CUMULANTS, THE RIEMANN HYPOTHESIS, AND SIMILAR PROBLEMS

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Abstract. In 1994 Conrey and Ghosh studied a function that may be written as $\Theta_1(u) = \exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right]\prod_{m=1}^{\infty}(1-e^{-2\pi m e^u})$. It is thought to be analogous to the function $\Phi(u)$ that is important in the theory of the Riemann Hypothesis. In particular, it is predicted that a doubly infinite sequence of Toeplitz determinants, consisting of normalized double moments of $\Theta_1(u)$, will all be positive.

We have pointed out that it will be possible to show that some of these requirements are satisfied if we can establish the existence of an infinite sequence of cumulants of $\Theta_1(u)$ (in the sense of Karlin) with increasing order r, each one sign-regular of type RR_r. This paper achieves the goal.

1. INTRODUCTION

1.1. **Can Karlin's analytic methods be applied to the Riemann Hypothesis ?** In 1968 the late S. Karlin published the book "Total Positivity" [4], which introduces and discusses a number of analytic methods that might be useful in finding a resolution of the Riemann Hypothesis (although Karlin did not mention this application). Apart from the pioneering work of Csordas, Norfolk, and Varga, [2] and [3], the possibility of applying these methods to the RH appears to have been largely overlooked. One reason for this may be that the preponderant focus of Number Theory specialists is on arithmetic methods, and another (as a leading expert declared) may be that those experts are largely unfamiliar with the area studied by Karlin.

The motivation behind this article is analgous to some old work on the Goldbach conjecture. In 1930 Schnirelmann proved (according to Wikipedia) that there is an integer $C < 800000$ such that <u>every</u> even number > 3 may be expressed as the sum of no more than C prime numbers. This is a step forward from the previous situation, in which there might have been even numbers that could not be written as the sum of primes, no matter how many of them are used. Nowadays it is thought by many that the lowest possible value is $C = 2$, but that has remained unproved for over 250 years.

1.2. **Important terms and concepts.** To explain the basis of our current work, we need to define a few terms and concepts, mostly found in Karlin's book. Where possible, we provide references to the text that follows.

- The function $\Phi(u)$ and its normalized double moments $\{\beta_n\}$. (2.1), (2.2).
- The family of determinants $D(n,r), \quad n = 0, 1, \ldots; \; r = 1, 2....$ (2.4)

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The use of the Sharcnet consortium computing resources has been very helpful.

- The positivity of these determinants is a necessary and sufficient condition for the validity of the RH. (2.5)
- The cumulants $\Psi_m(u),\ m = 0, 1, 2, \ldots$ of $\Phi(u)$, given by

$$\Psi_0(u) = \Phi(u); \quad \Psi_m(u) = \int_u^\infty dw \Psi_{m-1}(w), \ m = 1, 2, \ldots, \tag{1.1}$$

so that the derivative

$$\Psi_m^{(1)}(u) = -\Psi_{m-1}(u), \ m = 1, 2, \ldots \quad (3.18) \tag{1.2}$$

- The two-way Wronskian $W(r, m; u)$ corresponding to $\Psi_m(u)$ is

$$W(r, m; u) = \left| \Psi_m^{(i+j-2)}(u) \right|_{i,j=1}^r, \ u \geq 0. \quad (3.27)$$

- The cumulant $\Psi_m(u)$ is sign-regular (SR) of type RR_r provided that

$$\epsilon_p W(p, m; u) > 0, \ u \geq 0, \ p = 1, \ \ldots, r; \quad \epsilon(p) = (-1)^{p(p-1)/2} (3.28) \tag{1.3}$$

- If $\Psi_m(u)$ is SR of type RR_r, then so is $\Psi_{\overline{m}}(u)$ if $\overline{m} \geq m$, and there exists $m(r) \leq m$, the lowest value of m where SR applies for order r. Sec. 3.3.2
- If $r_2 > r_1$, then, if they exist, $m(r_2) \geq m(r_1)$.
- If $\Psi_m(u)$ is SR of type RR_r, the material in [4] leads to two different methods of proving that certain subsets of the determinants $D(n, r)$ are positive. [6], Documents Sec. 2.1.7 - 2.1.12 on http://publish.uwo.ca/ jnuttall
- Conrey and Ghosh [1] have described a set of eight functions that have properties in several ways analogous to those of $\Phi(u)$ for the RH. J. B. Conrey (private communication) pointed out that the above ideas might also be profitably applied to these functions. Sec. 2.2.

1.3. **Cumulant conjecture.** Before too much effort is expended on an investigation of the RH (and its CG analogs) based on the above information, it seems sensible to prove the analog of the Goldbach conjecture in the form below.

In [6, p. 11] the author proposed a conjecture on the application of cumulants (see Sec. 3.3.1) to the analytic study of the Riemann Hypothesis, and it is natural to extend this conjecture to the CG functions. For the first CG function, which we call $\Theta_1(u)$ (see Sec. 2.2), suppose that the cumulant function $\Psi_m(u)$ is given by (1.1), (1.2), with $\Phi(u)$ replaced by $\Theta_1(u)$. Then the conjecture reads

Conjecture For any order $r > 1$, there exists $M(r)$ such that the function $\Psi_{M(r)}(u)$ is sign-regular of type RR_r.

Sign-regularity is determined via (1.3).

This paper shows that the conjecture is correct. It is very likely that the conjecture holds for the other seven CG functions, and likely that the method could be extended to apply to the RH.

2. BACKGROUND INFORMATION

2.1. **The function $\Phi(u)$ and the Riemann Hypothesis.** Many years ago Pólya [8] suggested that an important function in the analytic study of the Riemann Hypothesis (RH) might be $\Phi(u)$, given in the notation of [2], by

$$\Phi(u) = \sum_{m=1}^{\infty} \left[2m^4\pi^2 e^{9u} - 3m^2\pi e^{5u}\right] \exp\left(-m^2\pi e^{4u}\right). \tag{2.1}$$

One method of attempting to implement this suggestion is through the following framework. Define the normalized double moments of $\Phi(u)$ by

$$\beta_n = \frac{1}{\Gamma(2n+1)} \int_0^{\infty} du \Phi(u) u^{2n}, \ \ n = 0, 1, \ldots. \tag{2.2}$$

and form a semi-infinite matrix B,

$$B_{i,j} = \begin{cases} \beta_{j-i}, & j \geq i; \\ 0, & j < i; \end{cases} \ \ i, j = 0,\ 1,\ 2,\ \ldots. \tag{2.3}$$

Then, if we define minors $D(n, r)$ of order r by

$$D(n, r) = \det[B_{i,j+n}]_{i,j=1,\ldots,r}, \tag{2.4}$$

the RH is true if and only if the following inequalities hold.

$$D(n, r) > 0, \quad n = 0, 1, \ldots; \ \ r = 1, 2.... \tag{2.5}$$

2.2. **The Conrey/Ghosh functions $\{\Theta_k(u)\}$.** In 1994 Conrey and Ghosh [1] (which we denote by CG) conjectured that there is a set of eight functions having properties analogous to the function $\Phi(u)$ of (2.1) . The eight CG cases are obtained from the Riemann case by replacing $\Phi(u)$ with $\Theta_k(u)$ given by

$$\Theta_k(u) = [\Theta_{24}(u)]^{k/24}, \ \ k = 1, 2, 3, 4, 6, 8, 12, 24, \tag{2.6}$$

where, in the notation of CG [1, p. 410],

$$F_\tau(u) = \Theta_{24}(u) = \left[\exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right] \prod_{m=1}^{\infty} (1 - e^{-2\pi m e^u})\right]^{24}. \tag{2.7}$$

Define the coefficients $\{P(j, k)\}$ by

$$\prod_{m=1}^{\infty} (1 - y^m)^k = \sum_{j=0}^{\infty} P(j, k) y^j, \ \ k = 1, 2, 3, 4, 6, 8, 12, 24, \tag{2.8}$$

and substitute (2.8) in (2.7). The result is a series for $\Theta_k(u)$ that is analgous in form to (2.1)for $\Phi(u)$.

The two simplest cases of the $P(j, k)$ are

- $k = 1$ The rules for finding the coefficients $\{P(j, 1)\}$ are
  - $P(0, 1) = 1$.
  - For each $n = 1, 2, \ldots$ let $p(n, 1) = \frac{3n^2-n}{2}$ and $p(n, -1) = p(n, 1) + n$. Then $P(p(n, 1), 1) = (-1)^n$ and $P(p(n, -1), 1) = (-1)^n$
  - For all other values of j, $P(j, 1) = 0$.
  - The quantities $p(n, 1)$ are the pentagonal numbers of Euler.
- $k = 3$ The rules for finding the coefficients $\{P(j, 3)\}$ are
  - For each $n = 0, 1, 2, \ldots$, then $P(n(n + 1)/2, 3) = (-1)^n(2n + 1)$
  - For all other values of j, $P(j, 3) = 0$.

If CG are correct, then replacing $\Phi(u)$ in (2.2) by one of the eight functions $\Theta_k(u)$, and following the procedure of (2.3) and (2.4), will lead to a set of determinants that satisfies (2.5). Thus we wish to study a set of nine presumably analogous functions

$$\{\Phi(u),\ \Theta_k(u), k = 1, 2, 3, 4, 6, 8, 12, 24\}, u \geq 0. \tag{2.9}$$

2.3. **Logarithmic concavity.** In the course of proving that the Turán inequalities apply to the RH and its analogs, Csordas, Norfolk and Varga [2], [3], and respectively CG [1, p. 410], show that the function $\Phi(u)$ and its analogs are strictly logarithmically concave. This is equivalent to the statements that

$$\frac{d^2 \log \Phi(u)}{du^2} < 0,\ \ u \geq 0, \tag{2.10}$$

and

$$\frac{d^2 \log \Theta_k(u)}{du^2} < 0,\ \ u \geq 0,\ \ k = 1, 2, 3, 4, 6, 8, 12, 24. \tag{2.11}$$

Using these results, techniques described in [2, pp. 523-528] may be adapted to show that, in all nine cases, the determinantal inequalities (2.5) hold for $r = 2$ and $n \geq 1$. This is easily extended to include $n = 0$, since $\Phi(u)$ and its analogs are positive for $u \geq 0$. Note that these inequalities are not the Turan inequalities, but a weaker version of them.

At one point in their proof Csordas, Norfolk and Varga [2, p. 525] refer to the 1968 book of Karlin [4] to substantiate [2, p. 528, (2.11)], what Karlin calls the 'basic composition formula'. The Karlin book contains a store of valuable information, which not only leads directly to a proof of the $r = 2$ inequalities (2.5), but also embeds the log concavity property in a more general framework that potentially includes all higher values of the order r.

In the next section we use some of the ideas in Karlin's book [4] to produce this framework in all nine cases. In Sec. 5 we prove the validity of our conjecture about the nine functions for the particular case of $\Theta_1(u)$.

3. SIGN-REGULARITY AND THE NINE FUNCTIONS

3.1. **Sign-regularity for order 2.**

3.1.1. *Two-way Wronskian.* The first observation is that the log concavity condition of (2.10) may be rewritten in terms of the two-way Wronskian

$$\det \begin{bmatrix} \Phi(u) & \Phi^{(1)}(u) \\ \Phi^{(1)}(u) & \Phi^{(2)}(u) \end{bmatrix} < 0,\ \ u \geq 0, \tag{3.1}$$

where the superscript indicates differentiation with respect to u. As Karlin [4, p. 12; p. 55, Thm. 2.6] shows, relation (3.1) and the condition that $\Phi(u) > 0,\ \ u \geq 0$, mean that the kernel $K(u, v) = \Phi(u + v)$, $u, v \geq 0$, is sign-regular of type RR_2. This statement is equivalent to

$$K_{[2]}(\underline{u}, \underline{v}) = \begin{vmatrix} K(u_1, v_1) & K(u_1, v_2) \\ K(u_2, v_1) & K(u_2, v_2) \end{vmatrix} < 0, \tag{3.2}$$

where

$$\underline{u} = (u_1, u_2), \quad 0 \leq u_1 < u_2, \tag{3.3}$$

and similarly for $\underline{v}$. Karlin calls the first expression in (3.2) a 'compound kernel of order 2'.

3.1.2. *Definitions and an important formula.* Following Karlin [4, p. 140] we define the function $\lambda(s)$, $s > 0$, as

$$\lambda(s) = \frac{1}{\Gamma(s)} \int_0^\infty du \Phi(u) u^{s-1}, \quad s > 0. \tag{3.4}$$

There is a corresponding kernel

$$\Lambda(s,t) = \lambda(s+t), \quad s,t > 0. \tag{3.5}$$

A third type of kernel is

$$\phi(u,s) = \frac{u^{s-1}}{\Gamma(s)}, \quad u > 0;\ s > 0. \tag{3.6}$$

A useful formula described by Karlin [4, p. 130, (5.15)] is

(3.7)

$$\Lambda(s,t) = \int_0^\infty du\, \phi(u, s+t)\Phi(u) \quad = \int_0^\infty du \int_0^\infty dv\, \phi(u,s) K(u,v) \phi(v,t), \quad s,t > 0.$$

3.1.3. *Compound kernels.* Now suppose that

$$\underline{u} = (u_1, u_2), \quad 0 \le u_1 < u_2, \quad d\underline{u} = du_1 du_2 \tag{3.8}$$

and similarly for $\underline{v}$. Also define

$$\underline{s} = (s_1, s_2), \quad 0 \le s_1 < s_2, \tag{3.9}$$

and similarly for $\underline{t}$. Then the compound kernel $\phi_{[2]}(\underline{u}, \underline{s})$ may be written as

$$\phi_{[2]}(\underline{u}, \underline{s}) = \frac{{u_1}^{s_1-1}{u_2}^{s_1-1}}{\Gamma(s_1)\Gamma(s_2)} \begin{vmatrix} 1 & {u_1}^{s_2-s_1} \\ 1 & {u_2}^{s_2-s_1} \end{vmatrix} \tag{3.10}$$

Also $\Lambda_{[2]}(\underline{s}, \underline{t})$ may be written as

$$\Lambda_{[2]}(\underline{s}, \underline{t}) = \begin{vmatrix} \lambda(s_1+t_1) & \lambda(s_1+t_2) \\ \lambda(s_2+t_1) & \lambda(s_2+t_2) \end{vmatrix}. \tag{3.11}$$

Finally, we twice use the Basic Composition Formula (BCF) [4, p. 17] on (3.7) to relate the three types of compound kernel discussed above, so that

$$\Lambda_{[2]}(\underline{s}, \underline{t}) = \int_0^\infty d\underline{u} \int_0^\infty d\underline{v}\, \phi_{[2]}(\underline{u}, \underline{s}) K_{[2]}(\underline{u}, \underline{v}) \phi_{[2]}(\underline{v}, \underline{t}). \tag{3.12}$$

3.1.4. *The determinants $D(n,2)$.* From (3.6) it is clear that $\phi_{[2]}(\underline{u}, \underline{s})$ and $\phi_{[2]}(\underline{v}, \underline{t})$ are always positive, and (3.2) shows that $K_{[2]}(\underline{x}, \underline{y}) < 0$, so that (3.12) proves that $\Lambda_{[2]}(\underline{x}, \underline{y}) < 0$ for all allowed values of $\underline{s}, \underline{t}$.

From (2.2) and (3.4) we have the formula

$$\beta_n = \lambda(2n+1), \tag{3.13}$$

so that, setting $s_1 = t_1 = n - 1/2$; $s_2 = t_2 = n + 1/2$, we find from (2.4) that

$$D(n,2) = -\begin{vmatrix} \lambda(2n-1) & \lambda(2n+1) \\ \lambda(2n+1) & \lambda(2n+3) \end{vmatrix} = -\Lambda_{[2]}(\underline{s}, \underline{t}) > 0, \quad n = 1, 2, \ldots. \tag{3.14}$$

as required by (2.5). The sole exception occurs for $n = 0$, but it is obvious that $D(0,2) > 0$.

3.1.5. *Conclusion.* For all nine cases, starting with the proven log concavity condition, we have used information from various parts of the Karlin book [4] to prove that all the $r = 2$ determinants of (2.5) are positive.

3.2. **Extension to order $r > 2$.** Suppose that we have a kernel of the form $K(u, v) = g(u + v)$, with corresponding two-way Wronskian

$$w(r; u) = \left| g^{(i+j-2)}(u) \right|_{i,j=1}^{r}. \tag{3.15}$$

It is said [4, p. 12; p. 55, Thm. 2.6] that the kernel $K(u, v) = g(u+v)$ is sign-regular of type RR_r if

$$\epsilon_p w(p, t) > 0, \;\; t \geq 0, \;\; p = 1, 2, \ldots, r. \tag{3.16}$$

Here the quantity $\epsilon(p) = (-1)^{p(p-1)/2}$.

Provided that a kernel corresponding to the function $g(u) = \Phi(u)$, or its analogs $\Theta_k(u)$, is sign-regular of type RR_r, $r > 2$, all the Karlin steps of Sec. 3.1 could be extended in an obvious manner. The result would be a proof that the appropriate $D(n, r) > 0, \;\; n \geq r - 1$. Next we discuss what is known or conjectured about the sign-regularity of the nine functions.

3.2.1. *Case of function $\Phi(u)$.*

- For the case of the RH function $\Phi(u)$, we have proved [5] that the kernel $K(u, v) = \Phi(u+v)$ is sign-regular of type RR_3. Consequently, in that case, $D(n, 3) > 0$, $n \geq 2$. Reliable numerical calculations show that $D(n, 3) > 0$, $n = 0, 1$.
- Again for the RH function only, the evidence is overwhelming that the above kernel is sign-regular of type RR_4, so that $D(n, 4) > 0$, $n \geq 3$, and again the gaps can be filled in numerically.
- For the RH function only, it is certain that that the above kernel is not sign-regular of type RR_5, which means, from the definition of sign-regularity (3.15), that it is not sign-regular of type RR_r, $r > 5$.

3.2.2. *Case of the eight functions $\Theta_k(u)$.* Numerical computations indicate that the kernels corresponding to $\Theta_k(u)$ have the following sign-regularity properties.

- k=1, 2, 3, 4 - Not sign-regular of order 3.
- k=6, 8, 12 - RR_3 but not RR_4.
- k= 24 - RR_4 but not RR_5.

3.3. **Cumulants.** Based on the concept of sign-regularity of the nine functions of (2.9), it appears that little or no progress beyond $r = 2$ can be made towards the goal of proving the $D(n, r)$ positivity relations (2.5). However, in 2011 the conjecture [6] of Sec.1.3 was made that promises, at least in part, to overcome this problem. The idea is to make use of kernels that are cumulants [1] of the kernels relating to the nine functions, (e.g. of $K(u, v) = \Phi(u + v)$) as described above. Below we outline the Karlin concepts behind the method, and then we report on calculations that support it.

[1]There appears to be some ambiguity about the meaning of 'Cumulant'. In Statistics the term means something other than Karlin's definition. Throughout we use Karlin's version.

3.3.1. *Cumulant definition.* Suppose we define the kernel $Q(u,v)$ by

$$Q(u,v) = \begin{cases} 1 & u < v \\ 0 & u \geq v \end{cases} . \tag{3.17}$$

Then the cumulant [4, p. 102, (1.17)] of the kernel $K(u,v)$ is the kernel $K_1(u,v)$, where

$$K_1(u,v) = \int_0^\infty dw Q(u,w) K(w,v) = \int_u^\infty dw K(w,v). \tag{3.18}$$

Alternatively, if $K(u,v) = \Phi(u+v))$, define

$$\Psi_1(u) = \int_u^\infty dw \Phi(w), \tag{3.19}$$

which we call the cumulant of $\Phi(u)$. Then we have

$$K_1(u,v) = \Psi_1(u+v). \tag{3.20}$$

The cumulant operation may be repeated indefinitely, leading to

$$\Psi_m(u) = \int_u^\infty dw \Psi_{m-1}(w), \;\; m = 2,3,\ldots, \tag{3.21}$$

and

$$K_m(u,v) = \Psi_m(u+v), \;\; m = 2,3,\ldots, \tag{3.22}$$

for the mth cumulant kernel of $K(u,v)$.

3.3.2. *Cumulant conjecture.* As Karlin [4, p. 102] remarks, if $K(u,v)$ is sign-regular of type RR_r, then $K_1(u,v)$ is sign-regular of type RR_ρ for some $\rho \geq r$. Since $Q(u,v)$ is TP [4, p. 46], this follows by applying the BCF to (3.18). Similarly, the order r of the type of sign-regularity of $K_m(u,v)$ never decreases as m increases. There are two possibilities:

(1) As m increases without limit, there is an upper limit to ρ, where the type of sign-regularity of $K_m(u,v)$ is RR_ρ.
(2) There is no upper limit to ρ as m increases indefinitely.

For simplicity in the following we restrict attention to one of the nine cases (2.9), and define the kernel

$$K(u,v) = \Theta_1(u+v), \;\; u,v \geq 0. \tag{3.23}$$

We then have

$$\Psi_1(u) = \int_u^\infty dw \Theta_1(w), \tag{3.24}$$

and

$$\Psi_m(u) = \int_u^\infty dw \Psi_{m-1}(w), \;\; m = 2,3,\ldots, \tag{3.25}$$

with

$$K_m(u,v) = \Psi_m(u+v), \;\; m = 1,2,\ldots. \tag{3.26}$$

We favor the second alternative, and propose, as in Sec. 1.3,

Conjecture For any order $r > 1$, there exists M such that the kernel $K_M(u,v)$ is sign-regular of type RR_r.

This condition is expressed in terms of $W(r,m;u)$, the two-way Wronskian for the case under discussion, given by, as in the Introduction,

$$W(r,m;u) = \left|\Psi_m^{(i+j-2)}(u)\right|_{i,j=1}^{r}, \tag{3.27}$$

where, say, $m > 2r$.

As in (3.16), the kernel $K_m(u,v) = \Psi_m(u+v)$ is sign-regular of type RR_r provided that

$$\epsilon_p W(p,m;t) > 0, \;\; t \geq 0, \;\; p = 1,2,\ldots,r. \tag{3.28}$$

The next two sections develop some ideas, which lead to a proof of the conjecture.

4. SOME RESULTS ABOUT WRONSKIANS AND CUMULANTS

4.1. **The basic Wronskian for $\Theta_1(u)$.** From (2.6), (2.7) and (2.8) we may write $\Theta_1(u)$ as

$$\Theta_1(u) = \exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right]\left\{1 + \sum_{n=1}^{\infty}(-1)^n[y^{(3n^2-n)/2} + y^{(3n^2+n)/2}]\right\}, \;\; u \geq 0, \tag{4.1}$$

where $y = \exp(-2\pi e^u)$.

The series in (4.1) converges rapidly, the more so for larger u. It is useful to obtain an explicit form for the two-way Wronskian $w(r,u)$, $r = 1,2,\ldots$ of the first term of (4.1),

$$w(r,u) = \left|a_1^{(i+j-2)}(u)\right|_{i,j=1}^{r}, \tag{4.2}$$

where

$$a_1(u) = \exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right]. \tag{4.3}$$

Karlin [4, p. 99, Thm. 1.1. (a)] states correctly that it is trivial to see that the factor $e^{(u/4)}$ in (4.3) does not affect the type of sign-regularity associated with $a_1(u)$. The same applies to the Wronskian (4.2), so that, for Sec. 4.1, we replace $a_1(u)$ in (4.3) by $a(u) = \exp(-\beta e^u)$, where $\beta = \pi/12$.

Lemma 4.1. *The basic two-way Wronskian $w(r,u)$ is given by*

$$w(r,u) = c_r z^{\sigma_r} e^{-rz}, \;\; u \geq 0, \tag{4.4}$$

where $\sigma_r = [r(r-1)]/2$ and $z = \beta e^u$. The coefficients $\{c_r\}$ are given by

$$c_{r+1} = \frac{-2rc_r^2}{c_{r-1}}, \;\; r = 2,3,\ldots, \tag{4.5}$$

where

$$c_1 = 1, \;\; c_2 = -1. \tag{4.6}$$

Proof. The proof proceeds by induction on r, starting with the relations

$$\begin{aligned} w(1,u) &= e^{-z} \\ w(2,u) &= -e^{2z}z, \end{aligned} \tag{4.7}$$

so that $c_1 = 1, \;\; c_2 = -1, \;\; \sigma_1 = 0, \;\; \sigma_2 = 1.$

J. B. Conrey (private communication) has kindly pointed out a useful general relation between two-way Wronskians for different orders r, which is

$$w(r-1,u)w(r+1,u) = w(r,u)w^{(2)}(r,u) - [w^{(1)}(r,u)]^2, \;\; r = 1,2,\ldots. \tag{4.8}$$

Given $w(r-1,u), w(r,u)$ we can solve (4.8) for $w(r+1,u)$ so that

$$w(r+1,u) = \left[w(r,u)w^{(2)}(r,u) - [w^{(1)}(r,u)]^2\right]/w(r-1,u) \;\; r = 1,2,\ldots \tag{4.9}$$

The relation (4.8) is a special case of Karlin [4, p. 60, (4.2)] with the parameter m chosen to be 2.

Now suppose that $w(r-1,u), w(r,u)$ are given by formula (4.4). Using the fact that

$$\frac{dz}{du} = z, \tag{4.10}$$

we find that

$$\begin{aligned} w(1,u)^{(1)} &= e^{-rz}z^{\sigma_r}c_r[\sigma_r - rz] \\ w(2,u)^{(2)} &= e^{-rz}z^{\sigma_r}c_r[{\sigma_r}^2 - 2rz(\sigma_r+1) + r^2z^2]. \end{aligned} \tag{4.11}$$

Substituting (4.11) in (4.9) leads to

$$w(r+1,u) = \frac{-2rc_r^2e^{-(r+1)z}z^{2\sigma_r-\sigma_{r-1}+1}}{c_{r-1}}, \tag{4.12}$$

which coincides with the expression arising from (4.4) if r is replaced by $r+1$. □

4.1.1. *The sign of* c_r. From (4.5) we have $c_{r-1}c_{r+1} = -2rc_r^2 < 0$, so that c_{r+1} and c_{r-1} have opposite signs. It follows from (4.6) that

$$c_r/|c_r| = \epsilon(r) = (-1)^{r(r-1)/2}, \;\; r = 1,2,\ldots. \tag{4.13}$$

4.2. **Representations of the Wronskian of Cumulants.**

4.2.1. *A formula for the cumulant.* Starting from (3.24) and iterating (3.25), we obtain an expression for the cumulant $\Psi_m(u)$ as a multiple integral. Reversing the order of the integrals produces the formula

$$\Psi_m(u) = \frac{1}{\Gamma(m+1)}\int_u^\infty dv(v-u)^m\Theta_1(v) = \frac{1}{\Gamma(m+1)}\int_0^\infty dv v^m\Theta_1(v+u). \tag{4.14}$$

This result may also be proved by differentiating with respect to u, to obtain

$$\Psi_m^{(1)}(u) = -\Psi_{m-1}(u), \tag{4.15}$$

which leads immediately to a proof of (4.14).

Formula (4.14) appears prominantly in the Karlin book [4, p. 193], but not, apparently, in the present context.

4.2.2. *An expression for the Wronskian of cumulants.* A useful representation of $W(r,m;u)$ in (3.27) is provided by

Lemma 4.2. *The two-way Wronskian $W(r,m;u)$ associated with $\Theta_1(u)$ is given by*

$$W(r,m;u) = \int_0^\infty d\underline{v} G(m) V(\underline{v}) T(\underline{v}) \tag{4.16}$$

where

$$V(\underline{v}) = \begin{vmatrix} 1 & v_1 & \dots & v_1^{r-1} \\ 1 & v_2 & \dots & v_2^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & v_r & \dots & v_r^{r-1} \end{vmatrix}, \tag{4.17}$$

and

$$T(\underline{v}) = \begin{vmatrix} \Theta_1(v_1+u) & \Theta_1^{(1)}(v_1+u) & \dots & \Theta_1^{(r-1)}(v_1+u) \\ \Theta_1(v_2+u) & \Theta_1^{(1)}(v_2+u) & \dots & \Theta_1^{(r-1)}(v_2+u) \\ \vdots & \vdots & & \vdots \\ \Theta_1(v_{r-1}+u) & \Theta_1^{(1)}(v_{r-1}+u) & \dots & \Theta_1^{(r-1)}(v_{r-1}+u) \end{vmatrix} \tag{4.18}$$

with

$$G(m) = \frac{(v_1 v_2 \dots v_r)^{m-r+1}}{\Gamma(r+1)\prod_{i=1}^r \Gamma(m-i+2)} \tag{4.19}$$

and

$$\underline{v} = (v_1, v_2, \dots, v_r), \quad 0 \le v_j, \ j = 1, \dots, r, \quad d\underline{v} = dv_1 dv_2, \dots, dv_r. \tag{4.20}$$

Proof. The method of proof follows the development in Karlin [4, p. 200 - 201], where Formula (4.24) is deduced from Formula (4.23), but Karlin had no need to include the factor $G(m)$. Also, for convenience, we have removed the restriction $v_1 < v_2 < \dots < v_r$, which accounts for the divisor $\Gamma(r+1)$ in $G(m)$.

We begin with (3.27). Repeatedly using (4.15) gives

$$W(r,m;u) = |\Psi_{m-i-j+2}(u)|_{i,j=1}^r, \tag{4.21}$$

after multiplying even rows and columns by -1, which does not change the value of $W(r,m;u)$.

Substituting using (4.14) leads to

$$W(r,m;u) = \left| \frac{1}{\Gamma(m-i-j+3)} \int_0^\infty dv v^{m-i-j+2} \Theta_1(v+u) \right|_{i,j=1}^r. \tag{4.22}$$

We now use the fact that $\Theta_1(v+u)$, and its derivatives with respect to v, tend to zero at an exponential rate for $v \to \infty$. Each element (i,j) of (4.22) is integrated by parts $j-1$ times (with the Θ_1 term differentiated), so that

$$W(r,m;u) = \left| \frac{(-1)^j}{\Gamma(m-i+2)} \int_0^\infty dv v^{m-i+1} {\Theta_1}^{(j-1)}(v+u) \right|_{i,j=1}^r. \tag{4.23}$$

Because

$$\prod_{j=1}^r (-1)^j = \epsilon(r), \tag{4.24}$$

(4.23) may be rewritten as

$$W(r,m;u) = \epsilon(r) \left| \frac{1}{\Gamma(m-i+2)} \int_0^\infty dv v^{m-i+1} {\Theta_1}^{(j-1)}(v+u) \right|_{i,j=1}^r . \tag{4.25}$$

Applying the basic composition formula [4, p. 98, (1.2)], we arrive at an expression analogous to [4, p. 201, (4.24)]. Finally, we invert the order of the rows in the determinant containing the coordinates $\{v_i\}$, which multiplies the result by $\epsilon(r)$, and then transpose rows and columns. Since $[\epsilon(r)]^2 = 1$, we end up with (4.16). □

4.3. **Representation of the Basic Wronskian of Cumulants.** In the formula (4.16) we now replace $\Theta_1(u)$ by $a_1(u)$ of (4.3), the first term in the expansion for $\Theta_1(u)$. This leads to

Lemma 4.3. *If in (4.16) the function $\Theta_1(u)$ is replaced by $a_1(u)$ of (4.3), i.e.*

$$a_1(u) = \exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right], \tag{4.26}$$

then the two-way Wronskian $\overline{W}(r,m;u)$ associated with $a_1(u)$ is given by

$$\overline{W}(r,m;u) = \int_0^\infty d\underline{v} \overline{G}(m) V(\underline{v}) \overline{T}(\underline{v}), \tag{4.27}$$

with $V(\underline{v})$ as in (4.17),

$$\overline{T}(\underline{v}) = \begin{vmatrix} 1 & y_1 & \dots & {y_1}^{r-1} \\ 1 & y_2 & \dots & {y_2}^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_r & \dots & {y_r}^{r-1} \end{vmatrix}, \tag{4.28}$$

and

$$\overline{G}(m) = \frac{\epsilon(r) \prod_{i=1}^r \left(v_i^{m-r+1} \exp[\gamma(v_i+u) - y_i]\right)}{\Gamma(r+1) \prod_{i=1}^r \Gamma(m-i+2)}. \tag{4.29}$$

Here

$$\gamma = 1/4, \quad Q(u) = \frac{\pi e^u}{12}, \quad \textit{and} \quad y_i = Q(u) e^{v_i}, \quad i = 1, 2, \dots, r. \tag{4.30}$$

Proof. The replacement of $\Theta_1(v+u)$ is

$$a_1(v+u) = e^{\gamma u} e^{\gamma v} \exp[-y], \tag{4.31}$$

so that

$$\frac{d}{dv} a_1(v+u) = (-y + 1/4) a_1(v+u). \tag{4.32}$$

Similarly it follows that

$$\frac{d^j}{dv^j} a_1(v+u) = (-y^j + P_{j-1}(y)) a_1(v+u), \tag{4.33}$$

where $P_j(y)$ is a polynomial in y of degree j, with $P_0(y) = 0$. Thus the replacement for the determinant (4.18) becomes

$$\left| [(-y_i)^{j-1} + P_{j-1}(y_i)] a_1(v_i+u) \right|_{i,j=1}^r . \tag{4.34}$$

Each row i has a common factor $a_1(v_i+u)$, which is removed and included in the numerator of $\overline{G}(m)$. Similarly, we remove a factor $(-1)^{(j-1)}$ from each column of (4.34), and consequently add a factor $\epsilon(r)$ to the numerator of $\overline{G}(m)$. The remaining determinant

$$\left|(y_i)^{j-1} + (-1)^{(j-1)} P_{j-1}(y_i)\right|_{i,j=1}^{r} \tag{4.35}$$

is such that column j is a linear combination of columns $\underline{Y_k}$, $k = 1, \ldots, j$, where the elements of column $\underline{Y_k}$ are $(y_i)^k$, $i = 1, \ldots, r$. This verifies the expression for $\overline{T}(\underline{v})$ in (4.18), and the lemma is proved. □

5. SIGN-REGULARITY OF WRONSKIANS RELATED TO $\Theta_1(u)$

5.1. **Bounds on $\Theta_1(u)$ and its component.** Using (4.1), with (4.3) for $a_1(u)$, we have

$$\Theta_1(u) = a_1(u) + \Phi_1(u), \tag{5.1}$$

where

$$\Phi_1(u) = a_1(u) \left\{ \sum_{n=1}^{\infty} (-1)^n [y^{(3n^2-n)/2} + y^{(3n^2+n)/2}] \right\}, \ \ u \geq 0, \tag{5.2}$$

with

$$y = \exp(-2\pi e^u) = K^{-1}(u). \tag{5.3}$$

Lemma 5.1. *An upper bound on $|\Phi_1(u)|$ is*

$$|\Phi_1(u)| < 1.01\, a_1(u) y, \ \ u \geq 0. \tag{5.4}$$

Proof. In a manner analogous to [3, Lemma 3.1, p 185] we see that

$$(3n-1)\pi e^u \geq \log(K(u)), u \geq 0; \ \ n = 1, 2, \ldots. \tag{5.5}$$

Thus

$$y^{(3n^2-n)/2} \leq K^{-n}(u) \tag{5.6}$$

so that

$$\left\{ \sum_{n=1}^{\infty} [y^{(3n^2-n)/2}] \right\} < \left\{ \sum_{n=1}^{\infty} K^{-n}(u) \right\} = \frac{1}{K(u)-1}, \ \ u \geq 0. \tag{5.7}$$

The second term in (5.2) may be treated in a similar fashion, leading to

$$|\Phi_1(u)| < a_1(u) \left[\frac{1}{K(u)-1} + \frac{1}{K^2(u)-1} \right], \tag{5.8}$$

The result immediately gives rise to the upper bound on $|\Phi_1(u)|$.

□

5.2. **Bounds on cumulants of components.** Next we construct bounds on cumulants obtained by substituting $a_1(u)$ for $\Theta_1(u)$ into the last formula of (4.14) leading to

$$\Omega(u,m) = \frac{1}{\Gamma(m+1)} \int_0^\infty dv v^m a_1(v+u). \tag{5.9}$$

Define

$$\gamma = 0.25; \quad Q(u) = \frac{\pi e^u}{12}; \quad \mu(u,m) = m/Q(u); \quad \alpha = m(1+\nu). \tag{5.10}$$

and let $\nu = \nu(\mu)$ (or $\nu(u,m) = \nu(\mu(u,m))$) be the unique positive number that satisfies

$$\nu e^\nu = \mu \tag{5.11}$$

Then we have

Lemma 5.2. *An upper bound on $\Omega(u,m)$ is $I(m)$, which satisfies*

$$I(m) \sim \frac{1}{\Gamma(m+1)} \left(\frac{2\pi}{\alpha}\right)^{1/2} \exp[\gamma\nu + (m+1)\log(\nu) - m/\nu], \quad m \to \infty. \tag{5.12}$$

Proof. Using the expression for $a_1(u)$ given in (4.3), we see that $\Omega(u,m)$ may be written as

$$I(m) = \frac{1}{\Gamma(m+1)} \int_0^\infty dv \exp[m\log v - Qe^v]e^{\gamma v}, \ \ u \geq 0, \tag{5.13}$$

Now we apply Laplace's method as described by Olver [7, Sec. 7.3, p.81]. Write the integral in (5.13) as

$$\int_0^\infty dv \exp[m\log v - Qe^v]e^{\gamma v} = \int_0^\infty dv \exp(g(v))f(v), \tag{5.14}$$

where

$$g(v) = [m\log v - Qe^v]; \quad f(v) = e^{\gamma v}, \ \ v \geq 0. \tag{5.15}$$

We see that

$$g^{(1)}(v) = [m/v - Qe^v], \ \ v \geq 0. \tag{5.16}$$

The function $g^{(1)}(v)$ has a unique zero at $v = \nu$.

Following Olver [7, Sec. 7.3, p.81] we change variable in the integral (5.14) from v to $z = v/\nu$. Using the new variable, the original integral takes the form

$$\int_0^\infty dz \exp(p(z))q(z), \tag{5.17}$$

where

$$p(z) = [m\log(\nu z) - Qe^{\nu z}]; \quad q(z) = \nu e^{Q\mu\log\nu} e^{\gamma\nu z}. \tag{5.18}$$

In this case

$$\frac{dp(z)}{dz} = [m/z - Q(u)\nu e^{\nu z}]; \quad \frac{d^2p(z)}{dz^2} = -[m/z^2 + Q(u)\nu^2 e^{\nu z}], \ \ v \geq 0, \tag{5.19}$$

and the zero of $p(z)$ occurs at $z = 1$, irrespective of the values of m and $Q(u)$.

We are concerned with the asymptotic behavior of $I(m)$ as $m \to \infty$, and Olver's method is designed for a structure such as that of (5.17). At $z = 1$, where $p(z)$

reaches its peak, $\frac{d^2p(z)}{dz^2}$ is large negative for large m, no matter what are the values of $Q(u)$, ν. Using Olver's analysis, it may be proved that

$$I(m) \sim \exp[p(1)]q(1)\int_{-\infty}^{\infty} dz \exp[-(\alpha/2)(z-1)^2], \quad m \to \infty, \tag{5.20}$$

where the integral in (5.20) is

$$\left(\frac{2\pi}{\alpha}\right)^{1/2}. \tag{5.21}$$

□

Likewise, set

$$\Omega_1(u,m) = \frac{1}{\Gamma(m+1)}\int_0^{\infty} dv v^m \Phi_1(v+u), \tag{5.22}$$

$$Q_1(u) = \frac{25\pi e^u}{12}; \quad \mu_1(u,m) = m/Q_1(u); \quad \alpha_1 = m(1+\nu_1), \tag{5.23}$$

and let $\nu_1 = \nu_1(\mu) = \nu(\mu_1)$ be the unique positive number that satisfies

$$\nu_1 e^{\nu_1} = \mu_1. \tag{5.24}$$

Then we have

Lemma 5.3. *An upper bound on $\Omega_1(u,m)$ is $I_1(m)$, which satisfies*

$$I_1(m) \sim \frac{1.01}{\Gamma(m+1)}\left(\frac{2\pi}{\alpha_1}\right)^{1/2} \exp[\gamma\nu_1 + (m+1)\log(\nu_1) - m/\nu_1], \quad m \to \infty, \tag{5.25}$$

where

$$\alpha_1 = m + \nu_1, \tag{5.26}$$

Proof. Substitute (5.4) in place of Φ_1 in (5.22), and proceed as in Lemma 4.2. □

5.3. **Lower bound on the basic Wronskian $\overline{W}(r,m;u)$.** In Sec. 4.1 we derived a simple formula giving $w(r,u)$, the Wronskian of $a_1(u)$, which showed that the function (and the corresponding kernel) is sign-regular of type RR_r, $r = 1,2,\ldots$. It therefore follows from the statement in Sec. 3.3.2 that any cumulant of $a_1(u)$ is also sign-regular of type RR_r.

The result of Lemma 4.3. provides an explicit formula (4.27) for the Wronskian of the cumulant of order m of the function $a_1(u)$, called $\overline{W}(r,m;u)$, the basic Wronskian. Using that formula we have

Lemma 5.4. *A lower bound $I_0(m)$ on $\epsilon(r)\overline{W}(r,m;u)$, the basic Wronskian, satisfies*

$$I_0(m) \sim \frac{C\left[\nu^{m+1}e^{-m/\nu}e^{\nu/4}\right]^r}{(2\alpha)^{r(r-1)/2}}, \quad m \to \infty, \tag{5.27}$$

where

$$C = \frac{Q(u)^{r(r-1)/2}e^{r\gamma u}}{\Gamma(r+1)\Gamma(m+1)^r}. \tag{5.28}$$

Here

$$\mu = m/Q(u), \quad \gamma = 1/4, \quad Q(u) = \frac{\pi e^u}{12}. \tag{5.29}$$

Proof. From (4.27) we have

$$\overline{W}(r,m;u) = \int_0^\infty d\underline{v}\, \overline{G}(m) V(\underline{v}) \overline{T}(\underline{v}), \tag{5.30}$$

with $V(\underline{v})$ as in (4.17), $\overline{T}(\underline{v})$ as in (4.28), so that, with $y_i = Q(u)e^{v_i},\ i = 1, 2, \ldots, r$,

$$V(\underline{v})\overline{T}(\underline{v}) = \prod_{i>j=1}^{r} (v_i - v_j)(y_i - y_j) \geq Q(u)^{r(r-1)/2} V(\underline{v})^2, \tag{5.31}$$

since

$$(v_i - v_j)(e^{v_i} - e^{v_j}) \geq (v_i - v_j)^2. \tag{5.32}$$

Using a notation based on that of Sec. 5.2, the function $\overline{G}(m)$ may be written as

$$\overline{G}(m) = \frac{\epsilon(r)}{\Gamma(r+1)} \left\{ \prod_{i=1}^{r} \frac{e^{\gamma u}}{\Gamma(m-i+2)} \right\} \left\{ \prod_{i=1}^{r} \exp(g(v_i)) f(v_i)) \right\} \tag{5.33}$$

where

$$g(v) = Q[\mu \log v - e^v]; \quad f(v) = v^{1-r} e^{\gamma v},\ \ v \geq 0. \tag{5.34}$$

Thus

$$\epsilon(r)\overline{W}(r,m;u) \geq C \left\{ \int_0^\infty d\underline{v} \left[\prod_{i=1}^{r} \exp(g(v_i)) f(v_i) \right] V(\underline{v})^2 \right\} \tag{5.35}$$

where

$$C = \frac{Q(u)^{r(r-1)/2} e^{r\gamma u}}{\Gamma(r+1)\Gamma(m+1)^r}. \tag{5.36}$$

To apply the Olver technique used in Lemma 5.2 to this case, it is convenient to change variables by setting

$$v_i = \nu(t_i + 1),\ \ i = 1, 2, \ldots, r. \tag{5.37}$$

The integral in (5.35) becomes

$$I_V = \nu^{r^2} \int_{-1}^{\infty} d\underline{t} \left[\prod_{i=1}^{r} \exp(g(\nu(t_i+1))) f(\nu(t_i+1)) \right] V(\underline{t})^2 \tag{5.38}$$

The quantity $V(\underline{t})^2$ is a homogeneous polynomial of degree $r(r-1)$ that we write as

$$V(\underline{t})^2 = \sum_{j_1 j_2 \ldots j_r = 1}^{r(r-1)/2} D(\underline{j}) t_1^{2j_1} t_2^{2j_2} \ldots t_r^{2j_r}, \tag{5.39}$$

with the coefficients $D(\underline{j}) = 0$ unless $j_1 + j_2 + \ldots + j_r = r(r-1)/2$. Note that we have used the observation that application of the Laplace method ensures that only even powers of t_i contribute to the leading terms.

Therefore (5.38) is equivalent to

$$I_V = \nu^{r^2} \sum_{j_1 j_2 \ldots j_r = 1}^{r(r-1)/2} D(\underline{j}) \left\{ \prod_{i=1}^{r} \int_{-1}^{\infty} dt_i t_i^{2j_i} \left[\exp(g(\nu(t_i+1))) f(\nu(t_i+1)) \right] \right\}. \tag{5.40}$$

Each integral in (5.40) is of the form

$$I_S = \nu^m \nu^{1-r} e^{\gamma\nu} \int_{-1}^{\infty} dt\ \left[\exp(m \log(t+1)) - Q e^{\nu(t+1)} \right] t^{2j} (t+1)^{1-r} e^{\gamma\nu t} \tag{5.41}$$

or

$$I_S = \nu^m \nu^{1-r} e^{\gamma\nu} \exp(-Qe^\nu) \int_{-1}^{\infty} dt \, \exp\left[-\alpha t^2/2 + O(t^3)\right] t^{2j}(1+O(t)), \tag{5.42}$$

where

$$\alpha(u,m) = m(1+\nu(u,m)). \tag{5.43}$$

Now we apply the Laplace method of Lemma 5.2 to (5.42), so that, for large m, we conclude that

$$I_S = \nu^m \nu^{1-r} e^{\gamma\nu} \exp(-Qe^\nu) \int_{-1}^{\infty} dt \, \exp\left[-\alpha t^2/2 + O(t^3)\right] t^{2j}(1+O(t)), \tag{5.44}$$

which means that, asymptotically for large m,

$$I_S \sim \nu^{m+1-r} e^{\gamma\nu} \exp(-Qe^\nu) \frac{1.3.\ldots(2j-1)}{(2\alpha)^j} \left(\frac{\pi}{\alpha}\right)^{1/2}. \tag{5.45}$$

Substituting the above relations into (5.35), we see that, asymptotically for large m,

$$\overline{W}(r,m;u) \geq \frac{\left[\nu^{m+1} e^{-m/\nu} e^{\gamma\nu}\right]^r}{(2\alpha)^{r(r-1)/2}} \left(\frac{\pi}{\alpha}\right)^{r/2} \sum_{j_1 j_2 \ldots j_r = 1}^{r(r-1)/2} D(\underline{j}) \left\{ \prod_{i=1}^{r} 1.3.\ldots(2j_i - 1) \right\}. \tag{5.46}$$

Now the quantities $D(\underline{j})$ are all integers, some possibly negative, but the expression

$$I_D = \sum_{j_1 j_2 \ldots j_r = 1}^{r(r-1)/2} D(\underline{j}) \left\{ \prod_{i=1}^{r} 1.3.\ldots(2j_i - 1) \right\} \tag{5.47}$$

appears as a factor in the formula for

$$J = \int_{-\infty}^{\infty} d\underline{t} \, \exp[-\underline{t}^2/2\,)[V(\underline{t}]^2\,, \tag{5.48}$$

which we know is positive. Thus I_D is a positive integer and therefore $I_D \geq 1$

Consequently we find that, asymptotically for large m,

$$\overline{W}(r,m;u) \geq C \frac{\left[\nu^{m+1} e^{-m/\nu} e^{\gamma\nu}\right]^r}{(2\alpha)^{r(r-1)/2}}. \tag{5.49}$$

as claimed above.

□

5.4. **Properties of the function ν.** To complete the analysis we need

Lemma 5.5. *With given $\rho > 1$, define $\nu(\mu)$, $\nu_1(\mu)$ by*

$$\nu(\mu) e^{\nu(\mu)} = \mu; \quad \nu_1(\mu) e^{\nu_1(\mu)} = \mu/\rho. \tag{5.50}$$

We prove the following relations.

(1) $\frac{\nu(\mu)}{\nu_1(\mu)} \to 1+, \ \mu \to \infty.$
(2) $\nu(\mu) - \nu_1(\mu) \to \log(\rho), \ \mu \to \infty.$
(3) $\nu(\rho^k \mu) \sim k \log(\rho) + \nu(\mu), \ \mu, \ k \to \infty.$

Proof. A simple, but key, observation is that $v(\mu_2) > v(\mu_1)$ if $\mu_2 > \mu_1$, so that, for instance,

$$\nu(\mu) > \nu_1(\mu). \tag{5.51}$$

Taking the logarithms of the relations in (5.50) and subtracting leads to

$$\frac{\nu(\mu)}{\nu_1(\mu)} - 1 = \frac{\log(\rho)}{\nu_1(\mu)} - \frac{1}{\nu_1(\mu)} \log\left[\frac{\nu(\mu)}{\nu_1(\mu)}\right]. \tag{5.52}$$

On account of (5.51) the LHS of (5.52) is positive, which implies that

$$\frac{\log(\rho)}{\nu_1(\mu)} > \frac{1}{\nu_1(\mu)} \log\left[\frac{\nu(\mu)}{\nu_1(\mu)}\right], \tag{5.53}$$

As $\mu \to \infty$ the LHS of (5.53) $\to 0+$, which means that

$$\frac{1}{\nu_1(\mu)} \log\left[\frac{\nu(\mu)}{\nu_1(\mu)}\right] \to 0+, \;\; \mu \to \infty. \tag{5.54}$$

From (5.52) we may now deduce Relation 1 above, i.e. that

$$\frac{\nu(\mu)}{\nu_1(\mu)} \to 1+, \;\; \mu \to \infty. \tag{5.55}$$

Another version of (5.52) is

$$\nu(\mu) - \nu_1(\mu) = \log(\rho) - \log\left[\frac{\nu(\mu)}{\nu_1(\mu)}\right], \tag{5.56}$$

from which Relation 2 follows, that, with the help of (5.55),

$$\nu(\mu) - \nu_1(\mu) \to \log(\rho), \;\; \mu \to \infty. \tag{5.57}$$

Applying (5.57) several times we confirm Relation 3 that

$$\nu(\rho^k \mu) \sim k \log(\rho) + \nu(\mu), \;\; \mu, \; k \to \infty. \tag{5.58}$$

□

5.5. **Sign-regularity of the Wronskian for $\Theta_1(u)$.** We are now in a position to verify the following theorem - a proof of the Conjecture in Sec. 1.3.

Theorem 5.6. *For any given value of order $r > 1$, there exists a sufficiently large value of m (say M) such that the two-way Wronskian W(p,m;u), based on the function $\Theta_1(u)$, satisfies*

$$\epsilon_p W(p, M; u) > 0, \;\; u \geq 0, \;\; p = 1, 2, \ldots, r. \tag{5.59}$$

This means that the kernel $K_M(u, v)$ is RR_r as the Conjecture proposes.

Proof. We choose an arbitrary value of $r > 1$ and show that $\exists$ M such that

$$\epsilon_r W(r, M; u) > 0, \;\; u \geq 0. \tag{5.60}$$

5.5.1. *Structure of* $W(r,m;u)$. To begin we note that, from (3.27), an element of $W(r,M;u)$ is given by

$$\Psi_m^{(i+j-2)}(u) = (-1)^{i+j-2}\Psi_{m-i-j+2}(u), \tag{5.61}$$

so that, reversing the sign of even rows and columns,

$$W(r,m;u) = |\Psi_{m-i-j+2}(u)|_{i,j=1}^r\,. \tag{5.62}$$

Using (5.9) and (5.22) we have

$$W(r,m;u) = |\Omega(u,m-i-j+2) + \Omega_1(u,m-i-j+2)|_{i,j=1}^r\,, \tag{5.63}$$

and also, for the basic Wronskian,

$$\overline{W(r,m;u)} = |\Omega(u,m-i-j+2)|_{i,j=1}^r\,. \tag{5.64}$$

Each column of the determinant (5.63) is the sum of two columns based on a_1 and Φ_1. The determinant may be written as the sum of 2^r determinants, each consisting of single columns of one or other of the two types. The choice of all the single columns of type a_1 leads to $\overline{W(r,m;u)}$, and we have shown that $\epsilon(r)\overline{W(r,m;u)} > 0,\ u \geq 0$.

As m becomes larger it is found that type a_1 terms are increasingly greater than the corresponding type Φ_1 terms. Using the bounds described in Lemmas 5.2 - 5.4, we show below that, for all u, $\epsilon(r)\overline{W(r,m;u)}$ is greater than the sum of the other 2^r-1 single-column determinants, provided that a large enough value of m is chosen. This means that, as $m \to \infty$,

$$\epsilon(r)W(r,m;u) > 0,\ \ u \geq 0. \tag{5.65}$$

5.5.2. *Upper bound on remainder* $R = W(r,m;u) - \overline{W(r,m;u)}$. Each of the 2^r-1 determinants in the remainder has its elements chosen from

$$\{\Omega(u,m-k),\ \Omega_1(u,m-k),\ \ k=0,\ldots,2r-2\}. \tag{5.66}$$

Such a determinant is the sum of $r!$ 'components', where a component is $\pm\prod_{i=1}^r \omega_i$, with elements ω_i being chosen appropriately from (5.66).

We divide the remainder determinants into sets characterized by the number n of elements in a component that are of the form $\Omega_1(u,m-k)$ for some value of k, i.e. elements of type Φ. The number of determinants corresponding to the value n is $C_n^r,\ \ n=1,\ldots,r$. (The determinant $\overline{W(r,m;u)}$ corresponds to $n=0$.)

The formula (5.12) provides an asymptotic approximation for upper bounds to $\Omega(u,m)$ for large m. To extend this relation to the case where m is replaced by $m-k$, we replace (5.15) by

$$g(v) = [m\log v - Qe^v]; \quad f(v) = v^{-k}e^{\gamma v},\ \ v \geq 0, \tag{5.67}$$

and apply the Laplace method as before. For large m the two methods are asymptotically the same.

This leads to the approximation

(5.68)

$$I(m-k) \sim \frac{1}{\Gamma(m-k+1)}\left(\frac{2\pi}{m(1+\nu)}\right)^{1/2}\exp[\gamma\nu+(m-k+1)\log(\nu)-m/\nu], m\to\infty.$$

where $\nu = \nu(\mu)$, with μ defined by $\mu = m/Q(u)$. Similarly (5.25) is replaced by (5.69)

$$I_1(m-k) \sim \frac{1.01}{\Gamma(m-k+1)} \left(\frac{2\pi}{(m(1+\nu_1)} \right)^{1/2} \exp[\gamma\nu_1 + (m-k+1)\log(\nu_1) - m/\nu_1], m \to \infty,$$

where $\nu_1 = \nu_1(\mu)$, with μ defined by $\mu = m/Q_1(u)$.

Using (5.68) and (5.69), we find that an asymptotic upper bound to the absolute value of the component $|\prod_{i=1}^{r} \omega_i|$, containing n elements of Type Φ, is

$$\begin{aligned} R_n \sim \quad & (1.01(2\pi)^{1/2})^r (m^{1/2}\Gamma(m-2r+3))^{-r} \\ & \times \left\{ (1+\nu)^{-1/2} \exp[\gamma\nu + (m+1)\log(\nu) - m/\nu] \right\}^{r-n} \\ & \times \left\{ (1+\nu_1)^{-1/2} \exp[\gamma\nu_1 + (m+1)\log(\nu_1) - m/\nu_1] \right\}^{n}, \end{aligned} \tag{5.70}$$

where we have used the inequality $0 \le k \le 2r-2$.

5.5.3. *Sign-regularity of the Wronskian W(r,m;u).* An important quantity is S_n, the ratio of R_n to I_0, the lower bound of the absolute value of the basic Wronskian (5.27), where

$$\begin{aligned} S_n \sim \quad & [1.01(2\pi)^{1/2}]^r \, 2^{r(r-1)/2} \, \Gamma(r+1) \, e^{-ru/4} \, Q(u)^{-r(r-1)/2} \\ & \times \exp[mn(\nu^{-1} - \nu_1^{-1}] \; [\nu_1/\nu]^{n(m+1)} \\ & \times m^{r(r-2)/2} \, \Gamma(m+1)^r \, \Gamma(m-2r+3)^{-r} \\ & \times (1+\nu)^{(r^2-2r+n)/2} \, (1+\nu_1)^{-n/2} \\ & \times \exp[(\nu_1 - \nu)n/4]. \end{aligned} \tag{5.71}$$

The crucial observation is that, as $m \to \infty$, $S_n \to 0$.

- In the first line of we may replace u by 0 since $u \ge 0$. Thus the first line is constant, since we are assuming a given r.
- For the remaining lines, the results of Lemma 5.5 are helpful. In the second line, we see that both functions are < 1 for all $u \ge 0$. The logarithm of the first function may be written as

$$mn(\nu^{-1} - \nu_1^{-1}) = mn\frac{(\nu_1 - \nu)}{\nu\nu_1}. \tag{5.72}$$

  If $\mu = m/Q(u)$ is large, then so will be ν, ν_1, and $\nu - \nu_1 \sim \log(25)$. Also ν, ν_1 are approximately linear in $\log(\mu)$, so that the function in (5.73) will be proportional to $\mu(\log(\mu))^2$ for large μ. The behavior for small μ and large m is even more favorable to our aim.
- It is not difficult to see that all the other terms in (5.71) are much smaller (or at least in the case of $[\nu_1/\nu]^{n(m+1)}$ supportive) than the term (5.73).

The theorem is proved by the observation that the remainder bound R has the form

$$R \le \sum_{n=1}^{r} r!/[n!(r-n)!]R_n, \tag{5.73}$$

since the above remarks show that R/I_0 may be made as small as desired by choosing a large enough value for m.

□

References

1. J. B. Conrey and A. Ghosh, *Turán inequalities and zeros of Dirichlet series associated with certain cusp forms*, Trans. Amer. Math. Soc. **342** (1994), 407–419.
2. G. Csordas, T. S. Norfolk, and R. S. Varga, *The Riemann hypothesis and the Turán inequalities*, Trans. Amer. Math. Soc. **296** (1986), 521–541.
3. G. Csordas and R. S. Varga, *Moment inequalities and the Riemann hypothesis*, Constr. Approx. **4** (1988), 175–198.
4. S. Karlin, *Total Positivity*, Stanford University Press, Stanford, CA, 1968.
5. J. Nuttall, *Wronskians, cumulants and the Riemann hypothesis.*
6. ———, *Determinantal approach to a proof of the Riemann hypothesis*, arXiv:math/1111.1128, 2011.
7. F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.
8. G. Pólya, *On the zeros of certain trigonometric integrals*, J. London Math. Soc. **1** (1926), 98–99.

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