A MOMENT DETERMINANTAL FRAMEWORK TO STUDY THE RIEMANN HYPOTHESIS AND RELATED QUESTIONS

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Abstract. Our previous determinantal method for the Riemann Hypothesis is analyzed in terms of a set of minors of a determinant with elements consisting of single moments of the function $\Phi(u)$. Numerical calculations suggest some striking regularities in the outputs. These patterns are repeated, with different numbers, when the method is applied to the Conrey/Ghosh Dirichlet series functions with k=1 and 3. These features could provide a new method of studying the hypothesis and its generalizations.

1. INTRODUCTION

This report[1] extends our previous work[2] [JN 2.1.7] on using single moments to prove some of the determinantal inequalities that arise in our approach to the resolution of the Riemann Hypothesis (RH). After summarizing the relevant parts of the earlier discussion we go in Sec. 3 into more details of the application of Karlin's treatment to our problem. The essential tools are various minors of the determinant (3.1). Karlin showed how a typical minor may be expressed in terms of five other minors in such a way that, if all five minors are positive, then it follows that the first minor also has that property. We show how the relations between minors may be represented by a combined tree of connections where several 'known' inputs are linked by a path to the desired minor, which is one of the determinants required by the RH to be positive. In some cases all the minors involved are positive, so that the positivity of the final determinant is proved. However in other cases this does not happen, and we need to find a technique that overcomes this problem.

Apart from the introduction of the above structure of relations, the main result, described in Secs. 3.5, 3.6, is the product of numerical calculations that suggest the existence of an unexpected (to the author) but remarkably benign numerical behavior (a constant), in contrast to other numerical variables that appear in the RH. The most notable example is the apparently chaotic variation of the Riemann zeros. Another, less chaotic but still not smooth case, is the function $m(r)$ in (3.22).

The significance of our results is enhanced by the fact that the same type of good behavior appears in a functions which Conrey and Ghosh [1] believe have properties analogous to the Riemann function - see Sec. 3.6.

The assistance provided by Brian Conrey is much appreciated.

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[2] Previous work by the author is available on the web site http://publish.uwo.ca/ jnuttall. The following refers to a section on that web site.

2. PREVIOUS RESULTS

2.1. **Moment determinants.** In previous reports [JN 2.1.1 - 2.1.8] we have described an approach to the proof of the RH that involves showing the positivity of a set of determinants, i.e.

$$D(n,r) > 0, \quad n = 0,1,\ldots; \; r = 1,2...,\tag{2.1}$$

so the sign of these determinants is what has to be obtained to resolve the RH. If all the inequalities hold then the RH is true. Conversely, if even one determinant is negative then the RH is false.

The entire development is based on the function $\Phi(u)$ of [JN 2.1.6 (2.1), p.3],where

$$\Phi\left(u\right) = \sum_{m=1}^{\infty}\left[2m^4\pi^2 e^{9u} - 3m^2\pi e^{5u}\right]\exp\left(-m^2\pi e^{4u}\right).\tag{2.2}$$

The normalized even moments of $\Phi(u)$ are defined by

$$\beta_n = \frac{1}{\Gamma\left(2n+1\right)}\int_0^{\infty} du\Phi(u)u^{2n}, \quad n = 0,1,\ldots,\tag{2.3}$$

with the convention that $\beta_n = 0$ if $n < 0$. The determinant $D(n,r)$ of order r is given by the Toeplitz form

$$D(n,r) = \begin{vmatrix} \beta_n & \beta_{n+1} & \ldots & \beta_{n+r-1} \\ \beta_{n-1} & \beta_n & \ldots & \beta_{n+r-2} \\ \vdots & \vdots & & \vdots \\ \beta_{n-r+1} & \beta_{n-r+2} & \ldots & \beta_n \end{vmatrix}\tag{2.4}$$

Another set of normalized moments, both even and odd, is defined by

$$b_n = \frac{1}{\Gamma\left(n+1\right)}\int_0^{\infty} du\Phi(u)u^{n}, \quad n = 0,1,\ldots.\tag{2.5}$$

In this case we define the 'moments' for negative n by

$$b_n = (-1)^{n+1}\Phi^{(-n-1)}(0), \quad n = -1,-2,\ldots.\tag{2.6}$$

A corresponding set of determinants is

$$\Delta(n,r) = \begin{vmatrix} b_n & b_{n+1} & \ldots & b_{n+r-1} \\ b_{n-1} & b_n & \ldots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-r+1} & b_{n-r+2} & \ldots & b_n \end{vmatrix}\tag{2.7}$$

Note that, if $n = 2j, \; j = \ldots,-2,-1,0,1,2,\ldots$, then $\beta_j = b_n$. This relation applies even for negative j, since the function $\Phi(u)$ is even and $\beta_j = 0$.

2.2. **Cumulants.** The development relies heavily on the cumulants $\{\Psi_m(u)\}$, where

$$\Psi_m(u) = \int_u^{\infty} dt\Psi_{m-1}\left(t\right), \quad m = 1,2,\ldots\tag{2.8}$$

and

$$\Psi_0(u) = \Phi(u).\tag{2.9}$$

There is a corresponding set of kernels

$$K\left(u,v;m\right) = \Psi_m\left(u+v\right), \; 0 \le u,v; \quad m = 0,1,\ldots.\tag{2.10}$$

We shall assume that Conjecture 1 of [JN 2.1.6 Sec. 4, p. 10] is correct. This conjecture relates to the kernel $K(u,v;m)$. It asserts that there is a non-decreasing integer function $m(r)$, $r = 2,3,\ldots$ such that, if $m \geq m(r)$, the kernel has sign-regularity of type RR_r (see [2, p. 12]) for any given order r, and $m(r)$ is the smallest such integer for which the property holds for that r. (It might be possible by analyzing the form of (2.11) for large m to prove Conjecture 1.) In the following we assume the truth of Conjecture 1. In [JN 2.1.8] we presented a table showing $m(r)$ for values of r up to $r = 111$. (The calculations have not been independently checked.)

Karlin [2, p. 193] discussed cumulants (he called them modified kernels), and he showed that

$$\Psi_m(u) = \frac{1}{\Gamma(m)} \int_u^\infty dt \Phi(t)(t-u)^{m-1}, \quad m \geq 1. \tag{2.11}$$

This formula may be verified by differentiating with respect to u, which shows that

$$\Psi_m^{(1)}(u) = -\Psi_{m-1}(u), \quad m \geq 1, \tag{2.12}$$

consistent with (2.8) and (2.9).

Karlin [2, p. 128] also showed that, if $K(u,v;m)$ is RR_r, then so is $K(u,v;m+1)$. In [JN 2.1.6 Sec. 2.7, p. 3], following Karlin, we described a useful technique for proving that a kernel is sign-regular. Applied to the kernel $K(u,v;m)$ of (2.10) the requirement for being RR_r is that

$$\epsilon_p w(p,m;u) > 0, \;\; u \geq 0, \quad p = 1,2,\ldots,r, \tag{2.13}$$

where the two-way Wronskian is defined as

$$w(p,m;u) = \det \left| \Psi_m^{(i+j-2)}(u) \right|_{i,j=1}^{p}. \tag{2.14}$$

The quantity $\epsilon_p = (-1)^{p(p-1)/2}$.

We note that from (2.11)

$$\Psi_n(0) = \frac{1}{\Gamma(n)} \int_u^\infty dt \Phi(t) t^{n-1} = b_{n-1}, \quad n \geq 1, \tag{2.15}$$

so that from (2.12) and (2.15)

$$\Psi_m^{(n)}(0) = (-1)^n \Psi_{m-n}(0) = (-1)^n b_{m-n-1}, \quad n = 0,1,\ldots. \tag{2.16}$$

It follows that, for $m \geq 0$,

$$w(r,m;0) = \begin{vmatrix} b_{m-1} & -b_{m-2} & \cdots & (-1)^{r-1} b_{m-r} \\ -b_{m-2} & b_{m-3} & \cdots & (-1)^{r-2} b_{m-r-1} \\ \vdots & \vdots & & \vdots \\ (-1)^{r-1} b_{m-r} & (-1)^{r-2} b_{m-r-1} & \cdots & b_{m-2r+1} \end{vmatrix} \tag{2.17}$$

Suppose that we multiply the even rows of (2.17) by -1. The columns will then have alternating signs $+,-,+,-,\ldots$, so that if we now multiply the even columns by -1, all the elements will have a positive sign. The net effect of the multiplications will create no change in the value of the determinant. In addition, reversing the order of the columns in (2.17) will multiply $w(r,m;0)$ by ϵ_r. We have effectively proved

Lemma 2.1. *The Wronskian $w(r, m; 0)$ of (2.17) satisfies*

$$\epsilon_r w(r, m; 0) = \Delta(n, r) \tag{2.18}$$

where $n = m - r$. Thus, if the Conjecture 1 is correct, we have

$$m \geq m(r) \implies \Delta(m - r, r) > 0. \tag{2.19}$$

Proof. The conjecture asserts that, if $m \geq m(r)$, then the kernel $K(u, v; m)$ is RR_r, so that (2.13) shows that $\epsilon_r w(r, m; u) > 0$, □

3. Karlin identities

3.1. **Definition.** If we presume that Conjecture 1 is correct, then (2.19) demonstrates the positivity of a set of determinants $\Delta(n, r)$ corresponding to certain values of n, r. The elements b_k of $\Delta(n, r)$ are single moments of $\Phi(u)$, whereas for the RH we need to prove the positivity of similar determinants $D(n, r)$ whose elements are even moments of $\Phi(u)$. Karlin [2, p. 59] has described a set of identities that may be used to establish a useful relation between the two types of determinant. We now explain how the Karlin identities may be used in this instance.

To investigate $D(n, r)$ for a given n, r we consider the $(2r - 1) \times (2r - 1)$ matrix

$$B(n, r) = \begin{bmatrix} b_n & b_{n+1} & \dots & b_{n+2r-2} \\ b_{n-1} & b_n & \dots & b_{n+2r-3} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \dots & b_n \end{bmatrix} \tag{3.1}$$

Note for even n that the sole $r \times r$ minor of $B(n, r)$ with only odd rows and columns has determinant $D(n/2, r)$, and our aim is to determine the sign of this minor.

The process consists of two stages. In the first stage (ST1) we consider in turn the r sub-matrices of $B(n, r)$ with $2r - 1$ rows and r consecutive columns, so that the first sub-matrix is

$$B_{1,1}(n, r) = \begin{bmatrix} b_n & b_{n+1} & \dots & b_{n+r-1} \\ b_{n-1} & b_n & \dots & b_{n+r-2} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \dots & b_{n-r+1} \end{bmatrix} \tag{3.2}$$

Moving from left to right in (3.2) let us label the columns of the first sub-matrix $B_{1,1}(n, r)$ by $1, 2, \dots, r$. Label the rows $1, 2, \dots, 2r - 1$ from top to bottom.

We use the symbol $M(r, j, k)$ to denote a certain $r \times r$ minor determinant of the sub-matrix as follows.

$M(r, 0, k)$ contains the consecutive rows $k - r + 1, k - r + 2, \dots, k$, with $k = r, r + 1, \dots, 2r - 1$

$M(r, j, k)$, $j \geq 1$, uses consecutive rows of the sub-matrix ending in row k, except that, at the end of the list, there are j gaps of one row each.

For example, with $r = 4$,
$M(r, 0, 6)$ has rows $3, 4, 5, 6$.
$M(r, 1, 7)$ has rows $3, 4, 5, 7$
$M(r, 2, 7)$ has rows $2, 3, 5, 7$

Lemma 3.1. *With the minors $M(r,j,k)$ defined as above, and appropriate restrictions on the values of j,k, the following relations hold.*

(3.3)

$$\begin{aligned}
&M(r,1,k)M(r-1,0,k-1)\\
&= M(r,0,k-1)M(r-1,1,k) + M(r,0,k)M(r-1,0,k-2)\\
\\
&M(r,j,k)M(r-1,j-2,k-2)\\
&= M(r,j-2,k-2)M(r-1,jj,k) + M(r,j-1,k)M(r-1,j-1,k-2)\\
\\
&jj = j,\ j \le r-2; \quad jj = j-1,\ j = r-1.
\end{aligned}$$

Proof. With the notation for determinants used by Karlin [2, p. 59] we may write

$$M(r,j,k) = B_{1,1}\begin{pmatrix} i_1, i_2, \ldots, i_r \\ 1,\ 2, \ldots, r \end{pmatrix} \tag{3.4}$$

where the top set of indices refers to rows, the bottom to columns, and

$$\begin{aligned}
&i_r = k \le 2r-1\\
&i_1 < i_2 < i_3 \ldots < i_r\\
&i_{m+1} - i_m = 1,\ m = 1, k-j-1\\
&i_{m+1} - i_m = 2,\ m = k-j, k-1;\ j \ge 1
\end{aligned} \tag{3.5}$$

For a given choice of r,j,k we determine the values of $i_1, i_2, \ldots, i_r$ and substitute them into the Karlin identity [2, p. 59] . Taking account of the fact that some of the indices in the Karlin determinants are not in increasing order, it will be found that (3.3) is valid. □

For example consider the 4×4 minor $M(4,2,7)$, so that $i_1, i_2, i_3, i_4 = 2,3,5,7$. The Karlin relation involves an integer i^0 between i_1 and i_4, not equal to $2,3,5,7$. Choose $i^0 = 4$, in which case the relation becomes

$$M(4,2,7)M(3,0,5) = M(4,0,5)M(3,1,7) + M(4,1,7)M(3,1,5), \tag{3.6}$$

as (3.3) states. Note that the choice of $i^0 = 6$ would give another relation, not needed for our purposes, although such a formula could easily be obtained.

Let us call the value of j in $M(r,j,k)$ the layer number. Using several applications of (3.3) we can produce a set of recursion relations that connect minors of layer 0 to a single top minor $M(r,r-1,2r-1)$. In the process we need the following minors in the example above.

$$\begin{array}{llll}
M(4,0,4) & & & \\
M(4,0,5) & M(4,1,5) & & \\
M(4,0,6) & & & \\
M(4,0,7) & M(4,1,7) & M(4,2,7) & M(4,3,7)
\end{array} \tag{3.7}$$

The ST1 minors of form $M(4,0,k)$ may be written in terms of various $\Delta(n,4)$ using the formula

$$M(r,0,k) = \Delta(n+r-k, r). \tag{3.8}$$

We think of the information as flowing from left to right in this table. Given those minors in layer 0, we may deduce the value of the minors in layer 1, twice using the first equation in Lemma 3.1. Thus $M(4,1,7)$ is related to $M(4,0,7)$ and $M(4,0,6)$, while $M(4,1,5)$ is related to $M(4,0,5)$ and $M(4,0,4)$. Next, using the second equation in Lemma 3.1, the layer 2 minor $M(4,2,7)$ is related to $M(4,1,7)$

in layer 1 and $M(4,0,5)$ in layer 0. Finally the minor $M(4,3,7)$ depends on the minor in layer 2 $M(4,2,7)$ and $M(4,1,5)$ in layer 1. All these relations also involve minor factors for $r=3$.

This tree-like structure is may be generalized to all cases.

Continuing with the $r=4$ example we note that the $r\times r$ minor $M(4,3,7)$ may be written

$$M(4,3,7)=\begin{vmatrix} b_n & b_{n+1} & \dots & b_{n+r-1} \\ b_{n-2} & b_{n-1} & \dots & b_{n+r-3} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \dots & b_{n-r+1} \end{vmatrix} \tag{3.9}$$

where we use only the odd rows from the matrix $B_{1,1}(n,4)$. This is the result of the first part of Stage 1. We repeat the above steps $r-1$ times , each time increasing the value of n by 1. The result will be 4 versions called $M(4,3,7)^{(1)}_\nu$, $\nu = n, n+1, n+2, n+3$, given by (3.9) with the appropriate value of ν, where the superscript stands for ST1. Here the minor $M(4,3,7)^{(1)}_n$ is what we previously denoted by $M(4,3,7)$. The determinants $\{M(4,3,7)^{(1)}_\nu\}$ may be thought of as the 4 minors with consecutive columns of the $r\times(2r-1)$ matrix (with $r=4$)

$$B_2(n,r)=\begin{bmatrix} b_n & b_{n+1} & \dots & b_{n+2r-2} \\ b_{n-2} & b_{n-1} & \dots & b_{n+2r-4} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+3} & \dots & b_n \end{bmatrix} \tag{3.10}$$

To proceed to the second stage (ST2) of the process we bring $B_2(n,r)$ into a form analgous to $B_{1,1}$ of (3.2) by transposing, reversing the order of the columns, and then reversing the order of the rows to produce the $(2r-1)\times r$ matrix

$$B_{2,1}(n,r)=\begin{bmatrix} b_n & b_{n+2} & \dots & b_{n+2r-2} \\ b_{n-1} & b_{n+1} & \dots & b_{n+2r-3} \\ \vdots & \vdots & & \vdots \\ b_{n-2r+2} & b_{n-2r+4} & \dots & b_n \end{bmatrix} \tag{3.11}$$

As in (3.2) we label the columns of $B_{2,1}(n,r)$ left to right by $1,2,\dots,r$ and the rows top to bottom by $1,2,\dots 2r-1$. From this point we repeat the steps after (3.2) to construct $r\times r$ minors from $B_{2,1}(n,r)$ that we now call $M(r,j,k)^{(2)}_n$ for ST2. The key point to note is that

$$M(r,0,2r-1-i)^{(2)}_n = M(r,r-1,2r-1)^{(1)}_{n+i},\ \ i=0,1,r-1. \tag{3.12}$$

This serves as a starting point for ST2. We use the relations in the generalization of table (3.7) for minors $M(r,j,k)^{(2)}_n$, which leads to an expression for $M(r,r-1,2r-1)^{(2)}_n = D(n/2,r)$ for even n. The combined process involves two stages with $2r-1$ layers in all.

3.2. **Application of Karlin identities - Order 2.** When using (3.3) to calculate the value of a minor $M(r,j,k)$ we find the required values of other such minors from previous steps in the process. We also need to know the values of some minors of the form $M(r-1,j',k')$. We suppose that they have been previously determined by an iterative process that begins with $r=1$. In that case a minor is a single normalized moment given by (2.5) or (2.6) . It is clear that $b_n > 0,\ n \geq -1$, and because $\Phi(u)$ is an even function, $b_{-2n} = 0,\ n = 1, 2, \ldots$. Some other values of $b_n,\ n < -2$ are also negative.

For $r = 2$ it is known that the kernel $K(u, v; 0)$ is sign-regular of type RR_2, so that Lemma 1.1 tells us that

$$\Delta(m-2,2) = b_{m-2}^2 - b_{m-1}b_{m-3} > 0,\ \ m \geq 0. \tag{3.13}$$

The sole first stage relation in this case is

$$\begin{aligned} &M(2,1,3)_\nu^{(1)} M(1,0,2)_\nu^{(1)} \\ &= M(2,0,2)_\nu^{(1)} M(1,1,3)_\nu^{(1)} + M(2,0,3)_\nu^{(1)} M(1,0,1)_\nu^{(1)} \end{aligned} \tag{3.14}$$

with $\nu = n, n+1$. It is easily checked using (3.8) that $M(1,0,2)_\nu^{(1)}$ and all the minors on the right hand side of (3.14) are positive for $n \geq 0$, so that $M(2,1,3)_\nu^{(1)}$ is also positive. The sole second stage relation is

$$\begin{aligned} &M(2,1,3)_n^{(2)} M(1,0,2)_n^{(2)} \\ &= M(2,0,2)_n^{(2)} M(1,1,3)_n^{(2)} + M(2,0,3)_n^{(2)} M(1,0,1)_n^{(2)}. \end{aligned} \tag{3.15}$$

Using (3.12) we find that (3.15) is equivalent to

$$\begin{aligned} &M(2,1,3)_n^{(2)} M(1,0,2)_n^{(2)} \\ &= M(2,1,2)_{n+1}^{(1)} M(1,1,3)_n^{(2)} + M(2,1,3)_n^{(1)} M(1,0,1)_n^{(2)}. \end{aligned} \tag{3.16}$$

The $M(1, j, k)$ minors in (3.16) are, in order of appearance, b_{n-1}, b_{n-2}, b_n. For $n \geq 2$ these quantities are always positive, so that the last 5 minors in (3.16) are positive, which means the the first minor $M(2,1,3)_n^{(2)} = b_n^2 - b_{n-2}b_{n+2} = d(n/2, 2) > 0$ when n is even. The same is true for $n = 0$ even though $b_{n-2} = 0$, but this case is trivial anyway.

Thus we have shown how to use the Karlin identities and the RR_2 sign-regularity of $K(u, v; m),\ m \geq 0$, to show that $D(n, 2) > 0,\ n = 0, 1, 2, \ldots$.

3.3. General structure of the identities.

The general form of the tree of minors such as (3.7) is given below for the case when r is even. To save space we have used the notation $p_j = r + j;\ u_j = r - j;\ t_j = 2r - j$ The symbol (j, k) is short for $M(r, j, k)$.

$$\begin{array}{llllllll} (0,r) \\ (0,p_1) & (1,p_1) \\ (0,p_2) \\ (0,p_3) & (1,p_3) & (2,p_3) & (3,p_3) \\ \vdots & & & \vdots \\ (0,t_4) \\ (0,t_3) & (1,t_3) & (2,t_3) & (3,t_3) & \ldots & (u_3,t_3) \\ (0,t_2) \\ (0,t_1) & (1,t_1) & (2,t_1) & (3,t_1) & \ldots & (u_3,t_1) & (u_2,t_1) & (u_1,t_1) \end{array} \tag{3.17}$$

In this case the rows are denoted by $r, r+1, \ldots, 2r-1$ and the columns (layers) by $0, 1, 2, \ldots, (r-1)$. The formulas in Lemma 3.1 describe the other two $r \times r$ minors that are connected with $M(r, j, k)$ and similarly the three $(r-1) \times (r-1)$ minors.

When r is odd the corresponding table is

$$
\begin{array}{llllllll}
(0,r) \\
(0,p_1) \\
(0,p_2) & (1,p_2) & (1,p_3) \\
(0,p_3) \\
(0,p_4) & (1,p_4) & (2,p_4) & (3,p_4) & (3,p_5) \\
\vdots & & & \vdots \\
(0,t_4) \\
(0,t_3) & (1,t_3) & (2,t_3) & (3,t_3) & \dots & (u_3,t_3) \\
(0,t_2) \\
(0,t_1) & (1,t_1) & (2,t_1) & (3,t_1) & \dots & (u_3,t_1) & (u_2,t_1) & (u_1,t_1)
\end{array}
\tag{3.18}
$$

As an example of the combined tree, for $r = 4$ we have

$$
\begin{array}{llllll}
(0,4)^{(1)}_{n+3} \\
(0,5)^{(1)}_{n+3} & (1,5)^{(1)}_{n+3} \\
(0,6)^{(1)}_{n+3} \\
(0,7)^{(1)}_{n+3} & (1,7)^{(1)}_{n+3} & (2,7)^{(1)}_{n+3} & \left[(3,7)^{(1)}_{n+3} = (0,4)^{(2)}_n\right] \\
\\
(0,4)^{(1)}_{n+2} \\
(0,5)^{(1)}_{n+2} & (1,5)^{(1)}_{n+2} \\
(0,6)^{(1)}_{n+2} \\
(0,7)^{(1)}_{n+2} & (1,7)^{(1)}_{n+2} & (2,7)^{(1)}_{n+2} & \left[(3,7)^{(1)}_{n+2} = (0,5)^{(2)}_n\right] & (1,5)^{(2)}_n \\
\\
(0,4)^{(1)}_{n+1} \\
(0,5)^{(1)}_{n+1} & (1,5)^{(1)}_{n+1} \\
(0,6)^{(1)}_{n+1} \\
(0,7)^{(1)}_{n+1} & (1,7)^{(1)}_{n+1} & (2,7)^{(1)}_{n+1} & \left[(3,7)^{(1)}_{n+1} = (0,6)^{(2)}_n\right] \\
\\
(0,4)^{(1)}_{n} \\
(0,5)^{(1)}_{n} & (1,5)^{(1)}_{n} \\
(0,6)^{(1)}_{n} \\
(0,7)^{(1)}_{n} & (1,7)^{(1)}_{n} & (2,7)^{(1)}_{n} & \left[(3,7)^{(1)}_{n} = (1,7)^{(2)}_n\right] & (2,7)^{(2)}_n & (3,7)^{(2)}_n
\end{array}
\tag{3.19}
$$

The formula (3.8) shows that

$$M(r,0,k)^{(1)}_{n+i} = M(r,0,k-i)^{(1)}_n, \tag{3.20}$$

so that all the entries in the first column of (3.19) (the inputs) are contained in the set

$$S = \{M(r,0,i)^{(1)}_n,\ \ i = 1, 2r-1\} \tag{3.21}$$

3.4. **Application of Karlin identities - Order** > 2**.** We continue to assume that Conjecture 1 is correct, and that function $m(r)$ for the RH is as given in our calculations [JN 2.1.8 slide 26], so that

(3.22)
$$\begin{array}{cccccccccc} r & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ m(r) & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 4 & 6 \end{array}$$

For given order r, the inputs to the combined tree are

$$M(r,0,i)_n^{(1)} = \Delta(n+r-i,r), \;\; i = 1, 2r-1 \tag{3.23}$$

so that from (2.19) $M(r,0,i)_n^{(1)} > 0$ if $(n+r-i)+r-(2r-1) = (n+1) \geq m(r)$. The lowest non-trivial case corresponds to $n = 2$ leading to $D(1,r)$, which means that (3.22) indicates that, under our assumptions, we can prove that $D(\nu,r) > 0, \;\; \nu \geq 0; \;\; r \leq 7$. This follows since all the minors of order r appearing in the tree will be positive. It should be possible to prove that the required minors of order $r-1$ will also be positive, as is the case for $r = 2$.

For values of $r > 7$ there will be at least one of the input minors for the combined tree that is negative. The number of negative minors will generally increase with r.

3.5. **Numerical results.** We have obtained some interesting numerical results using the above framework. They relate to the RH problem and also to two of the eight functions that have been postulated by Conrey and Ghosh [1] to behave, in many respects, in a fashion analogous to the RH. This last case is discussed in the next subsection.

For the RH we have used our previous calculations to obtain values of the single moments defined in (2.5) and (2.6), accurate, we believe, to several hundred figures. These lead to the various minors referred to in Sec. 3. Some results are given in the file 'rhcomb', and we explain the meaning of the various numbers in that file. Each line of the file refers to one entry in the combined tree of minors for order r, given in the second column, and repeated in Col. 11. Cols. 3 and 4 give j, k in $M(r,j,k)_n^{(1)}$, while Cols. 12, 13 relate to $M(r,j,k)_n^{(2)}$. The value of n appears in the first column. Col. 5 corresponds to the particular Stage 1 minor $M(r,j,k)_n^{(1)}$ of order r under consideration. Cols. 6,7 are the other two Stage 1 minors of order r in (3.3), while Cols. 8-10 correspond to $M(r-1,j,k)_n^{(1)}$ order $r-1$ Stage 1 minors. A positive minor is denoted by 1 and a negative minor by -1. Cols. 11-19 give the same information for Stage 2 minors.

The various blocks in the document correspond to different values of r. Stage 2 of the last line in each block, Col.14, gives $M(r,r-1,2r-1)_n^{(2)}$ which is equal to $D(n/2,r)$, defined in (2.4). It will be seen that all these entries are positive, in accord with the RH of (2.1) .

The entries in the first column of Stage 1, such as those shown in (3.8), may be written in terms of the matrix $\Delta(\eta,r)$ of (2.7) . The positivity of these for appropriate values of the arguments follows from Conjecture 1. In the case at hand, for $r \leq 7$, the file shows that all involved are positive. Using the relations of Lemma 3.1 it may be shown that all the minors in the file with $r \leq 7$ are therefore positive. This argument also requires study of the minors of order $r-1$.

However, the above argument no longer applies for $r \geq 8$ and $n = 2$. The file shows that, with $r = 8, n = 2$, several of the Stage 1 minors are negative, and so are two in Stage 2, namely $M(7,0,14)_2^{(2)}$ and $M(7,1,15)_2^{(2)}$, i.e. entries 4,5 in Stage 2, row (8 1 15). Nevertheless we find that all the other Stage 2 minors are positive.

In fact, each of the six Stage 2 minors corresponding to the succeeding 15 entries is positive.

It may be seen from the file that, surprisingly, a similar behavior is found for higher values of r, up to $r = 16$ in the file. For each of these r values there are 15 consecutive entries (an entry consists of six related minors), ending at the end of the list, for Stage 2, $n = 2$, with all minors positive. (We call such a sequence of entries, ending at the last entry in Stage 2, a positive (entry) chain.)The entry above the first of these 15 usually has at least one negative minor. We have done further calculations showing that the existence of this property holds for all r studied up to well above 40. Let us call the property the Constant Length Positive Entry Chain - CLPEC.

For $n = 4$ we find a similar behavior, but with 15 replaced by 29. Similar results have been obtained for even n up to 30. The observed values of the chain length $l(n)$ are

(3.24)

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$l(n)$	15	29	41	63	80	99	109	131	168	195	209	239	271	305	379

3.6. **Numerical results for Conrey-Ghosh functions.** In 1994 Conrey and Ghosh [1] described 8 functions for which they expected the analogue of the RH to hold. Here we concentrate on one of the functions (call this case CG1) which is obtained from the Riemann case by replacing $\Phi(u)$ by $F(u)$ given by

$$F_1(u) = \exp\left[\frac{u}{4} - \frac{\pi e^u}{12}\right] \prod_{m=1}^{\infty} (1 - e^{-2\pi m e^u}) \tag{3.25}$$

After expanding $F_1(u)$ as a power series, we follow exactly the same steps as were used on the RH. We have calculated some single moments and then analyzed the combined moment tree just as in Subsec. 3.5. See files 'CGk1comb' and 'CGk1comba'. We do not have as many of the moments as before, but there are enough to point to some trends. Our data are consistent with the existence of the CLPEC property in the CG1 situation. Our limited investigation indicates that the CLPEC applies to values of $n = 2 - 7$, with chain lengths $l(r)$ as follows.

(3.26)

n	2	3	4	5	6	7	8
$l(n)$	8	11	15	19	24	29	35

Note that the pattern seems to apply to both even and odd values of r, whereas the case $r = 3$ does not have the property in the RH.

The third Conrey-Ghosh function $CG3$ is given by

$$F_3(u) = \exp\left[\frac{3u}{4} - \frac{\pi e^u}{4}\right] \prod_{m=1}^{\infty} (1 - e^{-2\pi m e^u})^3 \tag{3.27}$$

A similar numerical investigation suggests that the CLPEC property also applies to the CG3 case, with chain lengths as follows.

(3.28)

n	2	4	6	8	10	12
$l(n)$	5	7	8	11	15	19

4. Conclusion

More calculations to explore the CLPEC phenomenon are indicated, but an plausible conjecture would be that the property holds for all r, and even $n \geq 2$, the only cases that are needed to prove the RH and its analogues. This conjecture could apply to the RH and all eight CG functions. The knowledge that the positive chains are of a constant length for given n might lead to an idea for a proof. It could well be helpful to find an explanation for the numbers in (**??**), relating them to the form of the tree. If it could be proved that each positive chain for all r and any fixed n has a constant positive length, then the RH and its analogues would follow immediately.

It should be mentioned that almost all the techniques used in the approach were described by the late Samuel Karlin.

References

1. J. B. Conrey and A. Ghosh, *Turán inequalities and zeros of Dirichlet series associated with certain cusp forms*, Trans. Amer. Math. Soc. **342** (1994), 407–419.
2. S. Karlin, *Total Positivity*, Stanford University Press, Stanford, CA, 1968.

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