RIEMANN DETERMINANTAL INEQUALITIES OF ORDER 3

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Abstract. It is known that the Riemann Hypothesis is correct if a certain set of determinantal inequalities $D(n,r) > 0,\ n = 0, 1, \ldots$ hold for all positive integer values of the order r of the determinants. The condition has been previously verified for orders $1, 2, 3$. Here we present an improved method for the proof of the result for order $r = 3$.

1. INTRODUCTION

1.1. It has long been known that the validity for all orders of a infinite set of inequalities is sufficient to prove the Riemann hypothesis (RH). Previously (see [5][1] we described a method of proving the relations for order 3, except for one case that can be treated numerically. In this report[2] we describe an improved method for dealing with the case of order 3. The new method is simpler to apply and understand, and it should be possible to adapt it to treat the case of order 4.

In Sections 1 and 2 we repeat the description of the necessary background information. In Section 3 we present the new method of proving the key theorem that the Wronskian $W_3(t)| < 0,\ \ t \geq 1$.

About 25 years ago Csordas, Norfolk and Varga [1] reported the first progress in what might be called the determinantal method for proving the Riemann hypothesis (RH). This method involves the study of the coefficients in the Taylor expansion of the Riemann ζ-function, which in the notation of [1] (slightly modified) is written

$$F(z) = \sum_{n=0}^{\infty} \beta_n z^n \tag{1.1}$$

where the series coefficients are

$$\beta_n = \frac{1}{\Gamma\left(2n+1\right)} \int_0^{\infty} dt \Phi\left(t\right) t^{2n}, \tag{1.2}$$

with the function $\Phi\left(t\right)$ given by

$$\Phi\left(t\right) = \sum_{m=1}^{\infty} \left[2m^4\pi^2 e^{9t} - 3m^2\pi e^{5t}\right] \exp\left(-m^2\pi e^{4t}\right). \tag{1.3}$$

The RH is equivalent to the statement that all the zeros of the entire function $F(z)$ are real and negative.

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1.2. Karlin [3, Theorem 5.3, p. 412; p. 393; Lemma 9.3, p. 89] tells us that this condition is ensured by a requirement on the coefficients $(\beta_n, n = 0, 1, \ldots)$ of the series $F(z)$, from which is formed a semi-infinite matrix B,

$$B_{i,j} = \begin{cases} \beta_{j-i}, & j \geq i; \\ 0, & j < i; \end{cases} \quad i, j = 0,\ 1,\ 2,\ \ldots. \tag{1.4}$$

Thus, if we define the minors $D(n, r)$ of order r by

$$D(n, r) = \det[B_{i,j+n}]_{i,j=1,\ldots,r}, \tag{1.5}$$

the RH is equivalent to the condition that

$$D(n, r) > 0, \quad n = 0, 1, \ldots;\ r = 1, 2 \ldots \tag{1.6}$$

In this case the matrix B is said to be totally positive or TP (see [3]). The ultimate objective is to prove (1.6), which we call the determinantal inequalities, and so demonstrate the truth of the RH.

1.3. In [1] attention was restricted to the case of order $r = 2$ ($r = 1$ is trivial). In that case the condition (1.6) reduces to

$$D(n, 2) = \begin{vmatrix} \beta_n & \beta_{n+1} \\ \beta_{n-1} & \beta_n \end{vmatrix} > 0, \quad n = 1, 2, \ldots, \tag{1.7}$$

while for the case $n = 0$ we require

$$D(0, 2) = \begin{vmatrix} \beta_0 & \beta_1 \\ 0 & \beta_0 \end{vmatrix} > 0, \tag{1.8}$$

which is trivially true.

In the process of deriving some stronger results Csordas et al. [1] (see also Csordas and Varga [2]) proved (1.7). Their main technique was originally introduced by Karlin, Proschan and Barlow [4] and later described by Karlin [3] . In a simplified version suitable for our purposes we define the kernel $K(x, y,),\ 0 \leq x, y < \infty$, by

$$K(x, y) = \Phi(x + y). \tag{1.9}$$

The corresponding compound kernel of order 2 is given by

$$K_{[2]}\left(\underline{x}, \underline{y}\right) = \det \begin{bmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{bmatrix} \tag{1.10}$$

where

$$\underline{x} = (x_1, x_2)\,, \quad 0 \leq x_1 < x_2, \tag{1.11}$$

and similarly for $\underline{y}$.

In [2] (see (2.7) for details) it is shown that $\log[\Phi(x)]$ is strictly concave on $(0, \infty)$, which means that

$$\det \begin{bmatrix} \Phi(x) & \Phi^{(1)}(x) \\ \Phi^{(1)}(x) & \Phi^{(2)}(x) \end{bmatrix} < 0, \ \ x \geq 0. \tag{1.12}$$

It is shown in [1] that consequently $K_{[2]}\left(\underline{x}, \underline{y}\right) < 0$ where (1.11) holds. See Lemma 2.1 below for a more general proof of this result.

In the terminology of Karlin [3] we say that $K(x, y)$ is sign-reverse regular of order 2, i.e. RR_2. As explained by Csordas et al. [1, p. 526] , the RR_2 condition leads to a proof of (1.7).

1.4. It is natural to ask whether the above techniques can be generalized to prove (1.6) for values of order $r > 2$. Our response here is to show that the corresponding compound kernel of order 3, namely

$$K_{[3]}\left(\underline{x}, \underline{y}\right) = \det \begin{bmatrix} K(x_1, y_1) & K(x_1, y_2) & K(x_1, y_3) \\ K(x_2, y_1) & K(x_2, y_2) & K(x_2, y_3) \\ K(x_3, y_1) & K(x_3, y_2) & K(x_3, y_3) \end{bmatrix} \tag{1.13}$$

is RR_3. This is what is done below in Sect.3.

As we describe in Section 2 the method of [4] then shows, just as it did for $r = 2$, that (1.6) holds for $r = 3, \ n = 2, 3, \ldots$.

1.5. In Section 2 we show how to prove (1.7) and (2.3) given that the respective compound kernels are $RR_r, \ r = 2, 3$.

Section 3 uses a theorem of [3] (see Lemma 2.1) to prove that $K_{[3]}\left(\underline{x}, \underline{y}\right) < 0$ is RR_3. This involves a number of elementary estimates, for which the techniques and information in [2] Sect. 3 are helpful. The structure of these relations is well-suited to implementation on a computer, where we rigorously perform operations such as multiplying and adding polynomials with integer coefficients. We also use an alogrithm to determine the zeros of certain polynomials. These results are not precise, but they are accurate enough for our purposes - there is no lack of rigor.

2. SIGN-REGULARITY AND POSITIVE RIEMANN DETERMINANTS

2.1. We first review the precise general definition of sign-reverse regularity when applied to the kernel $K(x, y)$ of (1.9), as stated by [3, p. 12]. The set X is the non-negative real line. The open simplex $\Delta_p(X)$ is

$$\Delta_r(X) = \{\underline{x} = (x_1, x_2, \ldots, x_p) \,|\, x_1 < x_2 < \ldots < x_p : x_i \in X\} \tag{2.1}$$

The compound kernel $K_{[p]}\left(\underline{x}, \underline{y}\right)$ is defined by

$$K_{[p]}\left(\underline{x}, \underline{y}\right) = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \ldots & K(x_1, y_p) \\ K(x_2, y_1) & K(x_2, y_2) & \ldots & K(x_2, y_p) \\ \vdots & \vdots & & \vdots \\ K(x_p, y_1) & K(x_p, y_2) & \ldots & K(x_p, y_p) \end{vmatrix} \tag{2.2}$$

where

$$\underline{x} = (x_1, x_2, \ldots, x_p) \in X; \quad \underline{y} = (y_1, y_2, \ldots, y_p) \in X \tag{2.3}$$

With $\epsilon_p = (-1)^{p(p-1)/2}$, we say that $K(x, y)$ is RR_r if $\epsilon_p K_{[p]}\left(\underline{x}, \underline{y}\right)$ is a non-negative function on $\Delta_p(X) \times \Delta_p(X)$ for each $p = 1, 2, \ldots, r$.

2.2. The following lemma is important in the later development.

Lemma 2.1. *Suppose that* $\psi(x)$ *is analytic in a neighborhood of* $(0, \infty)$*, and that the kernel* $k(x, y) = \psi(x + y), \ x, y \in (0, \infty)$*. Define* $w_p(t) = \det \left|\psi^{(i+j-2)}(t)\right|_{i,j=1}^{p}$*. If* $\epsilon_p w_p(t) > 0, \ t \geq 0, \ \ p = 1, 2, \ldots, r$ *then* $k(x, y)$ *is* RR_r*.*

Proof. This result is a special case of Theorem 2.6 of [3, p. 55]. The analyticity of $\psi(x,y)$ ensures that the differentiability requirements of the theorem are satisfied. The relation

$$\det\left|\frac{\partial^{i+j-2}}{\partial x^{i-1}\partial y^{j-1}}k(x,y)\right|_{i,j=1}^{p} = \det\left|\psi^{(i+j-2)}(t)\right|_{i,j=1}^{p}, \quad t = x+y, \tag{2.4}$$

together with [3, (1.3), p. 48] demonstrates that the requirements on the compound kernel appearing in the statement of the theorem hold. □

Now define $W_p(t)$ as

$$W_p(t) = \det\left|\Phi^{(i+j-2)}(t)\right|_{i,j=1}^{p}. \tag{2.5}$$

Theorem 2.2. *The kernel $K(x,y)$ defined by (1.9) is sign-regular of type RR_3.*

Proof. From Lemma 2.1 we must show, for $t \geq 0$, that $W_1(t) > 0$, while for $p = 2, 3$, that $W_p(t) < 0$.

Since $\Phi(t) > 0,\ t \geq 0$ the relation $W_1(t) > 0,\ t \geq 0$ is true.

Next, from [1, p. 523] we have $\Phi^{(1)}(t) \leq 0, t \geq 0$, from [2, (3.18)] that $\Phi^{(2)}(0) < 0$, and from [2, p. 197] we have $g(t) > 0,\ t > 0$, where

$$g(t) = -t\left(\Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2\right) + \Phi(t)\Phi^{(1)}(t) \tag{2.6}$$

It follows that

$$W_2(t) = \Phi(t)\Phi^{(2)}(t) - \Phi^{(1)}(t)^2 < 0, \quad t \geq 0. \tag{2.7}$$

The proof that $W_3(t) < 0,\ t \geq 0$ is given in Theorem 3.5. □

2.3. In the remainder of this section we use the fact that $K(x,y)$ is RR_3, and demonstrate how to prove the determinantal inequalities (1.6) for order $r = 3$, using the techniques of [4] and [1]. We define the function $\lambda(s),\ s > -1$, as

$$\lambda(s) = \frac{1}{\Gamma(s+1)}\int_0^\infty dx \Phi(x)\, x^s. \tag{2.8}$$

There is a corresponding kernel

$$\Lambda(s,t) = \lambda(s+t), \quad s,t > -1/2. \tag{2.9}$$

A useful third type of kernel is

$$G(s,u) = \frac{u^{s-1/2}}{\Gamma(s+1/2)}, \quad u > 0;\ s > -1/2. \tag{2.10}$$

[1, p. 526], and previously [4], show that

$$\begin{aligned}\Lambda(s,t) &= \frac{1}{\Gamma(s+t+1)}\int_0^\infty dx \Phi(x)\, x^{s+t} \\ &= \int_0^\infty du \int_0^\infty dv\, G(s,u)\, K(u,v)\, G(t,v), \quad s,t > -1/2. \end{aligned} \tag{2.11}$$

2.4. The last component of the machinery of the proof is the application of what Karlin [3, p. 17] calls the Basic Composition Formula (BCF). In the present case it may be used to relate the compound kernels corresponding to the kernels appearing in (2.11), so that

$$\Lambda_{[3]}\left(\underline{s},\underline{t}\right) = \int_0^\infty d\underline{u} \int_0^\infty d\underline{v}\, G_{[3]}\left(\underline{s},\underline{u}\right) K_{[3]}\left(\underline{u},\underline{v}\right) G_{[3]}\left(\underline{t},\underline{v}\right). \tag{2.12}$$

In (2.12) we have

$$\underline{u} = \left(u_1, u_2, u_3\right), \;\; 0 \le u_1 < u_2 < u_3 \quad d\underline{u} = du_1 du_2 du_3, \tag{2.13}$$

and similarly for $\underline{v}$. Also we have

$$\underline{s} = \left(s_1, s_2, s_3\right), \;\; -1/2 \le s_1 < s_2 < s_3, \tag{2.14}$$

and similarly for $\underline{t}$.

The compound kernel $G_{[3]}\left(\underline{s},\underline{u}\right)$ may be written as

$$G_{[3]}\left(\underline{s},\underline{u}\right) = \frac{{u_1}^{s_1-1/2}{u_2}^{s_1-1/2}{u_3}^{s_1-1/2}}{\Gamma\left(s_1+1/2\right)\Gamma\left(s_2+1/2\right)\Gamma\left(s_3+1/2\right)} \begin{vmatrix} 1 & {u_1}^{s_2-s_1} & {u_1}^{s_3-s_1} \\ 1 & {u_2}^{s_2-s_1} & {u_2}^{s_3-s_1} \\ 1 & {u_3}^{s_2-s_1} & {u_3}^{s_3-s_1} \end{vmatrix} \tag{2.15}$$

Also $\Lambda_{[3]}\left(\underline{s},\underline{t}\right)$ may be written as

$$\Lambda_{[3]}\left(\underline{s},\underline{t}\right) = \begin{vmatrix} \lambda\left(s_1+t_1\right) & \lambda\left(s_1+t_2\right) & \lambda\left(s_1+t_3\right) \\ \lambda\left(s_2+t_1\right) & \lambda\left(s_2+t_2\right) & \lambda\left(s_2+t_3\right) \\ \lambda\left(s_3+t_1\right) & \lambda\left(s_3+t_2\right) & \lambda\left(s_3+t_3\right) \end{vmatrix}. \tag{2.16}$$

Relations (2.13) and (2.14) show that the two G compound kernels in (2.12) are positive for all arguments. Since the compound kernel $K_{[2]}$ is negative for all arguments, (2.12) demonstrates that $\Lambda_{[3]}\left(\underline{s},\underline{t}\right) < 0$ for all valid $\underline{s}, \underline{t}$.

2.5. With this in mind we have

Theorem 2.3. *For all integer $n \ge 2$*

$$D(n,3) = \begin{vmatrix} \beta_n & \beta_{n+1} & \beta_{n+2} \\ \beta_{n-1} & \beta_n & \beta_{n+1} \\ \beta_{n-2} & \beta_{n-1} & \beta_n \end{vmatrix} > 0, \tag{2.17}$$

Proof. To prove these inequalities we choose

$$s_1 = t_1 = n-2, \quad s_2 = t_2 = n, \quad s_3 = t_3 = n+2, \quad n = 2, 3, \ldots \tag{2.18}$$

and use the relation (2.12), which gives

$$\begin{vmatrix} \beta_{n-2} & \beta_{n-1} & \beta_n \\ \beta_{n-1} & \beta_n & \beta_{n+1} \\ \beta_n & \beta_{n+1} & \beta_{n+2} \end{vmatrix} = \Lambda_{[3]}\left(\underline{s},\underline{t}\right) < 0. \tag{2.19}$$

The restriction $n \ge 2$ arises from the need to satisfy the condition $s > -1/2$ in (2.10).

Interchanging the rows of (2.19) produces the determinants in (2.17) and also changes the sign of the original determinants, so that (2.17) is verified. □

Remark 2.4. Just as for order $r = 2$ the case $n = 0$ is trivial, but for $n = 1$ the relation

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_0 & \beta_1 & \beta_2 \\ 0 & \beta_0 & \beta_1 \end{vmatrix} > 0, \tag{2.20}$$

does not follow from the general method. Its validity may be checked by inserting the numerical values of β_j listed by [1, p. 540].

3. PROOF OF SIGN-REGULARITY OF ORDER 3

3.1. The aim of this section is to prove that

$$W_3(t) < 0, \;\; t \geq 0, \tag{3.1}$$

where $W_3(t)$ is defined by (2.5). Using Csordas and Varga [2, p. 184] we may write

$$\Phi^{(k)}(t) = \pi \exp(5t - y) \left\{ p_{k+1}(y) + 4 \exp(-3y) p_{k+1}(4y) + \Psi_{2,k}(t) \right\} \tag{3.2}$$

where

$$\Psi_k(t) = \exp(y) \sum_{n=3}^{\infty} n^2 p_{k+1}(n^2 y) \exp(-n^2 y). \tag{3.3}$$

We have used $y = \pi e^{4t}$, and the polynomials $p_k(y)$, $k = 0, \ldots, 4$ are defined in [2, p. 184] to be

$$\begin{aligned} p_1(y) &= 2y - 3 \\ p_2(y) &= -8y^2 + 30y - 15 \\ p_3(y) &= 32y^3 - 224y^2 + 330y - 75 \\ p_4(y) &= -128y^4 + 1440y^3 - 4232y^2 + 3270y - 375 \\ p_5(y) &= 512y^5 - 8448y^4 + 41408y^3 - 68096y^2 + 30930y - 1875. \end{aligned} \tag{3.4}$$

The formula (2.5) is equivalent to

$$W_3\,(t) = \begin{vmatrix} \Phi(t) & \Phi^{(1)}(t) & \Phi^{(2)}(t) \\ \Phi^{(1)}(t) & \Phi^{(2)}(t) & \Phi^{(3)}(t) \\ \Phi^{(2)}(t) & \Phi^{(3)}(t) & \Phi^{(4)}(t) \end{vmatrix} \tag{3.5}$$

We now substitute (3.2) into (3.5). Each column of (3.5) may be thought of as a sum of three columns corresponding to the three terms in (3.2), so that $W_3(t)$ may be written as the sum of $3 \times 3 \times 3 = 27$ determinants formed by choosing one from each of the three columns comprising a column of $W_3(t)$. We omit the factor $\pi \exp(5t - y)$ in each term, since we are concerned only with the sign of $W_3(t)$. The result is

$$\begin{aligned} w_3(t) = {} & F_0(y) + e^{-3y} F_{1,1}(y) + e^{-6y} F_{1,2}(y) + e^{-9y} F_{1,3}(y) \\ & + F_{2,1}(y) + e^{-3y} F_{2,2}(y) + e^{-6y} F_{2,3}(y) \\ & + F_3(y) + F_4(y), \end{aligned} \tag{3.6}$$

where

$$W_3(t) = [\pi \exp(5t - y)]^3 w_3(t). \tag{3.7}$$

Below we define and discuss the various terms in (3.6).

3.2. The first term $F_0(y)$ in (3.6) is obtained by choosing the first term $p_{k+1}(y)$ in each element of $W_3(t)$, so that

$$F_0(y) = \begin{vmatrix} p_1(y) & p_2(y) & p_3(y) \\ p_2(y) & p_3(y) & p_4(y) \\ p_3(y) & p_4(y) & p_5(y) \end{vmatrix} \tag{3.8}$$

With the help of (3.4) it is found that

$$F_0(y) = 860160y^3 - 737280y^4 + 294912y^5 - 65536y^6 \tag{3.9}$$

The three zeros of $F_0(y)$, apart from $y = 0$, are

$$1.192509... \pm i2.187155...,\ 2.114980... \tag{3.10}$$

We have $t \geq 0$, i.e. $y \geq \pi$, so that, since $\pi > 2.11498...$, it follows that

$$F_0(y) < 0,\ \ y \geq \pi. \tag{3.11}$$

With $a = 1.192509...$, $b = 2.187155...$, $c = 2.114980...$ we may write

$$F_0(y) = -65536y^3[(y-a)^2 + b^2](y-c), \tag{3.12}$$

so that $-F_0(y)$ increases steadily as y increases from π. Thus the least upper bound of $F_0(y),\ y > \pi$, is $F_0(\pi) = -1.7904{\times}10^7$.

3.3. The next term in (3.6), i.e. $e^{-3y}F_{1,1}(y)$, corresponds to the choice of two columns of type $p_{k+1}(y)$ and one column of type $4e^{-3y}p_{k+1}(4y)$. We may therefore write $F_{1,1}(y)$ as

$$\begin{aligned} F_{1,1}(y) \ &= \begin{vmatrix} 4p_1(4y) & p_2(y) & p_3(y) \\ 4p_2(4y) & p_3(y) & p_4(y) \\ 4p_3(4y) & p_4(y) & p_5(y) \end{vmatrix} + \begin{vmatrix} p_1(y) & 4p_2(4y) & p_3(y) \\ p_2(y) & 4p_3(4y) & p_4(y) \\ p_3(y) & 4p_4(4y) & p_5(y) \end{vmatrix} \\ &+ \begin{vmatrix} p_1(y) & p_2(y) & 4p_3(4y) \\ p_2(y) & p_3(y) & 4p_4(4y) \\ p_3(y) & p_4(y) & 4p_5(4y) \end{vmatrix} \end{aligned} \tag{3.13}$$

The polynomial $F_{1,1}(y)$ is

$$\begin{aligned} F_{1,1}(y) \ &= 72253440y^3 - 482181120y^4 + 792281088y^5 - 579403776y^6 \\ &+228261888y^7 - 42467328y^8, \end{aligned} \tag{3.14}$$

and its non-trivial zeros are

$$1.143552... \pm -i1.630193...,\ 0.214613...,\ 1.182386...,\ 1.690895... \tag{3.15}$$

Again it follows that

$$F_{1,1}(y) < 0,\ \ y \geq \pi. \tag{3.16}$$

which shows that an upper bound to $e^{-3y}F_{1,1}(y)$ is zero.

Moreover we find that $e^{-3y}F_{1,1}(y)$ decreases as y increases, so that the least upper bound to $e^{-3y}F_{1,1}(y)$ is $e^{-3y}F_{1,1}(\pi) = -5.8783{\times}10^6$. Thus $e^{-3y}F_{1,1}(y)$ is about 33 percent of $F_0(y)$ at $y = \pi$

3.4. The third term in (3.6), i.e. $e^{-6y}F_{1,2}(y)$, corresponds to the choice of one column of type $p_{k+1}(y)$ and two columns of type $4e^{-3y}p_{k+1}(4y)$. We may therefore write $F_{1,2}(y)$ as

(3.17)
$$F_{1,2}(y) = \begin{vmatrix} 4p_1(4y) & 4p_2(4y) & p_3(y) \\ 4p_2(4y) & 4p_3(4y) & p_4(y) \\ 4p_3(4y) & 4p_4(4y) & p_5(y) \end{vmatrix} + \begin{vmatrix} p_1(y) & 4p_2(4y) & 4p_3(4y) \\ p_2(y) & 4p_3(4y) & 4p_4(4y) \\ p_3(y) & 4p_4(4y) & 4p_5(4y) \end{vmatrix}$$
$$+ \begin{vmatrix} 4p_1(4y) & p_2(y) & 4p_3(4y) \\ 4p_2(4y) & p_3(y) & 4p_4(4y) \\ 4p_34(y) & p_4(y) & 4p_5(4y) \end{vmatrix}$$

The polynomial $F_{1,2}(y)$ is

$$\begin{aligned} F_{1,2}(y) &= 1156055040y^3 - 6794772480y^4 + 16356999168y^5 \\ &-17334534144y^6 + 9739173888y^7 - 2717908992y^8, \end{aligned} \tag{3.18}$$

and its non-trivial zeros are

$$0.344204... \pm -i0.205802...,\ 0.842359... \pm -i1.214801...,\ 1.210203..., \tag{3.19}$$

Again it follows that

$$F_{1,2}(y) < 0, \;\; y \geq \pi. \tag{3.20}$$

which shows that an upper bound to $F_{1,2}(y)$ is zero. Again we find that $e^{-6y}F_{1,2}(y)$ decreases as y increases, so that the least upper bound to $e^{-6y}F_{1,2}(y)$ is $e^{-6y}F_{1,2}(\pi) = -5.6349\times 10^4$.

3.5. The fourth term in (3.6), i.e. $e^{-9y}F_{1,3}(y)$, corresponds to the choice of three columns of type $4e^{-3y}p_{k+1}(4y)$. We may therefore write $F_{1,3}(y)$ as

$$F_{1,3}(y) = \begin{vmatrix} 4p_1(4y) & 4p_2(4y) & 4p_3(4y) \\ 4p_2(4y) & 4p_3(4y) & 4p_4(4y) \\ 4p_3(4y) & 4p_4(4y) & 4p_5(4y) \end{vmatrix} \tag{3.21}$$

The polynomial $F_{1,3}(y) = 64F_{1,1}(4y)$ is

$$\begin{aligned} F_{1,3}(y) &= 3523215360y^3 - 12079595520y^4 + 19327352832y^5 \\ &-17179869184y^6 \end{aligned} \tag{3.22}$$

and its non-trivial zeros are

$$0.298127... \pm -i0.546788...,\ 0.528745..., \tag{3.23}$$

Again it follows that

$$F_{1,3}(y) < 0, \;\; y \geq \pi. \tag{3.24}$$

which shows that an upper bound to $F_{1,3}(y)$ is zero. Again we find that $e^{-9y}F_{1,3}(y)$ decreases as y increases, so that the least upper bound to $e^{-9y}F_{1,3}(y)$ is $e^{-9y}F_{1,3}(\pi) = -6.1328$.

3.6. So far the calculations have been exact, but for the remaining 5 terms in (3.6) we prove only positive upper bounds, which are found to be all small compared to the absolute value of $F_0(y)$. The fifth term involving $F_{2,1}$ corresponds to the sum of three determinants each containing two columns of type $p_{k+1}(y)$ and one column of type $\Psi_k(t)$. We write $F_{2,1}(y)$ as

$$F_{2,1}(y) = \sum_{k=0}^{4} \Psi_k(t)\rho_k(y) \tag{3.25}$$

where

$$\begin{aligned}
\rho_0(y) &= p_3(y)p_5(y) - p_4(y)^2 \\
\rho_1(y) &= -2(p_2(y)p_5(y) - (p_3(y)p_4(y)) \\
\rho_2(y) &= 2p_2(y)p_4(y) + p_1(y)p_5(y) - 3p_3(y)^2 \\
\rho_3(y) &= -2(p_1(y)p_4(y) - (p_2(y)p_3(y)) \\
\rho_4(y) &= p_1(y)p_3(y) - p_2(y)^2.
\end{aligned} \tag{3.26}$$

Substituting the definitions (3.4) into (3.26) leads to

$$\begin{aligned}
\rho_0(y) &= -486000y + 1867200y^2 - 3808320y^3 + 3118080y^4 \\
&\quad - 1255424y^5 + 229376y^6 - 16384y^7 \\
\rho_1(y) &= 302400y - 967680y^2 + 1324800y^3 - 752640y^4 \\
&\quad + 184320y^5 - 16384y^6 \\
\rho_2(y) &= -68640y + 167808y^2 - 151936y^3 + 52224y^4 - 6144y^5 \\
\rho_3(y) &= 6720y - 10752y^2 + 5888y^3 - 1024y^4 \\
\rho_4(y) &= -240y + 192y^2 - 64y^3
\end{aligned} \tag{3.27}$$

Lemma 3.1. *For $k = 0, \ldots, 4$ let $r(k)$ be the absolute value of the coefficient of y^{7-k} in (3.27), i.e. $r(0) = 16384,\ \ , r(1) = 16384,\ \ r(2) = 6144$, etc. Then*

$$|\rho_k(y)| < r(k)y^{7-k}, \ \ y \geq \pi; \ \ k = 0, \ldots, 4 \tag{3.28}$$

Proof. It is found that the non-trivial zero of $\rho_4(y) + 64y^3 = -240y + 192y^2$ is $< \pi$, so that $\rho_4(y) + 64y^3 > 0,\ \ y \geq \pi$ and $\rho_4(y) > -64y^3,\ \ y \geq \pi$. Similarly the non-trivial zeros of $\rho_4(y) - 64y^3 = -240y + 192y^2 - 128y^3$ have real parts $< \pi$, so that $\rho_4(y) < 64y^3,\ y \geq \pi$. Thus (3.28) holds for $k = 4$. A similar argument applies to the other values of k. □

We also need the following Lemma.

Lemma 3.2. *For $k = 0, \ldots, 4$ we have*

$$|\Psi_k(t)| < 2^{2k+1}3^{2k+4}y^{k+1}e^{-8y}C, \ \ y \geq \pi \tag{3.29}$$

where $y = \pi e^{4t}$ and $C = [1 - 81e^{-3\pi}]^{-1}$. XXXX

Proof. The case of $k = 6$ is proved in [2, p. 188], and the same method applies to the cases in the Lemma. □

To bound $F_{2,1}(y)$ we insert the inequalities (3.28) and (3.29) into (3.25), leading to

$$|F_{2,1}(y)| < \left[\sum_{k=0}^{4} 2^{2k+1}3^{2k+4}r(k)\right] Cy^8e^{-8y}, \;\; y \geq \pi. \tag{3.30}$$

Since the maximum value of y^8e^{-8y} occurs at $y = \pi$, it follows that

$$|F_{2,1}(y)| < \left[\sum_{k=0}^{4} 2^{2k+1}3^{2k+4}r(k)\right] C\pi^8e^{-8\pi}, \;\; y \geq \pi, \tag{3.31}$$

which means that $|F_{2,1}(y)| < 3.083\times10^3, \;\; y \geq \pi$.

3.7. The sixth term in (3.6) is $e^{-3y}F_{2,2}$, which corresponds to the sum of six determinants each containing one column of type $p_{k+1}(y)$, one column of type $4p_{k+1}(4y)$ and one column of type $\Psi_k(t)$. We write $F_{2,2}(y)$ as

$$F_{2,2}(y) = \sum_{k=0}^{4} \Psi_k(t)\eta_k(y) \tag{3.32}$$

where

$$\begin{aligned} \eta_0(y) &= 4e^{-3y}[p_3(y)p_5(4y) + p_3(4y)p_5(y) - 2p_4(y)p_4(4y)] \\ \eta_1(y) &= 4e^{-3y}[-2p_2(y)p_5(4y) - 2p_2(4y)p_5(y) + 2p_3(y)p_4(4y) \\ &\quad + 2p_3(4y)p_4(y)] \\ \eta_2(y) &= 4e^{-3y}[2p_1(y)p_4(4y) + 2p_1(4y)p_4(y) + p_1(y)p_5(4y) \\ &\quad + p_1(4y)p_5(y) - 6p_3(y)p_3(4y)] \\ \eta_3(y) &= 4e^{-3y}[-2p_2(y)p_5(4y) - 2p_2(4y)p_5(y) + 2p_2(y)p_3(4y) \\ &\quad + 2p_2(4y)p_3(y)] \\ \eta_4(y) &= 4e^{-3y}[p_1(y)p_3(4y) + p_1(4y)p_3(y) - 2p_2(y)p_2(4y)] \end{aligned} \tag{3.33}$$

Substituting the definitions (3.4) into (3.33) leads to

$$\begin{aligned} \eta_0(y) &= 4e^{-3y}[-9720000y + 144465600y^2 - 680073600y^3 + 1637053440y^4 \\ &\quad - 1951406080y^5 + 1191657472y^6 - 351272960y^7 + 37748736y^8] \\ \eta_1(y) &= 4e^{-3y}[6048000y - 7668864y^2 + 286387200y^3 - 497925120y^4 \\ &\quad + 421171200y^5 - 163807232y^6 + 23592960y^7] \\ \eta_2(y) &= 4e^{-3y}[-1372800y + 13881984y^2 - 39699200y^3 + 48193536y^4 \\ &\quad - 25466880y^5 + 4866048y^6] \\ \eta_3(y) &= 4e^{-3y}[134400y - 973056y^2 + 1853440y^3 - 1375232y^4 + 368640y^5] \\ \eta_4(y) &= 4e^{-3y}[-4800y + 21696y^2 - 22400y^3 + 9216y^4] \end{aligned} \tag{3.34}$$

Lemma 3.3. *For $k = 0, \ldots, 4$ let $q(k)$ be the value of the coefficient of y^{8-k} in (3.34), i.e. $r(0) = 37748736, \;, r(1) = 23592960, \;\; r(2) = 4866048$, etc. Then*

$$|\eta_k(y)| < q(k)y^{8-k}, \;\; y \geq \pi; \;\; k = 0, \ldots, 4 \tag{3.35}$$

Proof. The proof uses the same method as that of Lemma 3.1. □

Following the same procedure as in Sec. 3.6, we find that $|e^{-3y}F_{2,2}(y)| < 2.422\times 10^2$, $y \geq \pi$.

3.8. The seventh term $e^{-6y}F_{2,3}(y)$, corresponding to the sum of three determinants each containing two columns of type $4p_{k+1}(4y)$ and one column of type $\Psi_k(t)$, has a structure is similar to that of the fifth term $F_{2,1}(y)$ in Sec. 3.5. We replace $\rho_k(y)$ of (3.26) by $16\rho_k(4y)$, and the procedure of that section leads to

$$|F_{2,3}(y)| < \left[\sum_{k=0}^{4} 2^{19} 3^{2k+4} r(k)\right] C y^8 e^{-8y}, \;\; y \geq \pi. \tag{3.36}$$

As before we find that $|F_{2,3}(y)| < 1.379\times 10^1, \;\; y \geq \pi$.

3.9. For the next step we need

Lemma 3.4. *For $k = 0, \ldots, 4$ we have*

$$|p_{k+1}(y)| < 2^{2k+1} y^{k+1}, \;\; y \geq \pi \tag{3.37}$$

Proof. This Lemma is proved in the same manner as Lemma 3.1. □

The eighth term $F_3(y)$ corresponds to three determinants, each containing one column of type $p_{k+1}(y) + 4e^{-3y}p_{k+1}(4y)$ and two columns of type $\Psi_k(t)$. We write

$$\phi_k(y) = p_{k+1}(y) + e^{-3y} 4 p_{k+1}(4y) \tag{3.38}$$

so that, using (3.37), we have

$$|\phi_k(y)| \leq |p_{k+1}(y)| + |4e^{-3y}p_{k+1}(4y)| < 2^{2k+1}[1 + 2^{2k+4}e^{-3y}]y^{k+1} \tag{3.39}$$

To analyze $F_3(y))$ we use the structure of (3.25) and (3.26), with $\Psi_k(t)$ taking the place of $p_{k+1}(y)$ in (3.26), while $\phi_k(y)$ takes the place of $\Psi_k(t)$ in (3.25). Thus we have

$$F_3(y) = \sum_{k=0}^{4} \phi_k(y)(y)\sigma_k(y) \tag{3.40}$$

where

$$\begin{aligned}
\sigma_0(y) &= \Psi_2(t)\Psi_4(y) - \Psi_3(y)^2 \\
\sigma_1(y) &= -2(\Psi_1(y)\Psi_4(y) - (\Psi_2(y)\Psi_3(y)) \\
\sigma_2(y) &= 2\Psi_1(y)\Psi_3(y) + \Psi_0(y)\Psi_4(y) - 3\Psi_2(y)^2 \\
\sigma_3(y) &= -2(\Psi_0(y)\Psi_3(y) - (\Psi_1(y)\Psi_2(y)) \\
\sigma_4(y) &= \Psi_0(y)\Psi_2(y) - \Psi_1(y)^2.
\end{aligned} \tag{3.41}$$

We bound the functions $\sigma_k(y), \; k = 1, \ldots, 4$ by replacing each term on the right hand side of (3.41), and obtain an upper bound on $F_3(y)$ by using the inequalities (3.29) and (3.39) in (3.40). The result is that $|F_3(y)| < 3.497\times 10^{-2}, \;\; y \geq \pi$.

3.10. The ninth and last term in (3.6) is

$$F_4\,(t) = \begin{vmatrix} \Psi_0(t) & \Psi_1(t) & \Psi_2(t) \\ \Psi_1(t) & \Psi_2(t) & \Psi_3(t) \\ \Psi_2(t) & \Psi_3(t) & \Psi_4(t) \end{vmatrix} \tag{3.42}$$

The determinant may be written as the sum of 6 components, of which two are $\Psi_0(t)\Psi_2(t)\Psi_4(t)$ and $-\Psi_1(t)\Psi_1(t)\Psi_4(t)$. We obtain an upper bound to each component by replacing the products in these expressions by the product of the three bounds from (3.29). It turns out that each upper bound is the same, so that we find that

$$|F_4(y)| < 6 \times 2^{13}3^{16}y^7e^{-24y}C < 2^{14}3^{17}\pi^7e^{-24\pi}C =< 3.036\times10^{-12},\ \ y \geq \pi. \tag{3.43}$$

3.11. In summary we have found that, for $y \geq \pi$, the nine contributions to $w_3(t)$ in (3.6) have upper bounds as follows

(3.44)

1	$F_0(y)$	-1.7904×10^7
2	$e^{-3y}F_{1,1}(y)$	-5.8783×10^6
3	$e^{-6y}F_{1,2}(y)$	-5.6349×10^4
4	$e^{-9y}F_{1,3}(y)$	-6.1328×10^0
5	$F_{2,1}(y)$	3.083×10^3
6	$e^{-3y}F_{2,2}(y)$	2.422×10^2
7	$e^{-6y}F_{2,3}(y)$	1.379×10^1
8	$F_3(y)$	3.497×10^{-2}
9	$F_4(y)$	3.036×10^{-12}

We therefore have

Theorem 3.5. *For all $t \geq 0$, the function $W_3(t) < 0$, where $W_3(t)$ is defined in (2.5).*

Proof. For $y \geq \pi$ (i.e. $t \geq 0$) the sum of the upper bounds in (3.44) is negative , so that $w_3(t)$ and, in view of (3.7), also $W_3(t)$ are negative in the same range. □

This theorem achieves the aim of (3.1).

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