{n+1} = (0,6)^{(2)}_{n}\right]$

$(0,4)^{(1)}_{n}$
$(0,5)^{(1)}_{n}$ $(1,5)^{(1)}_{n}$
$(0,6)^{(1)}_{n}$
$(0,7)^{(1)}_{n}$ $(1,7)^{(1)}_{n}$ $(2,7)^{(1)}_{n}$ $\left[(3,7)^{(1)}_{n} = (0,7)^{(2)}_{n}\right]$ $(1,7)^{(2)}_{n}$ $(2,7)^{(2)}_{n}$ $(3,7)^{(2)}_{n}$

The formula (2.12) shows that

$$M(r,0,k)^{(1)}_{n+i} = M(r,0,k-i)^{(1)}_{n}, \tag{2.16}$$

so that all the entries in the first column of (2.15) (the inputs) are contained in the set

$$S = \{M(r,0,i)^{(1)}_{n},\ \ i = 1, 2r-1\}. \tag{2.17}$$

3. NUMERICAL CACLULATIONS OF MINORS

3.1. **Sign data for some minors.** Using high precision software, we have calculated the value of some minors related to five of the nine $\Phi(u)$ functions, namely

- $\Phi_0(u)$, the standard RH function.
- $\Phi_1(u)$, CG k=1
- $\Phi_3(u)$, CG k=3
- $\Phi_8(u)$, CG k=8
- $\Phi_{24}(u)$, CG k=24

It should be noted that no calculation has been independently verified, so that further study is indicated, but all the results so far appear to share a common structure, while much of the coding varies from case to case.

It is suggested that the interested reader download a 30 kb file
publish.uwo.ca/∼jnuttall/CGk1k24.txt

- The first half of the file relates to the Case CG k=1.

- The blocks of data correspond to $(r, n) = (4, 2), (4, 4), (5, 2), (5, 4)$, etc.
- In the first block the entries (4,1,7); (4,1,5); (4,2,7); (4,3,7) are the values of r, j, k in the four minors $M(r, j, k)_n$ needed for $r = 4$, $n = 2$.
- On the same line as a r, j, k entry there follows an indication of the sign of the 12 minors appearing in the formulas (2.3) for $M(r, j, k)_n^{(1)}$, $M(r, j, k)_n^{(2)}$. Positive is denoted by 1., negative by -1.
- The first sign is that of the minor $M(r, j, k)_n^{(1)}$, followed by the signs of the other two minors in (2.3) of order r.
- The next three signs are those of the three minors for order $r - 1$ that are used in (2.3).
- The remaining six signs are the same as the above, except they are related to $M(r, j, k)_n^{(2)}$.
- Note in particular that the minor corresponding to the seventh sign in the last row of a (r, n) block is $D(n/2, r)$. All these signs are positive, which is consistent with the validity of the GRH.
- The data in the second half of the file repeats the above for the Case CG k=24.

3.2. **Patterns in the observed minor sign data.** In the Case CG k=24 consider the signs in the last 3 rows of the data for $M(r, j, k)_n^{(2)}$, for $r = 4 - 10$, i.e. sign columns 7-12.

- With $n = 2$, in every case, all the signs are positive, but, for $r = 6 - 10$, at least one sign in the fourth row from the end is negative.
- Similarly, for $n = 4$ all the signs in the last 5 rows are positive, but, for $r = 6 - 10$, at least one sign in the sixth row from the end is negative

The situation is similar for the Case CG k=1, sign columns 7-12.

- With $n = 2$, in every case all the signs in the last 8 rows (to the extent that they exist) are positive, but, for $r = 6 - 10$, at least one sign in the ninth row from the end is negative.
- Similarly, for $n = 4$ all the signs in the last 15 rows are positive, but, for $r = 8 - 10$, at least one sign in the sixteenth row from the end is negative.

3.3. **Further patterns in the minor sign data.** The above calculations have been extended to significantly higher values of r.

- For CG k=24 with $n = 2$, in every case the last 6 columns in the final 3 rows have all positive signs. We say that the 'chain length' $l(2)$ for CG k=24 is has a value $l(2) = 3$.
- This behavior is repeated for CG k=24 with $n = 4$, with a chain length $l(4) = 5$.
- Similarly for CG k=1 we find, for the calculations carried out so far, that chain lengths exist with $l(2) = 8$ and $l(4) = 15$.
- The results also are similar for higher even values of n.
- Finally, the chain length phenomenon is also present in the other three cases listed in Sec. 3.1, i.e. CG cases k=3, 8 and the standard RH case, where our calculations have been most extensive.

The observed values of the chain lengths in the five cases studied so far are listed below.

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$l(n)$	15	29	41	63	80	99	109	131	168	195	209	239	271	305	379

TABLE 1. Computed values of the chain length $l(n)$ for even $n = 2, \ldots, 30$, in the case of the Riemann Hypothesis.

	n	2	4	6	8	10	12
$CG01$	$l(n)$	8	15	24	35		
$CG03$	$l(n)$	5	7	8	11	15	19
$CG08$	$l(n)$	5	7	8	11	15	19
$CG24$	$l(n)$	3	5	8	11	15	

TABLE 2. Computed values of the chain lengths $l(n)$ for several of the Conrey-Ghosh functions.

4. DISCUSSION

The nature of these results may be surprising. In particular the approximately 250,000 positive chain minors, with none negative, implied by the RH data in Table 1, may suggest the existence of a hidden mathematical structure. Note that all the values of $l(n)$ in that table are such that $l(n) + 1 = m^2$ or $l(n) + 1 = m(m + 1)$ for some value of integer m.

It appears that, in every case, the chain length increases with n. It is also suggested that the RH case and the 8 CG examples belong to a sequence with chain lengths that never increase for a given n.

In a sense the data suggest that the GRH is closest to failing for the case CG 24, and furthest from failing for the RH.

One clear conclusion is that the present calculations should be verified and extended. Some additional analytic observations, including the connection to cumulants, are to be found in [JN 2.1.10].

References

1. J. B. Conrey and A. Ghosh, *Turán inequalities and zeros of Dirichlet series associated with certain cusp forms*, Trans. Amer. Math. Soc. **342** (1994), 407–419.
2. S. Karlin, *Total Positivity*, Stanford University Press, Stanford, CA, 1968.

E-mail address: jnuttall@uwo.ca