BASIC WRONSKIANS IN THE RIEMANN HYPOTHESIS

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Abstract. We have previously proposed that the sign-regularity properties of a function $\Phi(t)$ related to the transform of the ζ-function could be the key to an analytic approach to the Riemann hypothesis. These properties are related to a corresponding Wronskian. In this report we find an explicit form for the Wronskian of a function based on the first term in the expansion of $\Phi(t)$ (case β). For a simplification (case α) of this case, the properties, in particular the location of the zeros, was soved by Szegö in 1924 in his study of partial sums of the exponential function.

1. INTRODUCTION

1.1. Pólya [6] in 1926 suggested that, when searching for an approach to the Riemann Hypothesis (RH), the properties of the function $\Phi(t)$ might be of considerable importance. With this in mind, following the lead of Csordas, Norfolk and Varga [1], [2], we have argued [4] that the sign-regularity behavior of $\Phi(t)$ and its cumulants could be useful in an analytic (rather than arithmetic) method. Recently we have suggested [5] [1] that, it might be helpful to study the properties of a simplified version of the function $\Phi(t)$, consisting of the first term in the infinite series that represents $\Phi(t)$.

As Karlin [3, p. 48,(1.3)] explains, the sign-regularity properties of a function $g(t)$ (or more precisely of the corresponding kernel $K(u,v) = g(u+v)$) are characterized by the two-way Wronskian

$$w(r,t) = \det \left| g^{(i+j-2)}(t) \right|_{i,j=1}^{r}, \quad t \geq 0; \quad r = 1, 2, \ldots, \tag{1.1}$$

where the superscript implies differentiation with respect to t. With an appropriate choice of scale for t and $y(t)$ we write the first term of $\Phi(t)$ as

$$g(t) = \exp[-y(t)]\beta(y(t)) \quad -\infty < t < \infty, \tag{1.2}$$

where $y(t) = e^t$. To correspond to the formula in [1, (1.14) p. 523] we use

$$\beta(y) = 2y - 3. \tag{1.3}$$

Note that we have omitted the factor $\exp(5t)$ from [1, (1.14) p. 523], since a statement in Karlin's book XXXX shows that it does not affect the results below.

As a useful preliminary exercise we first examine the case when $\beta(y)$ is replaced by $\alpha(y)$, where

$$\alpha(y) = y - 1. \tag{1.4}$$

We have found a simple analytic expression for $w(r,t)$ in this case. It is given by (2.23), where the key factor $W_r(y(t))$ is the partial sum of the power series for

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[1] Unpublished reports by the author are available at htpp://publish.uwo.ca/ jnuttall.

e^{-y}. The sign-regularity properties of $g(t)$ are related to the location of the zeros of $W_r(y(t))$, which for this case were determined by Szegö in 1924.

When we use $\beta(y)$ in (1.2) there is a different but still helpful analytic expression (3.2) for $W_r(y(t))$. A rigorous derivation of the properties of this version of $W_r(y(t))$ remains to be found, but numerical calculations [5] suggest that the two cases have similar structures.

1.2. In this note we use two methods that can provide useful information about $w(r,t)$. The first is based on [2, p. 184,(3.8)]. Let

$$g^{(k)}(t) = \exp(-y(t)q_{k+1}(y(t)), \ \ k = 0, 1, \dots, \tag{1.5}$$

so that

$$q_1(y) = y - 1 \ \text{ or } \ 2y - 3. \tag{1.6}$$

It follows that

$$q_{k+1}(y) = y(q_k^{[1]}(y) - q_k(y)), \ \ k = 1, 2, \dots, \tag{1.7}$$

where the derivative is taken with respect to y. Plainly $q_k(y)$ is a polynomial in y of degree k.

With our assumptions about $g(t)$ it follows from (1.5) that we may write

$$w(r,t) = \exp(-ry(t))\overline{W}_r(y) \tag{1.8}$$

where

$$\overline{W}_r\,(y) = \begin{vmatrix} q_1(y) & q_2(y) & \dots & q_r(y) \\ q_2(y) & q_3(y) & \dots & q_{r+1}(y) \\ \vdots & \vdots & & \vdots \\ q_r(y) & q_{r+1}(y) & \dots & q_{2r-1}(y) \end{vmatrix}, \tag{1.9}$$

and $y = y(t)$.

1.3. J. B. Conrey (private communication) has kindly pointed out a useful second source of information, an identity connecting $w(r-1,t), w(r,t), w(r+1,t), \ \ r = 1, 2, \dots$, that reads (we define $w(0,t) = 1$)

$$w(r-1,t)w(r+1,t) = w(r,t)w^{(2)}(r,t) - [w^{(1)}(r,t)]^2, \ \ r = 1, 2, \dots, \tag{1.10}$$

where the superscript implies differentiation with respect to t. Given $w(r-1,t), w(r,t)$ we can solve (1.10) for $w(r+1,t)$ so that

$$w(r+1,t) = \left[w(r,t)w^{(2)}(r,t) - [w^{(1)}(r,t)]^2\right]/w(r-1,t) \ \ \ r = 1, 2, \dots \tag{1.11}$$

The relation (1.10) is a special case of Karlin [3, p. 60,(4.2)] wth the parameter m chosen to be 2.

2. EXPLICIT FORM OF THE WRONSKIAN CASE $\alpha(y)$

2.1. From the representation (1.9) we see that $\overline{W}_r(y)$ is a polynomial in y of degree no more than $1 + 3 + \ldots + 2r - 1 = r^2$. In Lemma 6.1 of [4, p. 16] we described a method that is easily modified to prove, with $\epsilon(p) = (-1)^{\sigma(p)}$, that

$$\overline{W}_r(y) = \epsilon(r)y^{\sigma(r)}W_r(y), \tag{2.1}$$

where $\sigma(r) = r(r-1)/2$ and $W_r(y)$ is a polynomial of degree $\leq r^2 - \sigma(r) = \sigma(r+1)$. Substituting (2.1) and (1.8) into (1.10) leads after some manipulation to
(2.2)
$$e^{-2r}y^{(\sigma(r-1)+\sigma(r+1))}W_{r-1}W_{r+1} = e^{-2r}y^{2\sigma(r)}\left[r(W_r)^2y - W_rW_r^{[2]}y^2 - W_rW_r^{[1]}y + (W_r^{[1]})^2y^2\right],$$
which simplifies to

$$W_{r-1}(y)W_{r+1}(y) = \Delta_r(y), \ \ r = 1, 2, \ldots, \tag{2.3}$$

where $\Delta_r(y)$ is a polynomial given by

$$\Delta_r(y) = \left[r(W_r(y))^2 - yW_r(y)W_r^{[2]}(y) - W_r(y)W_r^{[1]}(y) + y(W_r^{[1]}(y))^2\right]. \tag{2.4}$$

We show below in Lemma 2.1 that the degrees of $W_r(y)$ and $\Delta_r(y)$ are r and $2r$ respectively.

2.2. From the relations of Sec. 2.1 we can deduce the dominant terms of $W_r(y)$ and $\Delta_r(y)$ as $y \to \infty$. Using $W_0(y) = 1$ and $W_1(y) = y - 1$ we have

Lemma 2.1. *In the case of $\alpha(y)$, for each $r \geq 1$, there exists a positive constant c_r such that*

$$W_r(y) = c_ry^r + O(y^{r-1}), \ \ y \to \infty, \tag{2.5}$$

and

$$\Delta_r(y) = (c_r)^2ry^{2r} + O(y^{2r-1}), \ \ y \to \infty. \tag{2.6}$$

The coefficient c_r is given by $c_0 = 1$; $c_1 = 1$, and

$$c_{r+1} = r(c_r)^2/c_{r-1}, \ \ r = 1, 2, \ldots. \tag{2.7}$$

Proof. Suppose that the relations (2.5) and (2.6) are correct for $r \leq \rho$. Let the degrees of polynomials $W_{\rho+1}(y)$ and $\Delta_{\rho+1}(y)$ be ρ_1, ρ_2, so that

$$W_{\rho+1}(y) = c_{\rho+1}y^{\rho_1} + O(y^{\rho_1-1}), \ \ y \to \infty, \tag{2.8}$$

$$W_{\rho+1}^{(1]}(y) = c_{\rho+1}\rho_1y^{\rho_1-1} + O(y^{\rho_1-2}), \ \ y \to \infty, \tag{2.9}$$

$$W_{\rho+1}^{[2]}(y) = c_{\rho+1}\rho_1(\rho_1-1)y^{\rho_1-2} + O(y^{\rho_1-3}), \ \ y \to \infty, \tag{2.10}$$

for some constant $c_{\rho+1}$.

When $r = \rho$ equation (2.3) may be written as

$$W_{\rho+1}(y) = \Delta_\rho(y)/W_{\rho-1}(y). \tag{2.11}$$

Choosing y to be large, and inserting the above expressions into (2.11), leads to

$$W_{\rho+1}(y) = \rho(c_\rho)^2y^{\rho+1}/c_{\rho-1} + O(y^\rho), \ \ y \to \infty. \tag{2.12}$$

This confirms (2.5) for $r = \rho + 1$ if we choose

$$c_{\rho+1} = \rho(c_\rho)^2/c_{\rho-1}, \tag{2.13}$$

Thus we have proved that c_r in (2.5) is positive for $r \geq 1$, since $c_0 = c_1 = 1$.

A similar argument verifies (2.6). $\square$

2.3. Following the above procedure in more detail leads to our main results. The key to the argument is the discovery that the partial sum of the exponential series (2.14) represents $W_r(y)$. This discovery was made by guessing a formula that fits the coefficients of a number of polynomials $W_r(y)$ obtained by calculation. The general result follows by induction as follows.

Theorem 2.2. *For $r = 0, 1, 2, \ldots$ in the case of $\alpha(y)$*

$$W_r(y) = c_r r! \sum_{j=0}^{r} (-1)^{r-j} y^j / j! \tag{2.14}$$

Proof. Suppose that (2.14) holds for $r = 2, 3, \ldots \rho$. Choosing $r = \rho$ in (2.14) and differentiating, we see that

$$W_\rho^{[1]}(y) = c_\rho \rho! \sum_{j=0}^{\rho-1} (-1)^{\rho-1-j} y^j / j! = [c_\rho \rho / c_{\rho-1}] W_{\rho-1}(y) = (r!) W_{\rho-1}(y). \tag{2.15}$$

Now from (2.4) we have

$$\Delta_\rho(y) = \left[\rho(W_\rho(y))^2 - yW_\rho(y)W_\rho^{[2]}(y) - W_\rho(y)W_\rho^{[1]}(y) + y(W_\rho^{[1]}(y))^2\right], \tag{2.16}$$

which we may rearrange to read

$$\Delta_\rho(y) = \left[W_\rho(y)T_1 + W_\rho^{[1]}(y)T_2\right] \tag{2.17}$$

where

$$\begin{aligned} T_1 = &\quad \rho W_\rho(y) - yW_\rho^{[2]}(y) \\ T_2 = &\quad yW_\rho^{[1]}(y) - W_\rho(y) \end{aligned} \tag{2.18}$$

On account of (2.11) we observe that the polynomial $\Delta_\rho(y)$ must be divisible by $W_{\rho-1}(y)$, or equivalently by $W_\rho^{[1]}(y)$ due to (2.15). The second term in (2.17) fulfills this requirement, so the first term must also have a factor $W_\rho^{[1]}(y)$, but $W_\rho(y)$ in that term does not. Consequently, we deduce that T_1 must have a factor $W_\rho^{[1]}(y)$.

The factor T_1 is of degree ρ, whereas $W_\rho^{[1]}(y)$ has degree $\rho - 1$, so that we postulate for the induction

$$\rho W_\rho(y) - yW_\rho^{[2]}(y) = (ay - b)W_\rho^{[1]}(y). \tag{2.19}$$

By inserting (2.14) in (2.19) and equating coefficients of each power of y, it is straightforward to check that (2.19) is correct if we choose $a = 1,\ b = \rho$. This means that

$$T_1 = \rho W_\rho(y) - yW_\rho^{[2]}(y) = (y - \rho)W_\rho^{[1]}(y), \tag{2.20}$$

so that

$$\begin{aligned} \Delta_\rho(y) = &\quad \left[W_\rho(y)(y-\rho)W_\rho^{[1]}(y) + W_\rho^{[1]}(y)(yW_\rho^{[1]}(y) - W_\rho(y))\right] \\ = &\quad \left[y(W_\rho(y) + W_\rho^{[1]}(y)) - (\rho+1)W_\rho(y)\right] W_\rho^{[1]}(y). \end{aligned} \tag{2.21}$$

Using (2.15) it is easy to see from (2.14) that

$$\Delta_\rho = W_{\rho+1}(y)W_{\rho-1}(y), \tag{2.22}$$

if we take account of (2.13).

The induction cycle is complete and we have proved the theorem. □

From (1.8) and (2.1) the form of the Wronskian for the case $\alpha(y)$ is

$$w(r,t) = \exp(-ry(t))\epsilon(r)y^{\sigma(r)}W_r(y(t)) \tag{2.23}$$

where $W_r(y)$ (called $W_r^{\alpha}(y)$ in the following) is given by (2.14).

3. EXPLICIT FORM OF THE WRONSKIAN CASE $\beta(y)$

3.1. We now repeat the discussion of Sec. 2.3, after making changes to account for the substitution of $\beta(y)$ for $\alpha(y)$. We make use of the following definitions.

(3.1)
$$\eta(k) = \frac{f(2k+2)}{f(k+1)f(k)}; \quad f(k) = k!; \quad f_1(k) = f(0)f(1)f(2)\ldots f(k), \;\; k = 0,1,2,\ldots.$$

Our main result is

Theorem 3.1. *When* $W_0(y) = 1$ *and* $W_1(y)=\beta(y)=2y-3$ *we have, for* $r = 0,1,2,\ldots,$

$$W_r(y) = \sum_{j=0}^{r} (-1)^{r-j} h(r,j) y^j \tag{3.2}$$

where

$$h(r,j) = 2^{(2j-r-1)} f_1(r)\eta(r-j)/f(j), \quad j = 0,1,\ldots,r. \tag{3.3}$$

Proof. Again we proceed by induction, after guessing the expression in (3.2). Suppose that (3.2) holds for $r = 2,3,\ldots \rho$. Choosing $r = \rho$ in (3.2) and differentiating, we find that

$$W_\rho^{[1]}(y) = f_1(\rho)\sum_{j=0}^{\rho-1} (-1)^{\rho-1-j} 2^{2j-\rho-1} y^j/f(j) = \frac{1}{2f(\rho)} W_{\rho-1}(y). \tag{3.4}$$

Now from (2.4) we have

$$\Delta_\rho(y) = \left[\rho(W_\rho(y))^2 - yW_\rho(y)W_\rho^{[2]}(y) - W_\rho(y)W_\rho^{[1]}(y) + y(W_\rho^{[1]}(y))^2\right], \tag{3.5}$$

which we may rearrange to read

$$\Delta_\rho(y) = \left[W_\rho(y)T_1 + W_\rho^{[1]}(y)T_2\right] \tag{3.6}$$

where

$$\begin{aligned} T_1 &= \rho W_\rho(y) - yW_\rho^{[2]}(y) \\ T_2 &= yW_\rho^{[1]}(y) - W_\rho(y) \end{aligned} \tag{3.7}$$

On account of (2.11) we observe that the polynomial $\Delta_\rho(y)$ must be divisible by $W_{\rho-1}(y)$, or equivalently by $W_\rho^{[1]}(y)$ due to (3.4). The second term in (3.6) fulfills this requirement, so the first term must also have a factor $W_\rho^{[1]}(y)$, but $W_\rho(y)$ in that term does not. Consequently, we deduce that T_1 must have a factor $W_\rho^{[1]}(y)$.

The factor T_1 is of degree ρ, whereas $W_\rho^{[1]}(y)$ has degree $\rho - 1$, so that we postulate for the induction

$$\rho W_\rho(y) - yW_\rho^{[2]}(y) = (ay - b)W_\rho^{[1]}(y). \tag{3.8}$$

By inserting (3.4) in (3.7) and equating coefficients of each power of y, it is straightforward to check that (3.7) is correct if we choose $a = 1,\ b = \rho + 0.5$. This means that

$$T_1 = \rho W_\rho(y) - yW_\rho^{[2]}(y) = (y - b)W_\rho^{[1]}(y), \tag{3.9}$$

so that

$$\begin{aligned} \Delta_\rho(y) = & \left[W_\rho(y)(y - \rho - 0.5)W_\rho^{[1]}(y) + W_\rho^{[1]}(y)(yW_\rho^{[1]}(y) - W_\rho(y))\right] \\ = & \left[y(W_\rho(y) + W_\rho^{[1]}(y)) - (\rho + 1.5)W_\rho(y)\right] W_\rho^{[1]}(y). \end{aligned} \tag{3.10}$$

Using (3.4) it follows from (3.2) that

$$\Delta_\rho = W_{\rho+1}(y)W_{\rho-1}(y), \tag{3.11}$$

if we take account of (3.3).

The induction cycle is complete and we have proved the theorem. □

From (1.8) and (2.1) the form of the Wronskian for the case $\beta(y)$ is

$$w(r,t) = \exp(-ry(t))\epsilon(r)y^{\sigma(r)}W_r(y(t)) \tag{3.12}$$

where $W_r(y)$ (called $W_r^\beta(y)$ in the following) is given by (3.2).

4. LOCATION OF THE ZEROS OF THE WRONSKIANS

4.1. In the application of the Wronskian $W_r^\beta(y)$ to the RH, the location of the zeros of $w(r,t)$, or equivalently $W_r^\beta(y)$, is of interest when considering sign-regularity. It turns out that the case of $W_r^\alpha(y)$ is simpler in ths regard, since this function is the partial sum of the power series for e^{-y} . Beginning with Szegö in 1924, the location of the zeros of functions such as $W_r^\alpha(y)$ has been extensively studied. A recent survey of the subject may be found in Vargas [7], and useful results appear in Zemyan [8].

Relevant information includes the following.

- If r is even then $W_r^\alpha(y)$ has no real zeros.
- If r is odd then $W_r^\alpha(y)$ has one real zero, say $z(r)$, and $z(r) > 0$.
- If $r_2 > r_1$ then $z(r_2) > z(r_1)$.
- The quantity $z(2r-1)/(2r-1) \to \hat{z},\ r \to \infty$, where $\hat{z}\exp(1+\hat{z}) = 1$, so that $\hat{z} = 0.278464\ldots$

4.2. Our numerical investigations [5] suggest that the structure of the zeros of $W_r^\beta(y)$ is similar to that of $W_r^\alpha(y)$. We have started to study this question - but perhaps there has been previous work on the problem.

5. DISCUSSION

It seems from calculations [5] that corresponding properties also apply to the cumulants of $g(t)$. An interesting project is to explore whether the method of this report may be extended to that case.

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