WRONSKIANS, CUMULANTS, AND THE RIEMANN HYPOTHESIS

JOHN NUTTALL

Abstract. This paper proposes a new, analytic approach to the resolution of the Riemann Hypothesis. The method has its origins in the 1986 results and techniques of Csordas, Norfolk and Varga (CNV) in their proof of the Turán inequalities. Here, the mathematical structure of their work has been significantly extended and generalized. We make frequent use of the ideas on the sign-regularity of kernels found in the book of Karlin. The notion of cumulants plays an important role. The final step is to prove that a doubly infinite set of determinants are all positive.

We present a conjecture, supported by computations, about the sign-regularity of a set of cumulants of the function called $\Phi(t)$ by CNV. To illustrate the ideas, the conjecture is proved for an early member of the set. We describe a new method, superior to the Karlin method used by CNV, for proving positivity of the determinants, but some cases remain to be treated.

1. INTRODUCTION

1.1. The functions $F(z)$ and $\Phi(t)$. [1]

About 25 years ago Csordas, Norfolk and Varga [1], while studying the Turán inequalities, analyzed the coefficients in the Taylor expansion of the Riemann ξ -function. In the notation of [1] (slightly modified) this is written

$$F(z)=\sum_{n=0}^{\infty}\beta_n z^n, \tag{1.1}$$

where the series coefficients are

$$\beta_n=\frac{1}{\Gamma\left(2n+1\right)}\int_0^{\infty}dt\Phi\left(t\right)t^{2n}, \tag{1.2}$$

with the entire function $\Phi\left(t\right)$ given by

$$\Phi\left(t\right)=\sum_{m=1}^{\infty}\left[2m^4\pi^2e^{9t}-3m^2\pi e^{5t}\right]\exp\left(-m^2\pi e^{4t}\right). \tag{1.3}$$

The Riemann hypothesis (RH) is equivalent to the statement that all the zeros of the entire function $F(z)$ are real and negative. The aim of this article is to describe the framework of a new, analytic approach to investigating the truth or otherwise of the RH.

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1.2. **The determinants $D(n,r)$ and the RH.** Karlin [3, Theorem 5.3, p. 412; p. 393; Theorem 9.1, p. 85] tells us that, one way to ensure that the zeros of $F(z)$ have this property for any such function $\Phi(t)$, is to require that the coefficients $(\beta_n, n = 0, 1, \ldots)$ of the series $F(z)$ satisfy the conditions (1.6) below.

Form a semi-infinite matrix B,

$$B_{i,j} = \begin{cases} \beta_{j-i}, & j \geq i; \\ 0, & j < i; \end{cases} \quad i, j = 0,\, 1,\, 2,\, \ldots. \tag{1.4}$$

Then, if we define minors $D(n,r)$ of order r by

$$D(n,r) = \det[B_{i,j+n}]_{i,j=1,\ldots,r}, \tag{1.5}$$

the RH is equivalent to the condition that

$$D(n,r) > 0, \quad n = 0, 1, \ldots; \;\; r = 1, 2.... \tag{1.6}$$

In this case the matrix B is said to be totally positive or TP (see [3]). The ultimate objective is to prove (1.6), and so demonstrate the truth of the RH.

1.3. **The case $r = 2$.** In the process of deriving some stronger results on the Turán inequalities (of no relevance to our approach), Csordas et al. [1] (see also Csordas and Varga [2] and Matiyasevich [8]) effectively proved that

$$D(n,2) = \begin{vmatrix} \beta_n & \beta_{n+1} \\ \beta_{n-1} & \beta_n \end{vmatrix} > 0, \quad n = 1, 2, \ldots. \tag{1.7}$$

Since

$$D(0,2) = \begin{vmatrix} \beta_0 & \beta_1 \\ 0 & \beta_0 \end{vmatrix} > 0,$$

it follows that the condition (1.6) is satisfied for $r = 2, \;\; n \geq 0$.

The main technique used by Csordas et al. [1] was originally introduced by Karlin, Proschan and Barlow [4] and later described by Karlin [3] . In a simplified version applicable to $r = 2$ we define the kernel $K(x,y)$, $0 \leq x, y < \infty$, by

$$K(x,y) = \Phi(x+y). \tag{1.8}$$

The corresponding compound kernel of order 2 is given by

$$K_{[2]}\left(\underline{x}, \underline{y}\right) = \det \begin{bmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{bmatrix},$$

where

$$\underline{x} = (x_1, x_2), \quad 0 \leq x_1 < x_2, \tag{1.9}$$

and similarly for $\underline{y}$.

In [2] (see (2.2) for details) it is shown that $\log[\Phi(x)]$ is strictly concave on $(0, \infty)$, which means that the two-way Wronskian of $\Phi(x)$

$$\det \begin{bmatrix} \Phi(x) & \Phi^{(1)}(x) \\ \Phi^{(1)}(x) & \Phi^{(2)}(x) \end{bmatrix} < 0, \;\; x \geq 0. \tag{1.10}$$

It is shown in [1] that, consequently, $K_{[2]}\left(\underline{x}, \underline{y}\right) < 0$, where (1.9) holds. See Lemma 2.1 below for a more general proof of this result.

In the terminology of Karlin [3], we say that $K(x,y)$ is sign-reverse regular of order 2, i.e. RR_2. As explained by Csordas et al. [1, p. 526], the RR_2 condition leads to a proof of (1.7).

1.4. **The case** $r = 3$**.** It is natural to ask whether the above techniques can be generalized to prove (1.6) for values of order $r > 2$. Our initial response is to show that the corresponding compound kernel of order 3, namely

$$K_{[3]}\left(\underline{x}, \underline{y}\right) = \det \begin{bmatrix} K(x_1, y_1) & K(x_1, y_2) & K(x_1, y_3) \\ K(x_2, y_1) & K(x_2, y_2) & K(x_2, y_3) \\ K(x_3, y_1) & K(x_3, y_2) & K(x_3, y_3) \end{bmatrix} \tag{1.11}$$

is RR_3. This is what is done below in Sec. 3. The key tool is derived from a theorem of [3] (see Lemma 2.1). In the spirit of [1] and [2], this proof involves a number of elementary estimates.

As we describe in Sec. 2, the method of [4] then shows, just as it did for $r = 2$, that the determinantal inequalities (1.6) also hold for $r = 3$, $n = 0, 1, \ldots$, bearing in mind Remark 2.4.

1.5. **Extension of the procedure to** $r > 3$**.** We see that the procedure used for the proof of the inequalities (1.6) for $r = 2, 3$ consists of two parts.

(1) Find a compound kernel based on $\Phi(t)$ that is RR_r.
(2) Show, with the help of identities described by Karlin in [3], that the inequalities (1.6) for the chosen value of r hold, except for $n < n_r$, where n_r is a positive integer, small for small r.

Since the relations taken from Karlin [3] are defined for all $r > 0$, we now enquire whether the method can be further extended to $r = 4$ and beyond. Such an extension is essential if we are to have any chance of solving the RH this way.

Numerical calculations of the Wronskian $W_4(t)$ for a range of values of t strongly suggest that the compound kernel corresponding to $\Phi(t)$ is RR_4. However they show that the equivalent function for $r = 5$ is definitely not RR_5. Note that, owing to the requirement on p in Lemma 2.1, the compound kernel of $\Phi(t)$ is therefore not RR_r for any $r \geq 5$.

We have proposed a method using cumulants of $\Phi(t)$ that overcomes the above difficulty, possibly for all higher values of r. In Sec. 4 we describe the mathematical relations that comprise the method. A key part is a conjecture about the sign-regularity of the cumulants of $\Phi(t)$, and we report numerical computations that support this conjecture. It must be emphasized that, even if the conjecture were to be proved, a gap in the procedure would remain to be closed (see Sec. 4.6) before the RH could be verified (or possibly rejected, if the data required for the conjecture is unsatisfactory) .

2. SIGN-REGULARITY AND POSITIVE $D(n, r)$ DETERMINANTS

2.1. **Sign-regularity.** We first review the general definition of sign-reverse regularity when applied to the kernel $K(x, y)$ of (1.8), as stated by [3, p. 12]. The set X is the non-negative real line. The open simplex $\Delta_p(X)$ is

$$\Delta_r(X) = \{\underline{x} = (x_1, x_2, \ldots, x_p) \,|x_1 < x_2 < \ldots < x_p : x_i \in X\}.$$

The compound kernel $K_{[p]}\left(\underline{x}, \underline{y}\right)$ is defined by

$$K_{[p]}\left(\underline{x}, \underline{y}\right) = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \ldots & K(x_1, y_p) \\ K(x_2, y_1) & K(x_2, y_2) & \ldots & K(x_2, y_p) \\ \vdots & \vdots & & \vdots \\ K(x_p, y_1) & K(x_p, y_2) & \ldots & K(x_p, y_p) \end{vmatrix},$$

where

$$\underline{x} = (x_1, x_2, \ldots, x_p) \in X; \quad \underline{y} = (y_1, y_2, \ldots, y_p) \in X.$$

With $\epsilon_p = (-1)^{p(p-1)/2}$, we say that $K(x,y)$ is RR_r if $\epsilon_p K_{[p]}\left(\underline{x}, \underline{y}\right)$ is a non-negative function on $\Delta_p(X) \times \Delta_p(X)$ for each $p = 1, 2, \ldots, r$.

2.2. **Wronskians and sign-regularity.** The following lemma is important in the later development.

Lemma 2.1. *Suppose that $\psi(x)$ is analytic in a neighborhood of $(0,\infty)$, and that the kernel $k(x,y) = \psi(x+y),\ x, y \in (0,\infty)$. Define $w_p(t) = \det\left|\psi^{(i+j-2)}(t)\right|_{i,j=1}^{p}$. If $\epsilon_p w_p(t) > 0,\ t \geq 0,\ \ p = 1, 2, \ldots, r$ then $k(x,y)$ is RR_r.*

Proof. This result is a special case of Theorem 2.6 of [3, p. 55]. The analyticity of $\psi(x,y)$ ensures that the differentiability requirements of the theorem are satisfied. The relation

$$\det\left|\frac{\partial^{i+j-2}}{\partial x^{i-1}\partial y^{j-1}} k(x,y)\right|_{i,j=1}^{p} = \det\left|\psi^{(i+j-2)}(t)\right|_{i,j=1}^{p}, \quad t = x+y,$$

together with [3, (1.3), p. 48] demonstrates that the requirements on the compound kernel appearing in the statement of the theorem hold. □

Now define $W_p(t)$ as

$$W_p(t) = \det\left|\Phi^{(i+j-2)}(t)\right|_{i,j=1}^{p}. \tag{2.1}$$

Theorem 2.2. *The kernel $K(x,y)$ defined by (1.8) is sign-regular of type RR_3.*

Proof. From Lemma 2.1 we must show, for $t \geq 0$, that $W_1(t) > 0$, while for $p = 2, 3$, that $W_p(t) < 0$. Since $\Phi(t) > 0,\ t \geq 0$ the relation $W_1(t) > 0,\ t \geq 0$ is true.

Next, from [9] we have $\Phi^{(1)}(t) \leq 0,\ t \geq 0$, from [2, (3.18)] that $\Phi^{(2)}(0) < 0$, and from [2, p. 197] we have $g(t) > 0,\ t > 0$, where

$$g(t) = -t\left(\Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2\right) + \Phi(t)\Phi^{(1)}(t)$$

It follows that

$$W_2(t) = \Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2 < 0, \quad t \geq 0. \tag{2.2}$$

The proof that $W_3(t) < 0,\ t \geq 0$ is given below in Theorem 3.5. □

2.3. **Some relations of Karlin.** In the remainder of this section, we use the fact that $K(x,y)$ is RR_3 to demonstrate how to prove the determinantal inequalities (1.6) for order $r = 3$, using the techniques of [4] and [1]. We define the function $\lambda(s),\ s > 0$, as

$$\lambda(s) = \frac{1}{\Gamma(s)}\int_0^\infty du\Phi(u)\, u^{s-1}, \ s > 0.$$

There is a corresponding kernel

$$\Lambda(s,t) = \lambda(s+t), \quad s, t > 0. \tag{2.3}$$

A useful third type of kernel is

$$\phi(u,s) = \frac{u^{s-1}}{\Gamma(s)}, \quad u > 0;\ s > 0. \tag{2.4}$$

[1, p. 526], and previously [4], show that

(2.5)
$$\Lambda\,(s,t) = \int_0^\infty du\,\phi(u,s+t)\Phi\,(u) \quad = \int_0^\infty du \int_0^\infty dv\,\phi\,(u,s)\,K\,(u,v)\,\phi\,(v,t)\,, \quad s,t>0.$$

2.4. **Sign-regularity and compound kernels.** The last component of the machinery of the proof is the application of what Karlin [3, p. 17] calls the Basic Composition Formula (BCF). In the present case it may be used to relate the compound kernels corresponding to the kernels appearing in (2.5), so that

$$\Lambda_{[3]}\,(\underline{s},\underline{t}) = \int_0^\infty d\underline{u} \int_0^\infty d\underline{v}\,\phi_{[3]}\,(\underline{u},\underline{s})\,K_{[3]}\,(\underline{u},\underline{v})\,\phi_{[3]}\,(\underline{v},\underline{t})\,. \tag{2.6}$$

In (2.6) we have

$$\underline{u} = (u_1,u_2,u_3)\,,\;\; 0\le u_1<u_2<u_3, \quad d\underline{u} = du_1du_2du_3, \tag{2.7}$$

and similarly for $\underline{v}$. Also we have

$$\underline{s} = (s_1,s_2,s_3)\,,\;\; 0\le s_1<s_2<s_3, \tag{2.8}$$

and similarly for $\underline{t}$.

The compound kernel $\phi_{[3]}\,(\underline{u},\underline{s})$ may be written as

$$\phi_{[3]}\,(\underline{u},\underline{s}) = \frac{{u_1}^{s_1-1}{u_2}^{s_1-1}{u_3}^{s_1-1}}{\Gamma\,(s_1)\Gamma\,(s_2)\Gamma\,(s_3)} \begin{vmatrix} 1 & {u_1}^{s_2-s_1} & {u_1}^{s_3-s_1} \\ 1 & {u_2}^{s_2-s_1} & {u_2}^{s_3-s_1} \\ 1 & {u_3}^{s_2-s_1} & {u_3}^{s_3-s_1} \end{vmatrix}$$

Also $\Lambda_{[3]}\,(\underline{s},\underline{t})$ may be written as

$$\Lambda_{[3]}\,(\underline{s},\underline{t}) = \begin{vmatrix} \lambda\,(s_1+t_1) & \lambda\,(s_1+t_2) & \lambda\,(s_1+t_3) \\ \lambda\,(s_2+t_1) & \lambda\,(s_2+t_2) & \lambda\,(s_2+t_3) \\ \lambda\,(s_3+t_1) & \lambda\,(s_3+t_2) & \lambda\,(s_3+t_3) \end{vmatrix}.$$

Relations (2.7) and (2.8) show that the two ϕ compound kernels in (2.6) are positive for all arguments. Since the compound kernel $K_{[3]}$ is negative for all arguments, (2.6) demonstrates that $\Lambda_{[3]}\,(\underline{s},\underline{t})<0$ for all valid $\underline{s},\underline{t}$.

2.5. **Positivity of $D(n,3)$.** With this in mind we have

Theorem 2.3. *For all integer $n\ge 2$*

$$D(n,3) = \begin{vmatrix} \beta_n & \beta_{n+1} & \beta_{n+2} \\ \beta_{n-1} & \beta_n & \beta_{n+1} \\ \beta_{n-2} & \beta_{n-1} & \beta_n \end{vmatrix} > 0, \tag{2.9}$$

where β_n is defined by (1.2).

Proof. To prove these inequalities we choose

$$s_1=t_1=n-3/2, \quad s_2=t_2=n+1/2, \quad s_3=t_3=n+5/2, \quad n=2,3,\dots,$$

and use the relation (2.6), along with

$$\beta_n = \lambda(2n+1), \tag{2.10}$$

which gives

$$\begin{vmatrix} \beta_{n-2} & \beta_{n-1} & \beta_n \\ \beta_{n-1} & \beta_n & \beta_{n+1} \\ \beta_n & \beta_{n+1} & \beta_{n+2} \end{vmatrix} = \Lambda_{[3]}\,(\underline{s},\underline{t}) < 0. \tag{2.11}$$

The restriction $n \geq 2$ arises from the need to satisfy the condition $s > 0$ in (2.4).

Reversing the order of the rows of (2.11) produces the determinants in (2.9), and also changes the sign of the original determinants, so that (2.9) is verified. □

Remark 2.4. Just as for order $r = 2$ the case $n = 0$ is trivial, but for $n = 1$ the relation

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_0 & \beta_1 & \beta_2 \\ 0 & \beta_0 & \beta_1 \end{vmatrix} > 0,$$

does not follow from the general method. Its validity may be checked by inserting the numerical values of β_j listed by [1, p. 540].

3. PROOF OF SIGN-REGULARITY OF ORDER 3

3.1. **Outline of proof for $r = 3$.** The aim of this section is to prove that

$$W_3(t) < 0, \;\; t \geq 0, \tag{3.1}$$

where $W_3(t)$ is defined by (2.1). Using Csordas and Varga [2, p. 184] we may write

$$\Phi^{(k)}(t) = \pi \exp(5t - y) \left\{ p_{k+1}(y) + 4 \exp(-3y) p_{k+1}(4y) + \Psi_k(t) \right\} \tag{3.2}$$

where

$$\Psi_k(t) = \exp(y) \sum_{n=3}^{\infty} n^2 p_{k+1}(n^2 y) \exp(-n^2 y).$$

We have used $y = \pi e^{4t}$, and the polynomials $p_k(y)$, $k = 0, \ldots, 4$ are defined in [2, p. 184] to be

$$\begin{aligned} p_1(y) &= 2y - 3, \\ p_2(y) &= -8y^2 + 30y - 15, \\ p_3(y) &= 32y^3 - 224y^2 + 330y - 75, \\ p_4(y) &= -128y^4 + 1440y^3 - 4232y^2 + 3270y - 375, \\ p_5(y) &= 512y^5 - 8448y^4 + 41408y^3 - 68096y^2 + 30930y - 1875. \end{aligned} \tag{3.3}$$

The formula (2.1) is equivalent to

$$W_3\,(t) = \begin{vmatrix} \Phi(t) & \Phi^{(1)}(t) & \Phi^{(2)}(t) \\ \Phi^{(1)}(t) & \Phi^{(2)}(t) & \Phi^{(3)}(t) \\ \Phi^{(2)}(t) & \Phi^{(3)}(t) & \Phi^{(4)}(t) \end{vmatrix}. \tag{3.4}$$

We now substitute (3.2) into (3.4). Each column of (3.4) may be thought of as a sum of three columns corresponding to the three terms in (3.2), so that $W_3(t)$ may be written as the sum of $3 \times 3 \times 3 = 27$ determinants formed by choosing one from each of the three columns comprising a column of $W_3(t)$. We omit the factor $\pi \exp(5t - y)$ in each term, since we are concerned only with the sign of $W_3(t)$. The result is

$$\begin{aligned} w_3(t) = F_0(y) &+ e^{-3y} F_{1,1}(y) + e^{-6y} F_{1,2}(y) + e^{-9y} F_{1,3}(y) \\ &+ F_{2,1}(y) + e^{-3y} F_{2,2}(y) + e^{-6y} F_{2,3}(y) \\ &+ F_3(y) + F_4(y), \end{aligned} \tag{3.5}$$

where

$$W_3(t) = [\pi \exp(5t - y)]^3 w_3(t). \tag{3.6}$$

Below we define and discuss the various terms in (3.5).

3.2. **The term $F_0(y)$.** The first term $F_0(y)$ in (3.5) is obtained by choosing the first term $p_{k+1}(y)$ in each element of $W_3(t)$, so that

$$F_0(y) = \begin{vmatrix} p_1(y) & p_2(y) & p_3(y) \\ p_2(y) & p_3(y) & p_4(y) \\ p_3(y) & p_4(y) & p_5(y) \end{vmatrix}.$$

With the help of (3.3) it is found that

$$F_0(y) = 860160y^3 - 737280y^4 + 294912y^5 - 65536y^6.$$

The three zeros of $F_0(y)$, apart from $y = 0$, are

$$1.192509... \pm i2.187155..., \; 2.114980....$$

We have $t \geq 0$, i.e. $y \geq \pi$, so that, since $\pi > 2.11498...$, it follows that

$$F_0(y) < 0, \;\; y \geq \pi.$$

With $a = 1.192509...$, $b = 2.187155...$, $c = 2.114980...$ we may write

$$F_0(y) = -65536y^3[(y-a)^2 + b^2](y-c),$$

so that $-F_0(y)$ increases steadily as y increases from π. Thus the least upper bound of $F_0(y), \;\; y > \pi$, is $F_0(\pi) = -1.7904\ldots \times 10^7$.

3.3. **The term $F_{1,1}(y)$.** The next term in (3.5), i.e. $e^{-3y}F_{1,1}(y)$, corresponds to the choice of two columns of type $p_{k+1}(y)$ and one column of type $4e^{-3y}p_{k+1}(4y)$. We may therefore write $F_{1,1}(y)$ as

$$\begin{aligned} F_{1,1}(y) \quad &= \begin{vmatrix} 4p_1(4y) & p_2(y) & p_3(y) \\ 4p_2(4y) & p_3(y) & p_4(y) \\ 4p_3(4y) & p_4(y) & p_5(y) \end{vmatrix} + \begin{vmatrix} p_1(y) & 4p_2(4y) & p_3(y) \\ p_2(y) & 4p_3(4y) & p_4(y) \\ p_3(y) & 4p_4(4y) & p_5(y) \end{vmatrix} \\ &+ \begin{vmatrix} p_1(y) & p_2(y) & 4p_3(4y) \\ p_2(y) & p_3(y) & 4p_4(4y) \\ p_3(y) & p_4(y) & 4p_5(4y) \end{vmatrix}. \end{aligned}$$

The polynomial $F_{1,1}(y)$ is

$$\begin{aligned} F_{1,1}(y) \quad &= 72253440y^3 - 482181120y^4 + 792281088y^5 - 579403776y^6 \\ &+228261888y^7 - 42467328y^8, \end{aligned}$$

and its non-trivial zeros are

$$1.143552... \pm -i1.630193..., \; 0.214613..., \; 1.182386..., \; 1.690895....$$

Again it follows that

$$F_{1,1}(y) < 0, \;\; y \geq \pi,$$

which shows that an upper bound to $e^{-3y}F_{1,1}(y)$ is zero.

Moreover we find that $e^{-3y}F_{1,1}(y)$ decreases as y increases, so that the least upper bound to $e^{-3y}F_{1,1}(y)$ is $e^{-3y}F_{1,1}(\pi) = -5.8783\ldots \times 10^6$. Thus $e^{-3y}F_{1,1}(y)$ is about 33 percent of $F_0(y)$ at $y = \pi$.

3.4. **The term $F_{1,2}(y)$.** The third term in (3.5), i.e. $e^{-6y}F_{1,2}(y)$, corresponds to the choice of one column of type $p_{k+1}(y)$ and two columns of type $4e^{-3y}p_{k+1}(4y)$. We may therefore write $F_{1,2}(y)$ as

$$F_{1,2}(y) = \begin{vmatrix} 4p_1(4y) & 4p_2(4y) & p_3(y) \\ 4p_2(4y) & 4p_3(4y) & p_4(y) \\ 4p_3(4y) & 4p_4(4y) & p_5(y) \end{vmatrix} + \begin{vmatrix} p_1(y) & 4p_2(4y) & 4p_3(4y) \\ p_2(y) & 4p_3(4y) & 4p_4(4y) \\ p_3(y) & 4p_4(4y) & 4p_5(4y) \end{vmatrix}$$
$$+ \begin{vmatrix} 4p_1(4y) & p_2(y) & 4p_3(4y) \\ 4p_2(4y) & p_3(y) & 4p_4(4y) \\ 4p_34(y) & p_4(y) & 4p_5(4y) \end{vmatrix}.$$

The polynomial $F_{1,2}(y)$ is

$$\begin{aligned} F_{1,2}(y) &= 1156055040y^3 - 6794772480y^4 + 16356999168y^5 \\ &-17334534144y^6 + 9739173888y^7 - 2717908992y^8, \end{aligned}$$

and its non-trivial zeros are

$$0.344204... \pm -i0.205802..., \ 0.842359... \pm -i1.214801..., \ 1.210203....$$

Again it follows that

$$F_{1,2}(y) < 0, \ \ y \geq \pi,$$

which shows that an upper bound to $F_{1,2}(y)$ is zero. Again we find that $e^{-6y}F_{1,2}(y)$ decreases as y increases, so that the least upper bound to $e^{-6y}F_{1,2}(y)$ is $e^{-6y}F_{1,2}(\pi) = -5.6349\ldots \times 10^4$.

3.5. **The term $F_{1,3}(y)$.** The fourth term in (3.5), i.e. $e^{-9y}F_{1,3}(y)$, corresponds to the choice of three columns of type $4e^{-3y}p_{k+1}(4y)$. We may therefore write $F_{1,3}(y)$ as

$$F_{1,3}(y) = \begin{vmatrix} 4p_1(4y) & 4p_2(4y) & 4p_3(4y) \\ 4p_2(4y) & 4p_3(4y) & 4p_4(4y) \\ 4p_3(4y) & 4p_4(4y) & 4p_5(4y) \end{vmatrix}.$$

The polynomial $F_{1,3}(y) = 64F_{1,1}(4y)$ is

$$\begin{aligned} F_{1,3}(y) &= 3523215360y^3 - 12079595520y^4 + 19327352832y^5 \\ &-17179869184y^6, \end{aligned}$$

and its non-trivial zeros are

$$0.298127... \pm -i0.546788..., \ 0.528745....$$

Again it follows that

$$F_{1,3}(y) < 0, \ \ y \geq \pi,$$

which shows that an upper bound to $F_{1,3}(y)$ is zero. Again we find that $e^{-9y}F_{1,3}(y)$ decreases as y increases, so that the least upper bound to $e^{-9y}F_{1,3}(y)$ is $e^{-9y}F_{1,3}(\pi) = -6.1328\ldots$.

3.6. **The term $F_{2,1}(y)$.** So far the calculations have been exact, but for the remaining 5 terms in (3.5) we prove only positive upper bounds, which are found to be all small compared to the absolute value of $F_0(y)$. The fifth term involving $F_{2,1}$ corresponds to the sum of three determinants each containing two columns of type $p_{k+1}(y)$ and one column of type $\Psi_k(t)$. We write $F_{2,1}(y)$ as

$$F_{2,1}(y) = \sum_{k=0}^{4} \Psi_k(t)\rho_k(y), \tag{3.7}$$

where

$$\begin{aligned} \rho_0(y) &= p_3(y)p_5(y) - p_4(y)^2, \\ \rho_1(y) &= -2(p_2(y)p_5(y) - (p_3(y)p_4(y)), \\ \rho_2(y) &= 2p_2(y)p_4(y) + p_1(y)p_5(y) - 3p_3(y)^2, \\ \rho_3(y) &= -2(p_1(y)p_4(y) - (p_2(y)p_3(y)), \\ \rho_4(y) &= p_1(y)p_3(y) - p_2(y)^2. \end{aligned} \tag{3.8}$$

Substituting the definitions (3.3) into (3.8) leads to

$$\begin{aligned} \rho_0(y) &= -486000y + 1867200y^2 - 3808320y^3 + 3118080y^4 \\ &\qquad - 1255424y^5 + 229376y^6 - 16384y^7, \\ \rho_1(y) &= 302400y - 967680y^2 + 1324800y^3 - 752640y^4 \\ &\qquad + 184320y^5 - 16384y^6, \\ \rho_2(y) &= -68640y + 167808y^2 - 151936y^3 + 52224y^4 - 6144y^5, \\ \rho_3(y) &= 6720y - 10752y^2 + 5888y^3 - 1024y^4, \\ \rho_4(y) &= -240y + 192y^2 - 64y^3. \end{aligned} \tag{3.9}$$

Lemma 3.1. *For $k = 0, \ldots, 4$ let $r(k)$ be the absolute value of the coefficient of y^{7-k} in (3.9), i.e. $r(0) = 16384,\ r(1) = 16384,\ r(2) = 6144$, etc. Then*

$$|\rho_k(y)| < r(k)y^{7-k},\ \ y \ge \pi;\ \ k = 0, \ldots, 4. \tag{3.10}$$

Proof. It is found that the non-trivial zero of $\rho_4(y) + 64y^3 = -240y + 192y^2$ is $< \pi$, so that $\rho_4(y) + 64y^3 > 0,\ y \ge \pi$, and $\rho_4(y) > -64y^3,\ y \ge \pi$. Similarly the non-trivial zeros of $\rho_4(y) - 64y^3 = -240y + 192y^2 - 128y^3$ have real parts $< \pi$, so that $\rho_4(y) < 64y^3,\ y \ge \pi$. Thus (3.10) holds for $k = 4$. A similar argument applies to the other values of k. □

We also need the following lemma.

Lemma 3.2. *For $k = 0, \ldots, 4$ we have*

$$|\Psi_k(t)| < 2^{2k+1}3^{2k+4}y^{k+1}e^{-8y}C,\ \ y \ge \pi, \tag{3.11}$$

where $y = \pi e^{4t}$ and $C = [1 - 81e^{-3\pi}]^{-1}$.

Proof. The case of $k = 6$ is proved in [2, p. 188], and the same method applies to the cases in the lemma. □

To bound $F_{2,1}(y)$ we insert the inequalities (3.10) and (3.11) into (3.7), leading to

$$|F_{2,1}(y)| < \left[\sum_{k=0}^{4} 2^{2k+1}3^{2k+4}r(k)\right] Cy^8e^{-8y}, \ \ y \geq \pi.$$

Since the maximum value of y^8e^{-8y} occurs at $y = \pi$, it follows that

$$|F_{2,1}(y)| < \left[\sum_{k=0}^{4} 2^{2k+1}3^{2k+4}r(k)\right] C\pi^8e^{-8\pi}, \ \ y \geq \pi,$$

which means that $|F_{2,1}(y)| < 3.083\ldots\times 10^3, \ \ y \geq \pi$.

3.7. **The term $F_{2,2}(y)$.** The sixth term in (3.5) is $e^{-3y}F_{2,2}$, which corresponds to the sum of six determinants each containing one column of type $p_{k+1}(y)$, one column of type $4p_{k+1}(4y)$ and one column of type $\Psi_k(t)$. We write $F_{2,2}(y)$ as

$$F_{2,2}(y) = \sum_{k=0}^{4} \Psi_k(t)\eta_k(y),$$

where

$$\begin{aligned}
\eta_0(y) &= 4e^{-3y}[p_3(y)p_5(4y) + p_3(4y)p_5(y) - 2p_4(y)p_4(4y)],\\
\eta_1(y) &= 4e^{-3y}[-2p_2(y)p_5(4y) - 2p_2(4y)p_5(y) + 2p_3(y)p_4(4y)\\
&\quad + 2p_3(4y)p_4(y)],\\
\eta_2(y) &= 4e^{-3y}[2p_1(y)p_4(4y) + 2p_1(4y)p_4(y) + p_1(y)p_5(4y)\\
&\quad + p_1(4y)p_5(y) - 6p_3(y)p_3(4y)],\\
\eta_3(y) &= 4e^{-3y}[-2p_2(y)p_5(4y) - 2p_2(4y)p_5(y) + 2p_2(y)p_3(4y)\\
&\quad + 2p_2(4y)p_3(y)],\\
\eta_4(y) &= 4e^{-3y}[p_1(y)p_3(4y) + p_1(4y)p_3(y) - 2p_2(y)p_2(4y)].
\end{aligned} \tag{3.12}$$

Substituting the definitions (3.3) into (3.12) leads to

$$\begin{aligned}
\eta_0(y) &= 4e^{-3y}[-9720000y + 144465600y^2 - 680073600y^3 + 1637053440y^4\\
&\quad - 1951406080y^5 + 1191657472y^6 - 351272960y^7 + 37748736y^8],\\
\eta_1(y) &= 4e^{-3y}[6048000y - 7668864y^2 + 286387200y^3 - 497925120y^4\\
&\quad + 421171200y^5 - 163807232y^6 + 23592960y^7],\\
\eta_2(y) &= 4e^{-3y}[-1372800y + 13881984y^2 - 39699200y^3 + 48193536y^4\\
&\quad - 25466880y^5 + 4866048y^6],\\
\eta_3(y) &= 4e^{-3y}[134400y - 973056y^2 + 1853440y^3 - 1375232y^4 + 368640y^5],\\
\eta_4(y) &= 4e^{-3y}[-4800y + 21696y^2 - 22400y^3 + 9216y^4].
\end{aligned} \tag{3.13}$$

Lemma 3.3. *For $k = 0, \ldots, 4$ let $q(k)$ be the value of the coefficient of y^{8-k} in (3.13), i.e. $r(0) = 37748736$, $r(1) = 23592960$, $r(2) = 4866048$, etc. Then*

$$|\eta_k(y)| < q(k)y^{8-k}, \ \ y \geq \pi; \ \ k = 0, \ldots, 4. \tag{3.14}$$

Proof. The proof uses the same method as that of Lemma 3.1. □

Following the same procedure as in Sec. 3.6, we find that $|e^{-3y}F_{2,2}(y)| < 2.422\ldots\times 10^2$, $y \geq \pi$.

3.8. **The term $F_{2,3}(y)$.** The seventh term $e^{-6y}F_{2,3}(y)$, corresponding to the sum of three determinants each containing two columns of type $4p_{k+1}(4y)$ and one column of type $\Psi_k(t)$, has a structure is similar to that of the fifth term $F_{2,1}(y)$ in Sec. 3.5. We replace $\rho_k(y)$ of (3.8) by $16\rho_k(4y)$, and the procedure of that section leads to

$$|F_{2,3}(y)| < \left[\sum_{k=0}^{4} 2^{19}3^{2k+4}r(k)\right] Cy^8e^{-8y}, \ \ y \geq \pi.$$

As before we find that $|F_{2,3}(y)| < 1.379\ldots\times 10^1, \ \ y \geq \pi$.

3.9. **The term $F_3(y)$.** For the next step we need

Lemma 3.4. *For $k = 0, \ldots, 4$ we have*

$$|p_{k+1}(y)| < 2^{2k+1}y^{k+1}, \ \ y \geq \pi. \tag{3.15}$$

Proof. This lemma is proved in the same manner as Lemma 3.1. □

The eighth term $F_3(y)$ corresponds to three determinants, each containing one column of type $p_{k+1}(y) + 4e^{-3y}p_{k+1}(4y)$ and two columns of type $\Psi_k(t)$. We write

$$\phi_k(y) = p_{k+1}(y) + e^{-3y}4p_{k+1}(4y),$$

so that, using (3.15), we have

$$|\phi_k(y)| \leq |p_{k+1}(y)| + |4e^{-3y}p_{k+1}(4y)| < 2^{2k+1}[1 + 2^{2k+4}e^{-3y}]y^{k+1}. \tag{3.16}$$

To analyze $F_3(y))$ we use the structure of (3.7) and (3.8), with $\Psi_k(t)$ taking the place of $p_{k+1}(y)$ in (3.8), while $\phi_k(y)$ takes the place of $\Psi_k(t)$ in (3.7). Thus we have

$$F_3(y) = \sum_{k=0}^{4} \phi_k(y)(y)\sigma_k(y), \tag{3.17}$$

where

$$\begin{aligned}
\sigma_0(y) &= \Psi_2(t)\Psi_4(y) - \Psi_3(y)^2,\\
\sigma_1(y) &= -2(\Psi_1(y)\Psi_4(y) - (\Psi_2(y)\Psi_3(y)),\\
\sigma_2(y) &= 2\Psi_1(y)\Psi_3(y) + \Psi_0(y)\Psi_4(y) - 3\Psi_2(y)^2,\\
\sigma_3(y) &= -2(\Psi_0(y)\Psi_3(y) - (\Psi_1(y)\Psi_2(y)),\\
\sigma_4(y) &= \Psi_0(y)\Psi_2(y) - \Psi_1(y)^2.
\end{aligned} \tag{3.18}$$

We bound the functions $\sigma_k(y), \ k = 1, \ldots, 4$, by replacing each term on the right hand side of (3.18), and obtain an upper bound on $F_3(y)$ by using the inequalities (3.11) and (3.16) in (3.17). The result is that $|F_3(y)| < 3.497\ldots\times 10^{-2}, \ \ y \geq \pi$.

3.10. **The term $F_4(y)$.** The ninth and last term in (3.5) is

$$F_4\,(t) = \begin{vmatrix} \Psi_0(t) & \Psi_1(t) & \Psi_2(t)\\ \Psi_1(t) & \Psi_2(t) & \Psi_3(t)\\ \Psi_2(t) & \Psi_3(t) & \Psi_4(t) \end{vmatrix}.$$

The determinant may be written as the sum of 6 components, of which two are $\Psi_0(t)\Psi_2(t)\Psi_4(t)$ and $-\Psi_1(t)\Psi_1(t)\Psi_4(t)$. We obtain an upper bound to each component by replacing the products in these expressions by the product of the three bounds from (3.11). It turns out that each upper bound is the same, so that we find that

$$|F_4(y)| < 6 \times 2^{13}3^{16}y^7e^{-24y}C < 2^{14}3^{17}\pi^7e^{-24\pi}C =< 3.036\ldots\times 10^{-12}, \;\; y \geq \pi.$$

3.11. **Theorem 3.5.** In summary we have found that, for $y \geq \pi$, the nine contributions to $w_3(t)$ in (3.5) have upper bounds as follows

$$\begin{array}{clr} 1 & F_0(y) & -1.7904\times 10^7 \\ 2 & e^{-3y}F_{1,1}(y) & -5.8783\times 10^6 \\ 3 & e^{-6y}F_{1,2}(y) & -5.6349\times 10^4 \\ 4 & e^{-9y}F_{1,3}(y) & -6.1328\times 10^0 \\ 5 & F_{2,1}(y) & 3.083\times 10^3 \\ 6 & e^{-3y}F_{2,2}(y) & 2.422\times 10^2 \\ 7 & e^{-6y}F_{2,3}(y) & 1.379\times 10^1 \\ 8 & F_3(y) & 3.497\times 10^{-2} \\ 9 & F_4(y) & 3.036\times 10^{-12}. \end{array} \tag{3.19}$$

We therefore have

Theorem 3.5. *For all $t \geq 0$, the function $W_3(t) < 0$, where $W_3(t)$ is defined in (2.1).*

Proof. For $y \geq \pi$ (i.e. $t \geq 0$) the sum of the upper bounds in (3.19) is negative , so that $w_3(t)$ and, in view of (3.6), also $W_3(t)$, are negative in the same range. □

This theorem achieves the aim of (3.1).

4. APPLICATION OF CUMULANTS FOR ARBITRARY ORDER

4.1. **Modification of the procedure.** In order to study the properties of cumulants, we modify the two parts of the procedure described in Sec. 1.5 as follows.

- In Part 1 replace $\Phi(t)$ by a cumulant function $\Psi_m(t)$ defined below. Here, m will depend on r.
- In Part 2 follow the steps described in Sec. 2.3 -2.5, with $\Phi(t)$ replaced by $\Psi_m(t)$, and with certain other modifications discussed below.

4.2. **Modifications for Part 1.** We introduce the cumulants $\{\Psi_m(u)\}$, where

$$\Psi_m(u) = \int_u^\infty du\Psi_{m-1}(u), \quad m = 1, 2, \ldots \tag{4.1}$$

and

$$\Psi_0(u) = \Phi(u). \tag{4.2}$$

There is a corresponding set of kernels

$$K(u, v; m) = \Psi_m(u + v), \;\; 0 \leq u, v; \quad m = 0, 1, \ldots.$$

In some applications it is helpful to have an explicit form for the cumulant $\Psi_m(u)$, rather than the iterative relations (4.1). Karlin [3] discussed cumulants (although in a different context), and he showed that (see also [5, Sec. 4])

$$\Psi_m(u) = \frac{1}{\Gamma(m)}\int_u^\infty dt\Phi(t)(t-u)^{m-1}, \quad m \geq 1. \tag{4.3}$$

Below we discuss the question of the sign-regularity of $\Psi_m(u)$.

4.3. **Modifications for Part 2.** In Secs. 2.3 - 2.4, as well as replacing $\Phi(t)$ by $\Psi_m(t)$, we substitute $K(u,v;m)$ for $K(u,v)$. We adjust the formulas for the change of order from 3 to a general value r.

As in (2.5) it may be shown that, for appropriate values of m, s, t,

$$\int_0^\infty du \int_0^\infty dv\phi(u,s)\Psi_m(u+v)\phi(v,t) = \int_0^\infty dv\phi(v,s+t)\Psi_m(v).$$

Integrating by parts m times leads to

$$\int_0^\infty dv\phi(v,s+t)\Psi_m(v) = \int_0^\infty dv\phi(v,s+t+m)\Phi(v) = \lambda(s+t+m),$$

where $t, s > 0$ and $m = 0, 1, 2, \ldots$. Thus we obtain

$$\Lambda\left(s+m/2, t+m/2\right) = \int_0^\infty du \int_0^\infty dv\phi\left(u,s\right) K\left(u,v;m\right) \phi\left(v,t\right).$$

As in (2.6), using the compound kernel $K_{[r]}(\underline{u}, \underline{v}; m)$ corresponding to $K(v,u;m)$, we have

$$\Lambda_{[r]}\left(\underline{s}+m\underline{w}/2, \underline{t}+m\underline{w}/2\right) = \int_0^\infty d\underline{u} \int_0^\infty d\underline{v}\, \phi_{[r]}\left(\underline{u},\underline{s}\right) K_{[r]}\left(\underline{u},\underline{v};m\right) \phi_{[r]}\left(\underline{v},\underline{t}\right), \tag{4.4}$$

where $\underline{u}$, $\underline{v}$, $\underline{s}$, $\underline{t}$ are defined as in (2.7) and (2.8), and $\underline{w}$ is a vector with all components equal to unity.

4.4. **The procedure with cumulants.** Now we adapt the procedure, used in Theorem 2.3 for $r = 3, m = 0$, to relate the compound kernel appearing in the LHS of (4.4) to some of the determinants needed in (1.6). In (4.4) we choose $\underline{s}$, $\underline{t}$ such that

$$s_j = t_j = \mu + 2j, \quad j = 1, \ldots, r.$$

The elements of the determinant $\Lambda_{[r]}\left(\underline{s}+m\underline{w}/2, \underline{t}+m\underline{w}/2\right)$ are then

$$\lambda(2\mu + 2i + 2j + m), \quad i, j = 1, \ldots, r. \tag{4.5}$$

If the argument of λ appearing in (4.2) is odd and positive for all entries, then we can use (2.3) to express all elements of $\Lambda_{[r]}\left(\underline{s}+m\underline{w}/2, \underline{t}+m\underline{w}/2\right)$ in terms of the coefficients β_n given by (2.10).

For use in (4.4), the inequality (2.4) requires that $\mu + 2 > 0$. It follows that the determinant with elements given by (4.5) is the same as $D(k,r)$ for some k, with the order of the rows reversed, and an appropriate choice of k. As before, the reversal of the rows changes the determinant by a factor of ϵ_r.

Thus all values of k are possible so long as $k \geq k_L$. It is found that

$$k_L = n_L + r - 1,$$

where $n_L = m/2$, m even: $(m+1)/2$, m odd.

4.5. **Conjecture on sign-regularity of $\Phi(t)$ cumulants.** We have proposed the following conjecture [5, Sec. 4.3].

CONJECTURE 1. For any given order $r > 1$, there is a lowest integer $m(r) < \infty$ such that $K(u,v;m(r))$ is RR_r.

Remark 4.1. Karlin [3, (1.17), p. 102; Remark 5.6, p. 128] shows that, because of the relation (4.1), if $K(u,v;m_1)$ is RR_r, then so is $K(u,v;m_2)$ for all $m_2 > m_1$. Thus the conjecture implies that, if $m \geq m(r)$, then $K(u,v;m)$ is RR_r. It also follows that, if $r_2 > r_1$, then $m\left(r_2\right) \geq m\left(r_1\right)$.

The conclusion is that, given Conjecture 1 and using the results of Sec. 4.4, we can prove the positivity of $D(n,r)$ for all values of n,r such that $n \geq k_L(m(r))$. Here

$$k_L(m) = n_L(m) + r - 1,$$

where $n_L(m) = m/2,\ m$ even: $(m+1)/2,\ m$ odd.

4.6. **Alternative method for Part 2 of the procedure.** Just as for the case $r = 3$, and assuming the truth of Conjecture 1, we have found that the above procedure for a given r will not apply to $D(n,r)$ for a finite, continuous set of values of n that includes $n = 1$. Let us call these cases 'exceptional'. We have discovered an alternative procedure [6][2] for proving the positivity of a set of $D(n,r)$, given Conjecture 1. It is superior to the above approach in the sense that, for most values of r, there are fewer exceptional cases. This method does not involve compound kernels, but uses only Wronskians and other determinants.

We extend the Wronskian definition of (2.1) to read

$$W(r,m;t) = \det \left| \Psi_m^{(i+j-2)}(t) \right|_{i,j=1}^{r}. \tag{4.6}$$

Setting $u = 0$ in (4.3) leads to

$$\Psi_m(0) = \frac{1}{\Gamma(m)} \int_0^\infty dt \Phi(t) t^{m-1} = b_{m-1}, \quad m \geq 1,$$

which serves as a definition of the single moments $b_n,\ n = 0,1,\ldots,$ of $\Phi(t)$, in contrast to the double moment β_n of (1.2). We define the single 'moments' for negative n by

$$b_n = (-1)^{n+1} \Phi^{(-n-1)}(0), \quad n = -1,-2,\ldots.$$

A corresponding set of determinants is

$$\Delta(n,r) = \begin{vmatrix} b_n & b_{n+1} & \ldots & b_{n+r-1} \\ b_{n-1} & b_n & \ldots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-r+1} & b_{n-r+2} & \ldots & b_n \end{vmatrix}.$$

Note that, if $n = 2j,\ j = \ldots,-2,-1,0,1,2,\ldots,$ then $\beta_j = b_n$. This relation also applies for negative j, since the function $\Phi(t)$ is even, and $\beta_j = 0$ for negative j .

It may be shown that

Lemma 4.1. *The determinant $W(r,m;0)$ of (4.6) satisfies*

$$\epsilon_r W(r,m;0) = \Delta(n,r), \tag{4.7}$$

where $n = m - r$.

As we have explained in [6], under the assumption of the appropriate sign-regularity, it follows that the various minors of $\Delta(n,r)$ with consecutive rows and columns will also be sign-regular. Karlin [3, pp. 58-9], following Fekete, has shown that then, the sole minor of $\Delta(n,r)$ formed by omitting all even-numbered rows and columns, will also be sign-regular. When n is even, this minor has the same determinant as $D(n/2,(r+1)/2)$.

We must stress that there remains an infinite number of exceptional cases that cannot be treated with this alternative single moment method. However, this

[2]Unpublished reports by the author are available at http://publish.uwo.ca/~jnuttall

method does not use all the information contained in a sign-regular Wronskian $W(r, m; t)$ - only values when $t = 0$ are involved. Left unanswered is the question of how to use Wronskian values when $t > 0$ (or some other information) to show that $D(n, r) > 0$ for the exceptional cases.

4.7. **Results of numerical calculations.** With the help of high precision software, we have calculated accurate values of the Wronskian $W(r, m; t)$ for $r \leq 142$, $m \leq 197$, and a set of values of t with a spacing of 0.0025. This spacing appears to be small enough to distinguish between Wronskians that are sign-regular and those that are not, but further independent calculations would be beneficial. With this information we can construct a table [7, Table 2] of the $m(r)$ of Conjecture 1 up to $r = 142$.

We list some computed values of $m(r)$ in the Table below.

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$m(r)$	0	0	0	1	1	1	2	4	6	7	7	7	9	11	13	15	15	15	16

The calculations involve the evaluation of the elements of the Wronskian determinants. This requires a numerical method of finding the cumulants using the formula (4.3). We take advantage of the fact [1, p. 523 Theorem A(ii)] that $\Phi(t)$ is analytic in a strip centered on the real axis. We approximate $\Phi(t)$ by expanding up to order 500 about the center of each of a set of intervals of length 0.0025, and then perfoming the resulting integrals analytically. It appears that precision of 700 figures is adequate.

4.8. **Discussion.** It appears that the approach to the RH summarized in this section has some promise. That view would be reinforced if the analytic relations could be checked, and the calculations repeated and extended independently. Barring the discovery of a serious flaw, there are two main problems to be solved.

(1) Prove the truth of Conjecture 1, including a suitable upper bound on $m(r)$.
(2) Extend the Single Moment Method Sec. 4.6 to cover all exceptional cases.

In relation to Problem 1, calculations suggest that, for every r, the Wronskian may be written as a sum of terms analogous to (3.5), which relates to $r = 3$. In that case (3.19) (which for the first 3 cases lists the value at $y = \pi$) shows that the first term is the largest, the second term is about 33% of the first, and the third is negligible. It is possible that a similar split will be useful for higher values of r.

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3278 McGuffin Hills Dr., Parkhill, Ontario N0M 2K0 Canada
E-mail address: jnuttall@uwo.ca