What is Summability Theory?
by Ken Roberts

The other day, a new acquaintance asked me, “What is summability theory?” Fair question. Although it’s been over three decades since I worked on summability theory, a person should always remain able to describe his first field of study upon inquiry.

First answer (joke): It’s a field of study whose name was devised so that one can state he is a student of summability!

Well, jokes aside, summability theory has quite a lot of interesting content. For a mathematician, at least. And, I suspect though I don’t really know, it has concepts and methods that might also be useful for physicists and philosophers, to provide some solid ground for calculations done in physics, and to provide an interesting example of definitional flexibility combined with rigour, for philosophy.

Summability theory fits within the broader mathematical topic grouping called analysis. It also has links to number theory. G. H. Hardy and J. E. Littlewood are perhaps the most well-known summabilists. The classic explanation of summability, and still the best, is Hardy’s book “Divergent Series”.

Suppose we have a series $a_1 + a_2 + ...$ which may or may not converge. Or equivalently, perhaps we have a sequence of partial sums $s_1, s_2, ...$ where $s_1 = a_1$, $s_2 = a_1 + a_2$, and so on.

Or, in a continuous analogue, we might have a function $S$ defined by an integral $S(T) = \int_1^T a(t)dt$. I’ll stick with series and sequences for the present. They’re a bit easier to think about; though summability of integrals has its own uniquely interesting features.

It is standard to say that the series $a_1 + a_2 + ...$ converges to the sum $s$ if its sequence of partial sums $s_1, s_2, ...$ approaches the limit $s$. That
simple definition resolves a deal of confusion about infinite processes, typified by some of Zeno’s paradoxes. But, that definition is not the only way to proceed.

Suppose, for instance, we are interested in the series $1 - 1 + 1 - 1 + 1 ...$. That series might arise from a description of the position of a ball which bounces back and forth between two walls. Or it might arise from the series $1 - x + x^2 - x^3 + ...$, say for $x$ between 0 and 1, and imagining what would be the limit of the series as $x$ tends to 1 (from the left).

One approach to summing $1 - 1 + 1 ...$ would be to look at the sequence of partial sums, namely 1, 0, 1, ..., and consider the average value of the first $n$ elements of the sequence. That average tends to $\frac{1}{2}$ as $n \to \infty$.

More precisely, the partial sum of the first $2k$ terms is 0, and the partial sum of the first $2k + 1$ terms is 1. The average of the first $2n$ partial sums is then $\frac{1}{2}$, and the average of the first $2n + 1$ partial sums is $\frac{n+1}{2n+1}$. Although the sequence of partial sums does not converge, the sequence of averages of the partial sums does converge.

An alternative approach would be to notice that for $x$ less than 1, $1 - x + x^2 ...$ equals $\frac{1}{1+x}$ and the limit of the latter expression, as $x$ tends to 1, is $\frac{1}{2}$.

In either approach, $\frac{1}{2}$ seems to be the “right” answer if we wish to assign a sum to $1 - 1 + 1 ....$

The above procedures can be formalized into two methods of summation, called the Cesàro and Abel methods. And of course we have ordinary convergence, which is another method of summation.

The Cesàro method considers the sequence $s_1, s_2, ...$ of partial sums of the series $a_1 + a_2 + ..., and defines another sequence $t_1, t_2, ...$ where each $t_n$ is the average of the first $n$ members of the $s_1, s_2, ...$ sequence. If
$t_n \to s$ as $n \to \infty$, then the series $a_1 + a_2 + ...$ is said to be Cesàro summable to $s$.

It turns out that an ordinarily convergent series is also Cesàro summable, and to the same sum value.

By repeating the Cesàro process of averaging the previous sequence of partial sums or averages, we obtain a scale of summability methods, Cesàro summation repeated $n$ times being called the $C_n$ method.\footnote{Technically, what I have here called the Cesàro method $C_n$ should be called the Hölder method $H_n$, which is equivalent to the $C_n$ method — see Hardy, pg 103. The Cesàro method is defined a bit differently, in such a way that some useful properties from more general methods are available to expedite study of the Cesàro methods.} Ordinary convergence is considered to be the $C_0$ method of summation of a series. Each $C_n$ method is a bit more powerful that its predecessor, that is, it can calculate a sum for some series which does not converge with a $C_k$ method for lesser $k$.

In contrast, the Abel method is typified by the power series example. It does not use a sequence of partial sums of the series, but considers all the series terms at once, assigning a diminishing weighting to the terms further out in the series.

Given the series $a_1 + a_2 + ...$, and a variable $x$ between 0 and 1, define a function $f(x)$ by $f(x) = a_1 + a_2 x + a_3 x^2 + ...$ and consider the limit of $f(x)$ as $x \to 1$ (from the left, ie values of $x$ less than 1). If $f(x) \to s$ as $x \to 1$, then the series $a_1 + a_2 + ...$ is said to be Abel summable to $s$. As with Cesàro summability, any ordinarily convergent series is Abel summable to its ordinary sum.

Moreover, the Abel method is more powerful than any of the $C_n$ Cesàro methods. It will sum any series which is $C_n$ summable, and furthermore there are series which are not $C_n$ summable for any $n$, but are Abel summable.

There is a nice scale of methods from $C_0$ (ordinary) summability, up thru the various $C_n$ Cesàro methods, capped by Abel summability.
And that’s not all. The range of Cesàro methods can be filled in by defining fractional Cesàro summability, $C_\lambda$ for any $\lambda > 0$. There are other methods of summability weaker than any $C_\lambda$ for positive $\lambda$, yet more powerful than $C_0$. There are methods more powerful than Abel summation. And there are methods which are not comparable; one method may sum some types of series which another cannot, and vice versa.

What’s all this useful for? Why do we study summability? There are several reasons.

- The various summability methods are interesting in themselves. Their taxonomy, their comparability and relative strength, the delineation of series which can be summed by one method but not by another.

- The efficient summation of slowly convergent series. Some methods are very rapid to determine a sum of a slowly convergent (perhaps irregularly oscillating) series. Some methods work better with series which arise in particular problem areas.

- Understanding series by using various methods as reagents to test them. Instead of using a powerful method to determine a sum, use a scale of weaker methods (like pH test strips) to characterize the behaviour of a series.

- Provide a solid ground for symbolic manipulations. Sometimes a calculation is done with a disregard for the risk that some intermediate stages may introduce infinities, or other non-convergent expressions. The infinities may, in the end result, cancel or become peripheral to the main goal of the derivation. For example, dividing an expression by $1 - x$, expanding that as $1 + x + x^2 \ldots$, collecting terms by rearrangement, and taking some limits.

Such approaches can produce interesting conclusions, but are the results to be trusted? Or have we obtained a false positive, comparable to the “proofs” that $1 = 2$ which can be obtained via a disguised division by zero? An understanding of summability methods can provide
the rationale to conclude that one has indeed found solid ground with the result, regardless of the necessity for treading water during the intermediate computations.

Abelian and Tauberian: Many results relating summability methods are spoke of as being either Abelian or Tauberian. That is an important distinction. There are of course many other results, neither Abelian nor Tauberian, but the key idea is the following:

Abelian results relate the summability of a series by method #1 to its summability by method #2. For example, the theorem that any series which is $C_n$ Cesàro summable for some $n$ is also Abel summable, is the prototypical Abelian result. An Abelian result says something like “method #2 is more powerful than method #1”.

Tauberian results work in the opposite direction, allowing us to reason that a series which is summable by a powerful method #2, and satisfies some additional conditions, is also summable by a less powerful method #1. The extra conditions on the series can be quite moderate. Tauberian theorems can be very interesting. In addition to the Hardy book, the 1932 paper by Norbert Wiener deserves a mention. (Annals of Math, vol 33, 1932, pp 1-100; reprinted by MIT Press in “Generalized Harmonic Analysis and Tauberian Theorems”).

Here’s an example of a problem that might be amenable to a Tauberian approach: Dirichlet’s number-theoretic result that, if $b$ and $c$ are relatively prime integers, than the arithmetic progression $b + cn$, for $n = 0, 1, 2, ...$, contains an infinity of prime numbers.

The basic idea is that, fixing $c$ and letting $b$ range over all the values between 1 and $c − 1$ which are relatively prime to $c$, one covers all the candidate prime numbers, which are an infinite set. As there’s nothing “special” about any particular $b$ value, provided that it is relatively prime to $c$, the progressions $b_1 + cn$ and $b_2 + cn$ presumably contain approximately the same density of primes, ie both have an infinity of primes.
The difficulty, of course, is how to capture the idea that “there’s nothing special” about the choice of offset $b$. That’s where summability and Tauberian ideas come in.

Define a series $S_b$ involving the primes in a particular one of those arithmetic progressions. For two different instances $S_{b_1}$ and $S_{b_2}$ the series will differ. But they might, for instance, represent a comparable density of primes.

Suppose we can hit the different series with a summability method which mashes the differences between the series sufficiently that we realize that the transformed series all have comparable densities, yet is not so powerful that we cannot back out via a Tauberian theorem to some conclusion about the properties of the original series. That could be a nice way of approaching the problem. The choice of method is determined by the problem’s requirements.

Being able to reason backwards from “the instances have some property on average after smoothing out differences between instances”, and “the instances are not specially different from one another, in relation to the smoothing method utilized”, to “hence particular instances individually have the property”; that is the Tauberian idea.

Something similar can be done in chemistry. We might have a variety of samples, perhaps obtained from plants, and we wonder if they are the same substance? We might grind up the various samples individually, and process them by some method which does not destroy their chemical properties, and assay the composition of the mashed-up samples. If we see the mashed-up samples have identical compositions, and we know that our mashing-up and assay method was sufficiently gentle that it would not destroy or increase the constituents we are interested in, then we can conclude the original samples were also identical for those constituents.

Well I’m not a chemist, and that description was quite naive, I’m sure. But so was the mathematical description! It does illustrate, however, how different sciences can utilize similar methodologies.
That’s all arm-waving. I just wanted to suggest the possibilities that thinking in terms of summability techniques can provide.

Summability theory draws upon a variety of concepts and theorems from functional analysis. As well, it has lots of heavy-duty symbolic computations. Those features are a part of its appeal for me and others who study summability.

For instance, my thesis concerned characterizing multiplier functions which have a certain property in relation to integrals evaluated with an extended Cesàro type of method. It is relatively easy to derive, using functional analytic methods, some necessary properties which those multiplier functions have to possess. However, proving that those properties are also sufficient to make a multiplier function “work” with the Cesàro method in question, is a complicated process. Only a person who really enjoys calculation would attempt it, but for such a person, the task can be a genuine pleasure.

One interesting sidenote which came about as a result of my thesis work, was in connection with fractional integration. Just as there are $C_\lambda$ Cesàro methods for series, there are $C_\lambda$ methods for evaluating integrals, where $\lambda$ is a positive real number; they involve a concept of fractional integration. With ordinary integrals, a function $a(t)$ whose integral $S(T) = \int_0^T a(t)dt$ is equal to a constant, must itself equal zero for almost all $t$. However, with fractional integration, and $\lambda$ between 0 and 1, there can be a nontrivial $a(t)$ whose integral $S(T)$ is constant for all $T$. Namely, $a(t) = ct^{-\lambda}$ for any constant $c$.

Ultimately, like most pure mathematics, summability theory provides a context within which one can enjoyably tinker with ideas.

There’s much more to summability theory than I’ve indicated above, but that should give you some flavour of the subject.