WRONSKIANS AND THE RIEMANN HYPOTHESIS

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Abstract. We have previously proposed that the sign-regularity properties of a function $\Phi(t)$ related to the transform of the ζ-function could be the key to an analytic approach to the Riemann hypothesis. This report suggests that these properties might be proved by relating them to the sign-regularity of a simpler function based on the first term in the expansion of $\Phi(t)$, presumably a easier problem to solve.

1. INTRODUCTION

1.1. In previous reports[1] [5], [6], [7], we have proposed the outline of an analytic approach for resolving the Riemann Hypothesis (RH) that may avoid the difficulties faced by many conventional methods due to the apparenly chaotic nature of the distribution of the Riemann zeros. Our work has been stimulated by the ideas of Csordas, Norfolk and Varga [1], [2], who in turn made extensive use of the concept of sign-regularity described in detail in the book of Karlin [3].

The method [5, Sec. 2.1] depends on the fact that the RH is true if (and only if) a set of determinants $D(n, r)$ of order r satisfies

$$D(n, r) > 0, \;\; n = 0, 1, \ldots; \;\; r = 1, 2, \ldots. \tag{1.1}$$

The elements of the determinants are normalized double moments of the function $\Phi(t)$ (see (3.1) below). Following Karlin and colleagues [4] we deduce that at least an infinite subset of the determinants will have the required positivity, provided that $\Phi(t)$ and its cumulants $\Psi_m(t)$ (see Sec. 3.3 below) are sign-regular of type RR_r.

To complete a proof of the RH along these lines two main tasks must be accomplished.

(1) Prove that for any $r > 1$ there is an integer $m(r)$ such that the cumulant $\Psi_m(t)$ is sign-regular of type RR_r if $m \geq m(r)$. We speculate that, to be useful, $m(r)$ will have to satisfy a bound of the form $m(r) < 2r - \delta(r)$, with $\delta(r)$ relatively small and positive.

(2) Given the completion of the first task, the method of [4] will suffice to prove (1.1) for all values of r, provided that $n > \eta(r)$, where $\eta(r)$ increases without limit as r increases. We have designed the single moment method [7], again based on information in the Karlin book, which extends the above results, but there is still a gap for lower n and higher r that needs to be filled. This question is not addressed here.

To achieve the aims of Task 1 above, we suggest that the first step is to prove the conjecture proposed in Sec. 3.3, which relates to the sign-regularity properties

Date: August 2012.

[1]Unpublished reports by the author available at http://publish.uwo.ca/∼jnuttall

of the first term in the expansion (3.1) (i.e. $a_1(t)$) and its cumulants. We expect that this proof should be easier than the proof of Task 1.

We hope to then make use of the observation that, for increasing m, the value of $\Psi_m(t)$ (a positive function, which is the sum of positive components) is increasingly dominated by the contribution of the first term in (3.1), which might lead to the desired result.

In Sec. 2 we present some basic information about Wronskians that underlies the further developments. In Sec. 3 we first prove the required properties of the two-way Wronskians based on the function $a_1(t)$. There follows in Sec. 3.3 a numerical investigation of the Wronskians of the corresponding cumulants. In Sec. 4 we give some numerical results that support our speculations on the proof of Task 1.

2. WRONSKIANS

2.1. The notion of a Wronskian plays a major role in the discussion. Given a suitably analytic function $g(t)$, the corresponding two-way[2] Wronskian of order r is defined as

$$w(r,t) = \det \left| g^{(i+j-2)}(t) \right|_{i,j=1}^{r}, \quad t \geq 0, \tag{2.1}$$

where the superscript implies differentiation with respect to t.

It is said that the kernel $K(u,v) = g(u+v)$ is sign-regular of type RR_r provided that

$$\epsilon_p w(p,t) > 0, \;\; t \geq 0, \;\; p = 1, 2, \ldots, r. \tag{2.2}$$

Here the quantity $\epsilon(p) = (-1)^{p(p-1)/2}$.

2.2. J. B. Conrey (private communication) has kindly pointed out the following relation

$$w(r-1,t)w(r+1,t) = w(r,t)w^{(2)}(r,t) - [w^{(1)}(r,t)]^2. \tag{2.3}$$

A straightforward proof of a more general form of (2.3) is given by Karlin [3, p. 60].

Equation (2.3) means that, by recursion, we can calculate $w(r+1,t)$ given the forms of $w(r-1,t)$ and $w(r,t)$.

3. SIMPLIFIED VERSION OF $\Phi(t)$

3.1. In the formalism of [2, p 184] the key function $\Phi(t)$ given by [2, p 175] is written as

$$\Phi(t) = \sum_{n=1}^{\infty} a_n(t). \tag{3.1}$$

Omitting an inessential factor $\pi^{-1/4}$ from [2] we define

$$p_1(y) = 2y - 3, \tag{3.2}$$

and

$$a_1(t) = y(t)^{5/4} p_1(y(t)) \exp(-y(t)) \tag{3.3}$$

[2]The term 'double' has also been suggested by J. B. Conrey (private communication) and Vine and Dale [8]. However recent articles on dynamical systems have attributed a different meaning to the term 'double Wronskian', so, to avoid confusion, we use the term 'two-way'.

where

$$y(t) = \pi e^{4t}. \tag{3.4}$$

Similarly we write $a_n(t)$ as

$$a_n(t) = n^2 y(t)^{5/4} p_1(n^2 y(t)) \exp(-n^2 y(t)), \;\; n = 1, 2, \ldots \tag{3.5}$$

Our initial aim in connection with Task 1 is to study the Wronskian (2.1) when $g(z) = \Phi(t)$. For now we simplify $\Phi(t)$ by using only the first term (3.3) in (3.1). In addition we remark that Karlin [3, p. 99] has shown that the type of sign-regularity exhibited by the Wronskian is unchanged by the omission of the factor $y(t)^{5/4}$ in (3.3). Thus we study the properties of $w(r, t)$ when

$$g(t) = \exp(-y(t)) q_1(y(t)) = \exp(-y(t))(2y(t) - 3). \tag{3.6}$$

Since

$$\frac{d}{dt} y(t) = 4y(t) \tag{3.7}$$

it follows as in [2, p 184] that

$$g^{(k)}(t) = \exp(-y(t)) q_{k+1}(y(t)), \;\; k = 0, 1, \ldots, \tag{3.8}$$

where

$$q_{k+1}(y) = 4y(q_k'(y) - q_k(y)). \tag{3.9}$$

In (3.9) the derivative is taken with respect to y. We see that $q_k(y)$ is a polynomial in y of degree k.

3.2. With our assumptions about $g(t)$ it follows from (2.1) that we may write

$$w(r, t) = \exp(-r y(t)) W_r(y) \tag{3.10}$$

where

$$W_r\,(y) = \begin{vmatrix} q_1(y) & q_2(y) & \ldots & q_r(y) \\ q_2(y) & q_3(y) & \ldots & q_{r+1}(y) \\ \vdots & \vdots & & \vdots \\ q_r(y) & q_{r+1}(y) & \ldots & q_{2r-1}(y) \end{vmatrix}, \tag{3.11}$$

and $y = y(t)$.

Clearly $W_r(y)$ is a polynomial in y of degree no higher than $1+3+5+\ldots+2r-1 = r^2$. It turns out that

- The degree of $W_r(y)$ is $r(r + 1)/2$.
- The polynomial $W_r(y)$ has a factor $y^{r(r-1)/2}$.
- Thus there are at most $r + 1$ non-zero coefficients in $W_r(y)$.

The second statement was proved in [5, Sec. 6.3], but all these statements may be proved more easily from (2.3). The first 15 figures of the $W_r(y)$ polynomial coefficients are listed in [5, (6.4), p. 16] for $r = 2, \ldots, 7$.

High precision numerical calculations of the zeros of the polynomials $W_r(y)$, $r = 2, \ldots, 30$ are consistent with the following general properties

- If r is even then $W_r(y)$ has no real zeros.
- If r is odd then $W_r(y)$ has one real zero, say $z(r)$, and $z(r) > 0$.
- If $r_2 > r_1$ then $z(r_2) > z(r_1)$.
- The sign of the highest coefficient in $W_r(y)$ is ϵ_r.

Some values of $z(r)$ are listed below in (3.12).

(3.12)

r	$z(r)$
1	1.5
3	2.1149807662939701427905564202948725204752413086 2102
5	2.715643269938116435585250717972025060259732318 53385
7	3.3078877851270639134618866104676576458600388600 3658
9	3.8945394772464520720838644553756632593522473795 0357
11	4.4771849909899170762811312208574767840553648647 9801
13	5.0568098078220175072956957140347203834312381876 9292
15	5.6340703487504748789780029489553120341026545512 7067
17	6.2094269969165542013620442300600502093859123627 1743

Since $t \geq 0$ means that $y \geq \pi = 3.14159\ldots$, the above properties show that $g(t)$ is sign-regular of type RR_6, because

$$\epsilon_r w(r,t) > 0; \;\; r \leq 6, \;\; t \geq 0. \tag{3.13}$$

The same applies when $g(t)$ is chosen to be $a_1(t)$. Note that our previous calculations and analysis imply that $\Phi(t)$ of (3.1) is sign-regular of type RR_4 (but not RR_5), a weaker condition.

3.3. To deal with sign-regularity of $a_1(t)$ for $r > 6$ we suggest the use of the technique introduced in [5]. The cumulants $\{h_m(t)\}$ of a function $h(t)$ are formed by (see [5, p. 10])

$$h_m(t) = \int_t^\infty du\, h_{m-1}(u), \quad m \geq 1, \tag{3.14}$$

and

$$h_0(t) = h(t). \tag{3.15}$$

The definition of the cumulant shows that

$$h_m^{(1)}(t) = -h_{m-1}(t), \quad m \geq 1, \tag{3.16}$$

where the superscript means differentiation with respect to t. In [5, p. 12] we note that an explicit formula for $h_m(t)$ is

$$h_m(t) = \frac{1}{\Gamma(m)} \int_t^\infty du\, h(u)(u-t)^{m-1}, \quad m \geq 1. \tag{3.17}$$

Now define

$$h(t) = a_1(t) = y(t)^{5/4}(2(y(t)-3)\exp(-y(t)), \tag{3.18}$$

and let the cumulants of $h(t)$ be $h_m(t)$, $m = 1, 2, \ldots$.

As in (2.1) define the two-way Wronskian $w(r,m;t)$ by

$$w(r,m;t) = \det \left| h_m^{(i+j-2)}(t) \right|_{i,j=1}^{r}, \;\; t \geq 0; \;\; m = 0, 1, 2, \ldots \tag{3.19}$$

Numerical calculations of the values of $w(r,m;t)$, $r = 2, \ldots, 30$ are consistent with the following general properties

- If r is even then $\epsilon_r w(r,m;t) > 0$, $t \geq 0$.
- If r is odd then $\epsilon_r w(r,m;t)$ has up to one real zero for $t \geq 0$, say at $z(r,m)$.
  1. If also $\epsilon_r w(r,m;t) > 0$ then $t > z(r,m)$
  2. If also $\epsilon_r w(r,m;t) < 0$ then $t < z(r,m)$
  3. $z(r,m_2) < z(r,m_1)$, $m_2 > m_1$

(4) If $z(r,m_1) > \pi$, $\exists\, m_2 > m_1$ such that $z(r,m_2) < \pi$

(5) $z(r_2,m) > z(r_1,m),\ \ r_2 > r_1$

The file 'wrona' provides information about the Wronskians $w(r,m;t),\ \ r = 2,\ldots,29;\ \ m = 0,\ldots,10$. It will be seen that the data has the properties listed above.

On the basis of the samples, for a given r the data in the file show that there is a number $\mu(r)$ with the following properties.

- If $p = 1,\ldots,r$ and $m \geq \mu(r)$ then $\epsilon_r w(p,m;t) > 0,\ \ t \geq 0$
- For no lower value of $\mu(r)$ is the above true.

For example, if we choose $r = 8$, then the file shows that $\mu(r) = 1$. This is in spite of the fact that $\epsilon_r w(8,1;t) > 0,\ \ t \geq 0$, but because of the values of $\epsilon_r w(7,1;t)$, which is negative for certain $t > 0$. Table 1 below lists the values of $\mu(r)$ for $r \leq 98$ according to our calculations.

For comparison, the corresponding information for the cumulants of $\Phi(t)$ is given in Table 2, where we list values of $m(r)$, the lower bound of m for $K(u,v;m) = \Psi_m(u+v)$ to be RR_r. Note that $\mu(r)$ increases at a considerably lower rate than $m(r)$.

The remarks in Sec. 3.3 may be sumarized in

CONJECTURE 1. For the Wronskian $w(r,m;t)$ of (3.19) based on the cumulant of $h(t) = a_1(t)$, there exists a lowest integer $\mu(r)$ such that $w(p,\mu(r);t) > 0,\ \ t \geq 0;\ \ p = 1,\ldots,r;\ \ r \geq 1$. We may say that $K_h(u,v;m) = h(u+v)$ is RR_r.

This conjecture is equivalent to the conjecture in [5, Sec. 4.3] (i.e. Task 1) with $\Phi(t)$ replaced by $h(t)$, the first term in the series (3.1).

4. COMPARISON OF WRONSKIANS

4.1. In the file 'wronb' we provide information about two sets of two-way Wronskians for $r = 4,\ldots,15;\ \ m = 0,\ldots,20$. In the first column headed 'complete' we list

$$\overline{w}(r,m;t) = \det\left|\Phi_m^{(i+j-2)}(t)\right|_{i,j=1}^{r},\quad t \geq 0;\ \ m = 0,1,2,\ldots, \tag{4.1}$$

i.e. the Wronskians directly needed for the RH case. In the second column headed 'first term' we list $w(r,m;t)$ from (3.19), and the third column contains the ratio of the first two.

Several trends appear in the data in file 'wronb'. These trends continue in further calculations we have made.

1. For all values of r,m the ratio of the two Wronskians approaches 1 as t increases.
2. For given r and $t = 0$ the ratio approaches 1 as m increases past $m(r)$.
3. For values of $t > 0$ the trend of (2) continues at an increasing rate.
4. As r increases the above behavior continues but the rate of the changes decreases.

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mu(r)$	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2
r	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$\mu(r)$	3	3	3	3	4	4	5	5	5	5	6	6	6	6	7
r	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
$\mu(r)$	7	7	7	8	8	8	8	9	9	10	10	10	10	11	11
r	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
$\mu(r)$	11	11	12	12	13	13	13	13	14	14	15	15	15	15	16
r	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
$\mu(r)$	16	16	16	17	17	18	18	18	18	19	19	20	20	20	20
r	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
$\mu(r)$	21	21	21	21	22	22	23	23	23	23	24	24	25	25	25
r	92	93	94	95	96	97	98								
$\mu(r)$	25	26	26	27	27	27	27								

TABLE 1. Values of $\mu(r)$, the lower bound of m for $K_h(u,v;m) = h(u+v)$ to be RR_r.

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(r)$	0	0	0	1	1	1	2	4	6	7	7	7	9	11	13
r	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$m(r)$	15	15	15	16	18	20	22	24	24	24	25	27	29	31	33
r	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
$m(r)$	34	34	34	36	38	40	42	44	45	45	45	47	49	51	53
r	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
$m(r)$	55	57	57	57	58	60	62	64	66	68	69	69	69	71	73
r	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
$m(r)$	75	77	79	81	81	81	82	84	86	88	90	92	94	94	94
r	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
$m(r)$	95	97	99	101	103	105	107	108	108	109	111	113	115	117	119
r	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106
$m(r)$	121	122	122	122	124	126	128	130	132	134	136	137	137	137	139
r	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121
$m(r)$	141	143	145	147	149	151	151	151	152	154	156	158	160	162	164
r	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136
$m(r)$	166	166	166	168	169	172	173	175	177	179	181	182	182	183	185
r	137	138	139	140	141	142									
$m(r)$	187	189	191	193	195	197									

TABLE 2. Values of $m(r)$, the lower bound of m for $K(u,v;m) = \Psi_m(u+v)$ to be RR_r.

References

1. G. Csordas, T. S. Norfolk, and R. S. Varga, *The Riemann hypothesis and the Turán inequalities*, Trans. Amer. Math. Soc. **296** (1986), 521–541.
2. G. Csordas and R. S. Varga, *Moment inequalities and the Riemann hypothesis*, Constr. Approx. **4** (1988), 175–198.
3. S. Karlin, *Total Positivity*, Stanford University Press, Stanford, CA, 1968.
4. S. Karlin, F. Proschan, and R. E. Barlow, *Moment inequalities of Pólya frequency functions*, Pacific J. Math. **11** (1961), 1023–1033.
5. J. Nuttall, *Determinantal inequalities and the Riemann hypothesis*, arXiv:math/1111.1128, 2011.
6. ———, *Lecture notes on the Riemann hypothesis*, unpublished, 2011.
7. ———, *The single moment method in the determinantal approach to the Riemann hypothesis*, unpublished, 2011.
8. R. Vein and P. Dale, *Determinants and their applications in mathematical physics*, Springer, New York, 1999.

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