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SHORT COMMUNICATION

Diversity partitions in 3-way sorting: functions, Venn diagram mappings, typical additive series, and examples

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Abstract: In this paper, logarithmic functions are described based on which the total diversity of a collection can be partitioned into components specific to factorial effects. The standard statistical *modus operandi* of testing hypotheses in a factorial design is applied, only the test criteria changed. The factor identities are chosen according to stated hypothesis, observations are made and sorted, the associated entropy and information quantities are calculated, and probabilistic tests of significance are performed regarding main effects and interaction terms. The basics are presented in the main text. The partition functions, their Venn diagram mappings, and a complete printout from the application program DIVPART.EXE are collected in a separate Appendix accessible with this article on the publisher's website.

Introduction

Logarithm based entropy and information quantities are the diversity scalars. Entropy is scalar for main effects and information for interaction effects. When diversity is analysed based on the main effects/interaction model, the factors associated with the effects enter the analysis as the sorting criteria. Thus, the basis of the analysis is a contingency vector (sorting observations by the states of one factor), a contingency table (sorting by the states of two factors) or a solid (sorting by the states of three or more factors). In this communication, I consider the case of three factors, formulate in all 34 diversity partitions for main effects, conditional effects, interactions, and conditional interactions, and give Venn diagram mappings to guide the choice of additive sequences of the partition functions. I point out the difference between proper as opposed to improper sums of partitions, give a worked numerical example, and introduce the application program DIVPART.EXE, which performs the rather tedious computations automatically. The first part of this communication introduces the basics functions. This is followed by presentation of a numerical example in the second part, and a discussion of salient points in the third. Key references that can lead readers to others depending on the detail of interest conclude the main text. An appendix available through the Internet is attached to the paper. It contains sections on elementary symbolism and some numerical identities, practice run of DIVPART.EXE, including complete set of results printed by the program, and list of partition functions with their Venn diagram mappings. DIVPART.EXE and sample data file are downloadable from the author's webpage http://ecoqua.ecologia.ufrgs.br/~lorloci/ at the link of the author's downloadable files in the DIVPART folder.

Basic entropy and information functions

We designate as *A, B, C* the three factors whose numbers of states are *a, b, c* by which the elementary observations are sorted. An elementary observation in the example given is a three-valued vector that identifies the climax type *(A),* functional type *(B),* and floristic domain or flora of origin *(C)* of a given species. A typical case taken from Orlóci and He (1996) is

Abies nephrolepis 111.

This is one of the 646 species in Li's (1993) list from the Heilong Jiang flora in China which we coded 111, implying a climatic climax tree from the Boreal-Subalpine flora. The 3-dimensional sorting *(AxBxC* design*)* shown in Table 1 is based on this code.

Rényi's (1961) generalised entropy of order α serves as a scalar for the entropy portion of total diversity in the *AxBxC* design. This function is written as

$$H(A,B,C) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} p_{ijk}^{\alpha};\ p_{ijk} = \frac{f_{ijk}}{f_{...}},$$

The basic definition of the information portion of diversity is scaled by Rényi's (1961) information of order α

$$I(A,B,C) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}};\ q_{ijk} = \frac{f_{ij.}\, f_{i.k}\, f_{.jk}}{f_{i..}\, f_{.j.}\, f_{..k}}$$

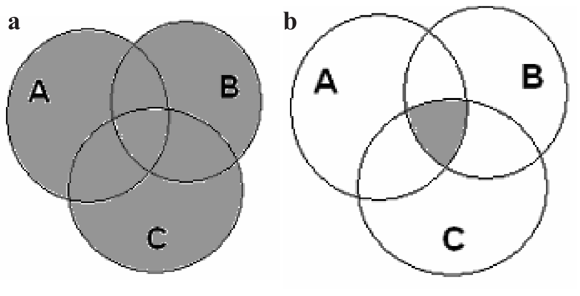
The partition functions corresponding to cases in between are listed in the Appendix. The following are the symbolic data elements:

a,b,c — the number of states of factors *A, B, C* based on which the observations are sorted;

f_{ijk} — a characteristic element in the three-way solid, a count or frequency. The version of DIVPAR.EXE presented in the paper takes the data in the f_{ijk} form;

$f_{ij.}, f_{i.k}, f_{.jk}, f_{i..}, f_{.j.}, f_{..k}, f_{...}$ — marginal totals. Dots in the subscripts indicate summation over the specific dimension whose subscript is replaced by a dot.

Inspection of the Venn diagrams in Figure 1a,b reveals that *H(A,B,C)* represents the total diversity (entropy in the design) attributable to the three-factor effects, and that *I(ABC)* represents the most specific term in the three-factor interaction partitions called shared or mutual information. The letters *A, B, C* in the Venn diagrams identify the area segments proportional to the factor effects. Perfect proportionality is implied that requires α in all cases to approach the limit 1. In practice, this requirement can be approximated rather closely by setting α equal to almost 1, say 0.99999. The "elegant" solution involves rewriting the equations for the case of α approaching 1 (Orlóci 1991), but this solution would leave the algorithm applicable only to order one of α.



Total entropy 3-way interaction

Figure 1. A Venn diagram representation of total entropy *H(A,B,C)* (**a**) and the 3-way interaction information *I(ABC)* (**b**). *A, B* and *C* are factor identities from which the diversity effect is assumed to derive, depending in magnitude on the distribution of frequencies over the factors' states *a, b, c* by which the observations are sorted. The diversity partition functions and numerical examples are given in the main text and in the Appendix.

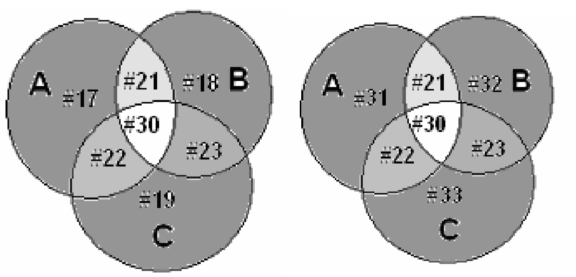
Example

The source data set is Li's (1993). The species of the original data set were sorted according to 3 climax types *(A)*, 5 functional types *(B)* and 3 flora types *(C)*. Table 1 contains the joint frequencies. The numerical values of selected partitions are given in Table 2 and for all 34 partitions in Table 3. The computations were performed by application program DIVPART.EXE. The Appendix contains a training run of DIVPART.EXE including the complete printout. The source code of DIVPART.EXE is in Kemény and Kurtz's *Tru Basic*.

What we can see in the numbers of Table 2 is the extreme influence of inheritance as a contributor to diversity, outweighing by far the effect of the other sorting factors on the regional scale. But when taking the sorting factors other than inheritance, flora element dominates. To be noted is that the sum on line 2 of Table 2, which is an imperfect sum because the terms repeat portions of the interaction information to the tune of 0.304742 bits in the manner of Figure 2 of the elementary partitions. The sum on line 6 is a perfect sum in the sense of

Joint = Conditional + Interaction + Mutual =
= 3.807155 + 0.304742 + 0.055280 =
= Principal marginal – Interaction – Mutual =
= 4.471926 – 0.304742 = 4.1672

Table 3 shows results for the 34 partition functions over six orders of α. Order one is the basis of perfect sums



Entropy Information

Figure 2. Venn diagram mappings of elementary partitions by type from which all partitions can be constructed. The partition functions are listed in the Appendix with function number labels used in this graph. Numerical results are given in the main text. The Appendix contains the complete list of partition functions and Venn diagram mappings.

Table 1. Three-way sorting of 646 species recorded in Heilongjiang flora of China. The source list is Li's (1993). The sorting is ours (Orlóci and He 1996, Orlóci et al. 2002). The numerical results are summarised in Table 2 and Table 3. The Appendix contains the partition functions, Venn diagram mappings, and the sample run of the application program DIVPART.EXE.

Climax type (A) i	Functional type (B) j	Flora (C) k		
		Boreal - Subalpine	Montane	Forest steppe
1. Climatic	1. Trees	4	5	8
	2. Shrubs	10	58	18
	3. Geophytes	9	79	2
	4. Other herbs	11	131	11
	5. Chamaephytes	7	10	2
2. Edaphic	1. Trees	1	1	0
	2. Shrubs	1	17	8
	3. Geophytes	1	58	2
	4. Other herbs	1	22	3
	5. Chamaephytes	3	0	0
3. Serial	1. Trees	0	0	1
	2. Shrubs	3	5	7
	3. Geophytes	0	16	13
	4. Other herbs	2	46	39
	5. Chamaephytes	7	18	5

Table 2. Selected diversity partitions for the sorting model adapted in Table 1. Partitions functions are given in the Appendix. Numerical results are listed for 34 partitions in Table 3. The percentages in the table are relative to total diversity. All values are in bits. See the main text for further details regarding rounding errors and additivity.

Sources of diversity #	Entropy or information of order one bits	% *
1. Inheritance ($\log_2 646$)**	9.333155	223.9714
2. Principal marginal	4.471926	107.3145
2.1 Functional type H(A)	1.413781	33.92705
2.2 Flora element H(B)	1.950734	46.81252
2.3 Climax type H(C)	1.107411	26.57497
3. Conditional (equivocation)	3.807155	91.36178
3.1 Functional type H(A|B,C)	1.182195	28.36959
3.2 Flora element H(B|A,C)	1.71701	41.20376
3.3 Climax type H(C|B,A)	0.90795	21.78843
4. Interaction I(A;B;C)	0.304742	7.313012
5. Mutual I(ABC)	0.05528	1.326576
6. Joint H(A,B,C)	4.16712	100

* Percent of *H(A,B,C)*, ** 646 is the number of species in the Li (1993) records.

when the logic of the Venn diagrams in Figure 2 is followed. Higher order partitions are noted for increased stability in the sense discussed by Orlóci et al. (2002). The proof of a sum being perfect or imperfect should come from the logic of the algebra rather than the numerical values. The numerical values are not completely reliable since they are affected by rounding errors that can be quite substantial.

Table 3. Numerical values of entropy and information partitions, corresponding to the partition functions listed in the Appendix. All values are in bits. This table shows results over six orders of α. Zero order results register the state richness of the sorting factors individually and jointly. Order one is the basis of perfect sums when the logic of the Venn diagrams in Figure 2 is followed. Higher order partitions are noted for increased stability (Orlóci et al. 2002). See the main text for further details regarding rounding errors and additivity.

Specific name	$\alpha = 0$	0.99999	2	3	4	5
1 H(A):	1.584962	1.413781	1.262624	1.149569	1.071864	1.019344
2 H(B):	2.321928	1.950734	1.765196	1.660521	1.591493	1.541643
3 H(C):	1.584962	1.107411	.824519	.690027	.62312	.585807
4 H(A|B):	1.584962	1.308497	1.147351	1.056232	1.000059	.962254
5 H(A|C):	1.584962	1.342761	1.153771	1.03064	.954617	.90653
6 H(B|A):	2.321928	1.84545	1.624049	1.497602	1.416013	1.359581
7 H(B|C):	2.321928	1.877575	1.693656	1.597911	1.538983	1.498668
8 H(C|A):	1.584962	1.03639	.770032	.665525	.618101	.591985
9 H(C|B):	1.584962	1.034252	.785253	.678228	.62341	.591084
10 H(A,B,C):	5.491853	4.167182	3.508687	3.166289	2.964988	2.833216
11 H(A,B):	3.90689	3.259231	2.940917	2.749465	2.621845	2.529873
12 H(A,C):	3.169925	2.450172	1.988109	1.722564	1.57223	1.482679
13 H(B,C):	3.90689	2.984987	2.464933	2.229622	2.106492	2.031434
14 H(A,B|C):	3.90689	3.059771	2.698942	2.483801	2.34555	2.251362
15 H(A,C|B):	3.169925	2.216447	1.798411	1.587019	1.470824	1.400582
16 H(B,C|A):	3.90689	2.753401	2.229741	1.998355	1.874227	1.796814
17 H(A|B,C):	1.584962	1.182195	.9874	.886004	.829188	.794158
18 H(B|A,C):	2.321928	1.71701	1.501884	1.381794	1.305588	1.253971
19 H(C|A,B):	1.584962	.90795	.702319	.627353	.590691	.569059
20 I(A;B;C):	0	.304742	.541518	.777559	1.00347	1.204176
21 I(AB):	0	.105283	.201785	.293636	.38164	.464268
22 I(AC):	0	.07102	.150061	.238508	.334022	.429946
23 I(BC):	0	.073158	.163339	.290869	.463964	.656811
24 I(AB|C):	0	.160564	.288991	.4266	.577656	.728372
25 I(AC|B}):	0	.1263	.254512	.412805	.576412	.714844
26 I(BC|A}):	0	.128438	.30672	.608171	.911874	1.162744
27 I(A{B,C}):	0	.231584	.413281	.582499	.733577	.859737
28 I(B{A,C}):	0	.233722	.476526	.80355	1.168917	1.480231
29 I(C;A,B)	0	.199459	.387444	.672283	1.103985	1.54906
30 I(ABC):	0	.05528	.110475	.196733	.341513	.553902
31 I(A):	0	.171181	.322337	.435392	.513097	.565618
32 I(B):	0	.371193	.556731	.661406	.730434	.780284
33 I(C):	0	.477551	.760443	.894935	.961842	.999155
34 I(A,B,C):	0	1.32467	1.983165	2.325563	2.526864	2.658636

Discussion

This communication is a follow up on two of our earlier papers on biodiversity (Orlóci and He 1996, Orlóci et al. 2002) with novel contents. We refer readers to the original publications for a broader list of relevant references, discussions, and typical examples. The novel materials are presented in both main text and Appendix, including partition functions and their Venn diagram mappings. While the paper limits itself to the case of 3 factors, the idea of diversity partitions is general and applicable to any number of factors. But since the number of partitions grows as a product with the number of factors considered, beyond a point the large number of possible partitions makes it impractical to derive all of them. But to define them all is really unnecessary, considering that well planned research is focused on a limited number of hypotheses about a limited number of factor interactions. So the practical thing to do is to derive diversity partitions that correspond to the specific hypotheses to be tested.

The derivation of the partitions that fit *a priori* stated hypotheses is the first step. The next step is the derivation of probability distributions for the partitions to facilitate the test of the hypotheses. I do not discuss this aspect since I have nothing novel to present in that regard. Kullback (1959) is a fundamental source for arguments that proof that when the regularity conditions that he specifies exist, twice the information of order 1 will have an asymptotically chi-squared distribution with given degrees of freedom that I give in the Appendix. My preference is, however, for empirical distributions derived in the spirit of some Monte Carlo procedure. In this regard reference is made to Pillar (1996, 1999) for ecological context and to Edgington (1987) for theory.

The introduction of the application program DIVPART.EXE is among our objectives. The version available free of charge for downloading (source address http://ecoqua.ecologia.ufrgs.br/~lorloci/) takes data from a three-dimensional contingency table like in Table 1. The number of states is not limited. The program can be used for the analysis of two dimensional contingency tables, but with dummy numbers for third dimension. If we were to use it for such a case, with factors limited to *A* and *B* in Table 1, we would interpret only the terms in Tables 2 and 3 that have the *A* and *B* labels.

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Appendix

The Appendix is downloadable from the Publishers' web site of this issue of *Community Ecology* at address www.akademiai.com.

Appendix

Please read the main text for details and important facts about the materials resented in this appendix.

1. Elementary symbols and some numerical identities.

$$\mathbf{F}_{AB1} = \begin{bmatrix} f_{111} & f_{121} & f_{131} & f_{141} & f_{151} \\ f_{211} & f_{221} & f_{231} & f_{241} & f_{251} \\ f_{311} & f_{321} & f_{331} & f_{341} & f_{511} \end{bmatrix} = \begin{bmatrix} 4 & 10 & 9 & 11 & 7 \\ 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & 0 & 2 & 7 \end{bmatrix}$$

$$\mathbf{F}_{A2C} = \begin{bmatrix} f_{121} & f_{122} & f_{123} \\ f_{221} & f_{222} & f_{223} \\ f_{321} & f_{322} & f_{323} \end{bmatrix} = \begin{bmatrix} 10 & 58 & 18 \\ 1 & 17 & 8 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\mathbf{F}_{2BC} = \begin{bmatrix} f_{211} & f_{212} & f_{213} \\ f_{221} & f_{222} & f_{223} \\ f_{231} & f_{232} & f_{233} \\ f_{241} & f_{242} & f_{243} \\ f_{251} & f_{252} & f_{253} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 17 & 8 \\ 1 & 58 & 2 \\ 1 & 22 & 3 \\ 3 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F}_{AB.} = \begin{bmatrix} f_{11.} & f_{12.} & f_{13.} & f_{14.} & f_{15.} \\ f_{21.} & f_{22.} & f_{23.} & f_{24.} & f_{25.} \\ f_{31.} & f_{32.} & f_{33.} & f_{34.} & f_{51.} \end{bmatrix} = \begin{bmatrix} 17 & 86 & 90 & 153 & 19 \\ 2 & 26 & 61 & 26 & 3 \\ 1 & 15 & 29 & 87 & 30 \end{bmatrix}$$

$$\mathbf{F}_{A.C} = \begin{bmatrix} f_{1.1} & f_{1.2} & f_{1.3} \\ f_{2.1} & f_{2.2} & f_{2.3} \\ f_{3.1} & f_{3.2} & f_{3.3} \end{bmatrix} = \begin{bmatrix} 41 & 238 & 41 \\ 7 & 98 & 13 \\ 12 & 85 & 65 \end{bmatrix}$$

Principle marginal totals:

$$\mathbf{F}_{A..} = \begin{bmatrix} f_{1..} \, f_{2..} \, f_{3..} \end{bmatrix} = \begin{bmatrix} 365 & 118 & 162 \end{bmatrix}$$

$$\mathbf{F}_{.B.} = \begin{bmatrix} f_{.1.} & f_{.2.} & f_{.3.} & f_{.4.} & f_{.5.} \end{bmatrix} = \begin{bmatrix} 20 & 127 & 180 & 266 & 52 \end{bmatrix}$$

$$\mathbf{F}_{..C} = \begin{bmatrix} f_{..1} \, f_{..2} \, f_{..3} \end{bmatrix} = \begin{bmatrix} 60 & 466 & 119 \end{bmatrix}$$

Characteristic elementary vectors:

$$\mathbf{F}_{2B3} = \begin{bmatrix} f_{213} \, f_{223} \, f_{233} \, f_{243} \, f_{253} \end{bmatrix} = \begin{bmatrix} 0 & 8 & 2 & 3 & 0 \end{bmatrix}$$

$$\mathbf{F}_{33C} = \begin{bmatrix} f_{331} \, f_{332} \, f_{333} \end{bmatrix} = \begin{bmatrix} 0 & 16 & 13 \end{bmatrix}$$

2. Practice run of DIVPART.EXE

The version of DIVPAR.EXE described takes the data in sorted (f_{ijk}) form as a string with each element separated by a return (paragraph) character through subscripts k, j and i. In the example:

f_{111}
f_{112}
...
f_{343}
The application program DIVPART.EXE is available free of charge for downloading on the website http://ecoqua.ecologia.ufrgs.br/~lorloci in folder Diversity partition program and data. The following is printed by the program:

Program: biodiversity analysis of a 3-dimensional contingency table.

Revised 2000.02.09 2001.01.22 2006.03.08 2006.10.27
Renyi's generalised entropy and information are partitioned into components.
The components contain additive terms when the order variable Alpha is approaching 1.
Maximum sorting dimension: 3 (A,B,C).
Input data: a x b x c frequencies presented in a column string.
Alpha starts with 0. An upper limit requested. Your response was 5 .
Number of partitions: 34
Data file name:
C:\Documents and Settings\Laszlo Orloci\Desktop\Compile diversity prog\DATAHEILO
NGJIANG 3X5X3.TRU
Date and time:20061029 21:16:06
Printda file name: printda.tru
Number of sorting criteria: 3
Number of factor states: a= 3 b= 5 c= 3
Number of iterations: 1
Data and marginal totals:
Layers of the a x b x c contingency solid:
4 5 8
10 58 18
9 79 2
11 131 11
7 10 2

1 1 0
1 17 8
1 58 2
1 22 3
3 0 0

0 0 1
3 5 7
0 16 13
2 46 39
7 18 5
axb principal marginal totals:
17 86 90 153 19
2 26 61 26 3
1 15 29 87 30

axc principal marginal totals:
41 283 41
7 98 13
12 85 65
bxc principal marginal totals:
5 6 9
14 80 33
10 153 17
14 199 53
17 28 7
a principal marginal totals:
365 118 162
b principal marginal totals:
20 127 180 266 52
c principal marginal totals:
60 466 119

NOTE –
I( ) Renyi's information of order alpha
H( ) Renyi's entropy of order alpha
Richness: 6.4692503 nats 9.3331554 bits
Alpha runs from 0 to upper limit chosen.
All logs in nats. Divide nats by ln 2 to convert into bits.
Values in rows in order of Alpha from zero to upper limit:

1 H(A):** 1.584962 ** 1.413781 ** 1.262624 ** 1.149569 ** 1.071864 ** 1.019344
2 H(B):** 2.321928 ** 1.950734 ** 1.765196 ** 1.660521 ** 1.591493 ** 1.541643
3 H(C):** 1.584962 ** 1.107411 ** .824519 ** .690027 ** .62312 ** .585807
4 H(A|B):** 1.584962 ** 1.308497 ** 1.147351 ** 1.056232 ** 1.000059 ** .962254
5 H(A|C):** 1.584962 ** 1.342761 ** 1.153771 ** 1.03064 ** .954617 ** .90653
6 H(B|A):** 2.321928 ** 1.84545 ** 1.624049 ** 1.497602 ** 1.416013 ** 1.359581
7 H(B|C):** 2.321928 ** 1.877575 ** 1.693656 ** 1.597911 ** 1.538983 ** 1.498668
8 H(C|A):** 1.584962 ** 1.03639 ** .770032 ** .665525 ** .618101 ** .591985
9 H(C|B):** 1.584962 ** 1.034252 ** .785253 ** .678228 ** .62341 ** .591084
10 H(A,B,C):** 5.491853 ** 4.167182 ** 3.508687 ** 3.166289 ** 2.964988 ** 2.833216
11 H(A,B):** 3.90689 ** 3.259231 ** 2.940917 ** 2.749465 ** 2.621845 ** 2.529873
12 H(A,C):** 3.169925 ** 2.450172 ** 1.988109 ** 1.722564 ** 1.57223 ** 1.482679
13 H(B,C):** 3.90689 ** 2.984987 ** 2.464933 ** 2.229622 ** 2.106492 ** 2.031434
14 H(A,B|C):** 3.90689 ** 3.059771 ** 2.698942 ** 2.483801 ** 2.34555 ** 2.251362
15 H(A,C|B):** 3.169925 ** 2.216447 ** 1.798411 ** 1.587019 ** 1.470824 ** 1.400582
16 H(B,C|A):** 3.90689 ** 2.753401 ** 2.229741 ** 1.998355 ** 1.874227 ** 1.796814
17 H(A|B,C):** 1.584962 ** 1.182195 ** .9874 ** .886004 ** .829188 ** .794158
18 H(B|A,C):** 2.321928 ** 1.71701 ** 1.501884 ** 1.381794 ** 1.305588 ** 1.253971
19 H(C|A,B):** 1.584962 ** .90795 ** .702319 ** .627353 ** .590691 ** .569059
20 I(A;B;C):** 0 ** .304742 ** .541518 ** .777559 ** 1.00347 ** 1.204176
21 I(AB):** 0 ** .105283 ** .201785 ** .293636 ** .38164 ** .464268
22 I(AC):** 0 ** .07102 ** .150061 ** .238508 ** .334022 ** .429946
23 I(BC):** 0 ** .073158 ** .163339 ** .290869 ** .463964 ** .656811
24 I(AB|C):** 0 ** .160564 ** .288991 ** .4266 ** .577656 ** .728372
25 I(AC|B}):** 0 ** .1263 ** .254512 ** .412805 ** .576412 ** .714844
26 I(BC|A}):** 0 ** .128438 ** .30672 ** .608171 ** .911874 ** 1.162744
27 I(A{B,C}):** 0 ** .231584 ** .413281 ** .582499 ** .733577 ** .859737

28 I(B{A,C}):** 0 ** .233722 ** .476526 ** .80355 ** 1.168917 ** 1.480231
29 I(C;A,B)** 0 ** .199459 ** .387444 ** .672283 ** 1.103985 ** 1.54906
30 I(ABC):** 0 ** .05528 ** .110475 ** .196733 ** .341513 ** .553902
31 I(A):** 0 ** .171181 ** .322337 ** .435392 ** .513097 ** .565618
32 I(B):** 0 ** .371193 ** .556731 ** .661406 ** .730434 ** .780284
33 I(C):** 0 ** .477551 ** .760443 ** .894935 ** .961842 ** .999155
34 I(A,B,C):** 0 ** 1.32467 ** 1.983165 ** 2.325563 ** 2.526864 ** 2.658636

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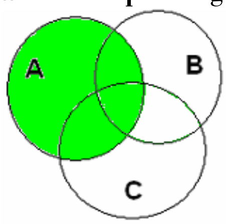
3. Complete set of partition functions

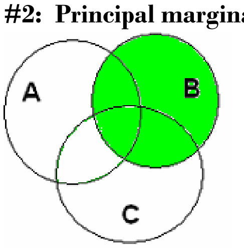
A. Entropy of order α

 # 1: Principal marginal H(A)



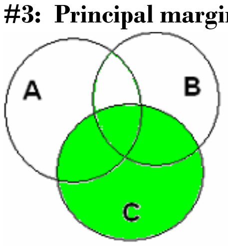
$$H(A) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{a} p_i^{\alpha} \; ; \quad p_i = \frac{f_{i..}}{f_{...}}$$

#2: Principal marginal H(B)



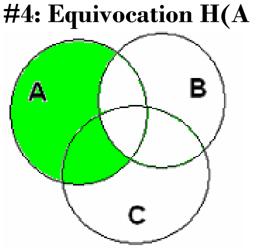
$$H(B) = \frac{1}{1-\alpha} \ln \sum_{j=1}^{b} p_j^{\alpha} \; ; \; p_j = \frac{f_{.j.}}{f_{...}}$$

#3: Principal marginal H(C)



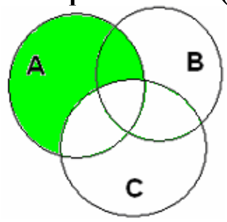
$$H(C) = \frac{1}{1-\alpha} \ln \sum_{k=1}^{c} p_k^{\alpha} \; ; \; p_k = \frac{f_{..k}}{f_{...}}$$

#4: Equivocation H(A|B) = #11 - #2



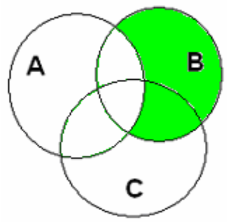
$$H(A \mid B) = \frac{1}{(1-\alpha) f_{...}} \sum_{j=1}^{b} f_{.j.} \ln \sum_{i=1}^{a} p_{i \mid j}^{\alpha} \; ; \quad p_{i \mid j} = \frac{f_{ij.}}{f_{.j.}}$$

#5: Equivocation H(A | C) = #12 - #3



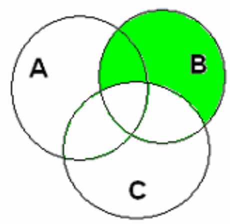
$$H(A\,|\,C)=\frac{1}{(1-\alpha)f_{...}}\sum_{k=1}^{c} f_{..k}\ln\sum_{i=1}^{a} p_{i\,|\,k}^{\alpha}\;;\quad p_{i\,|\,k}=\frac{f_{i.k}}{f_{..k}}$$

#6: Equivocation H(B | A) = #11 - #1



$$H(B\,|\,A)=\frac{1}{(1-\alpha)f_{...}}\sum_{i=1}^{a} f_{i..}\ln\sum_{j=1}^{b} p_{j\,|\,i}^{\alpha}\;;\quad p_{j\,|\,i}=\frac{f_{ij.}}{f_{i..}}$$

#7: Equivocation H(B | C) = #13 - #3

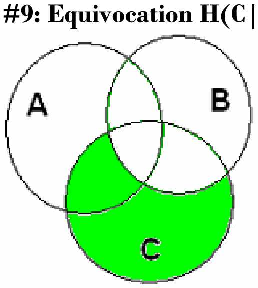


$$H(B\,|\,C)=\frac{1}{(1-\alpha)f_{...}}\sum_{k=1}^{c} f_{..k}\ln\sum_{j=1}^{b} p_{j\,|\,k}^{\alpha}\;;\quad p_{j\,|\,k}=\frac{f_{.jk}}{f_{..k}}$$

#8: Equivocation H(C | A) = #12 - #1

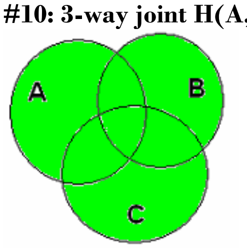
$$H(C\,|\,A)=\frac{1}{(1-\alpha)f_{...}}\sum_{i=1}^{a} f_{i..}\ln\sum_{k=1}^{c} p_{k\,|\,i}^{\alpha}\;;\quad p_{k\,|\,i}=\frac{f_{i.k}}{f_{i..}}$$

#9: Equivocation H(C | B) = #13- #2



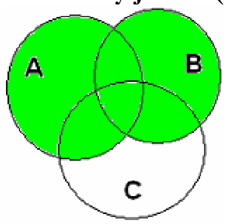
$$H(C\,|\,B)=\frac{1}{(1-\alpha)f_{...}}\sum_{j=1}^{b} f_{.j.}\ln\sum_{k=1}^{c} p_{k\,|\,j}^{\alpha}\;;\quad p_{k\,|\,j}=\frac{f_{.jk}}{f_{.j.}}$$

#10: 3-way joint H(A,B,C) = #1 + #2 + #3 - #20



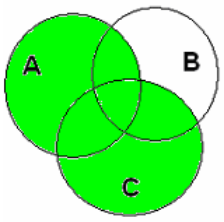
$$H(A,B,C)=\frac{1}{1-\alpha}\ln\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} p_{ijk}^{\alpha}\;;\quad p_{ijk}=\frac{f_{ijk}}{f_{...}}$$

#11: 2-way joint H(A,B) = #1 + #2 - #21



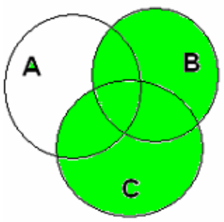
$$H(A,B) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} p_{ij}^{\alpha}\ ; \quad p_{ij} = \frac{f_{ij.}}{f_{...}}$$

#12: 2-way joint H(A,C) = #1 + #3 - #22



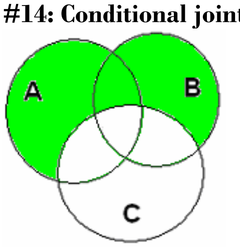
$$H(A,C) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{a} \sum_{k=1}^{c} p_{ik}^{\alpha}\ ; \quad p_{ik} = \frac{f_{i.k}}{f_{...}}$$

#13: 2-way joint H(B,C) = #2 + #3 - #23



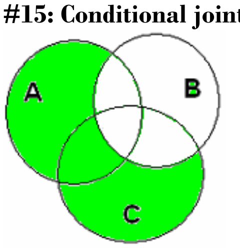
$$H(B,C) = \frac{1}{1-\alpha} \ln \sum_{j=1}^{b} \sum_{k=1}^{c} p_{ik}^{\alpha}\ ; \quad p_{jk} = \frac{f_{.jk}}{f_{...}}$$

#14: Conditional joint H(A,B|C) = #10 - #3



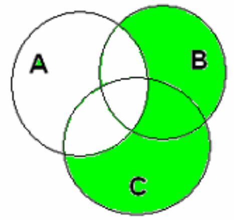
$$H(A,B\,|\,C) = \frac{1}{(1-\alpha) f_{...}} \ln \sum_{k=1}^{c} f_{..k} \sum_{i=1}^{a} \sum_{j=1}^{b} p_{ij|k}^{\alpha}\ ; \, p_{ij\,|\,k} = \frac{f_{ijk}}{f_{..k}}$$

#15: Conditional joint H(A,C|B) = #10 - #2



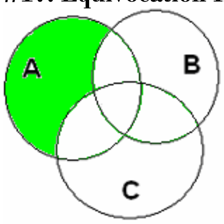
$$H(A,C\,|\,B) = \frac{1}{(1-\alpha) f_{...}} \ln \sum_{j=1}^{b} f_{.j.} \sum_{i=1}^{a} \sum_{k=1}^{c} p_{ik|j}^{\alpha}\ ; \, p_{ik\,|\,j} = \frac{f_{ijk}}{f_{.j.}}$$

#16: Conditional joint H(B,C|A) = #10 - #1



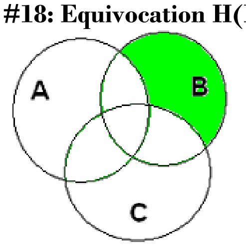
$$H(B,C \mid A) = \frac{1}{(1-\alpha)\, f_{...}} \ln \sum_{i=1}^{a} f_{i..} \sum_{j=1}^{b} \sum_{k=1}^{c} p_{jk|i}^{\alpha}\,;\; p_{jk\,|\,i} = \frac{f_{ijk}}{f_{i..}}$$

#17: Equivocation H(A|B,C) = #10 - #13



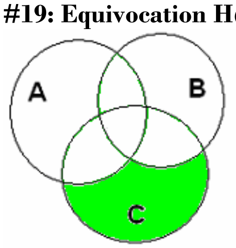
$$H(A \mid B,C) = \frac{1}{(1-\alpha)\, f_{...}} \ln \sum_{j=1}^{b} \sum_{k=1}^{c} f_{.jk} \sum_{i=1}^{a} p_{i|jk}^{\alpha}\,;\; p_{i\,|\,jk} = \frac{f_{ijk}}{f_{.jk}}$$

#18: Equivocation H(B|A,C) = #10 - #12



$$H(B \mid A,C) = \frac{1}{(1-\alpha)\, f_{...}} \ln \sum_{i=1}^{a} \sum_{k=1}^{c} f_{i.k} \sum_{j=1}^{b} p_{j|ik}^{\alpha}\,;\; p_{j\,|\,ik} = \frac{f_{ijk}}{f_{i.k}}$$

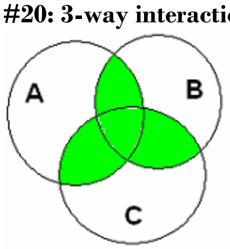
#19: Equivocation H(C|A,B) = #10 - #11



$$H(C \mid A,B) = \frac{1}{(1-\alpha)\, f_{...}} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} f_{ij.} \sum_{k=1}^{c} p_{k|ij}^{\alpha}\,;\; p_{k\,|\,ij} = \frac{f_{ijk}}{f_{ij.}}$$

B. Information of order α

#20: 3-way interaction I(A;B;C) = #10 - #17- #18 - #19 - #30

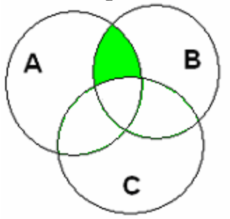


$$I(A;B;C) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}};\; p_{ijk} = \frac{f_{ijk}}{f_{...}},$$

$$q_{ijk} = \frac{f_{i..} f_{.j.} f_{..k}}{f_{...}^{3}}$$

DF = abc – a – b – c + 2

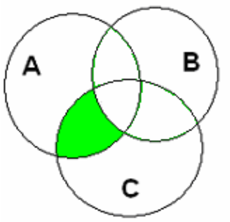
#21: Marginal interaction I(AB) = #24 - #30



$$I(AB) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{p_{ij}^{\alpha}}{q_{ij}^{\alpha-1}}; \quad p_{ij} = \frac{f_{ij.}}{f_{...}}, \quad q_{ij} = \frac{f_{i..} f_{.j.}}{f_{...}^{2}}$$

DF = ab – a – b +1

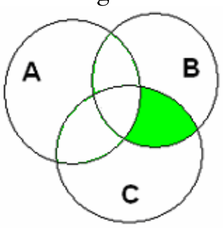
#22: Marginal interaction I(AC) = #25 - #30



$$I(AC) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{a} \sum_{k=1}^{c} \frac{p_{ik}^{\alpha}}{q_{ik}^{\alpha-1}}; \quad p_{ik} = \frac{f_{i.k}}{f_{...}}, \quad q_{ik} = \frac{f_{i..} f_{..k}}{f_{...}^{2}}$$

DF = ac – a – c +1

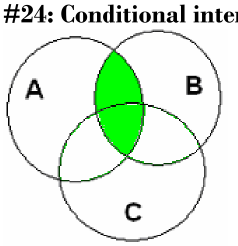
#23: Marginal interaction I(BC) = #26 - #30



$$I(BC) = \frac{1}{\alpha - 1} \ln \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{jk}^{\alpha}}{q_{jk}^{\alpha-1}}; \quad p_{jk} = \frac{f_{.jk}}{f_{...}}, \quad q_{jk} = \frac{f_{.j.} f_{..k}}{f_{...}^{2}}$$

DF = bc – b – c + 1

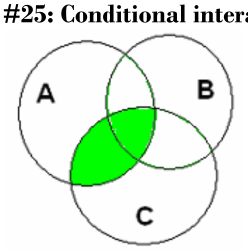
#24: Conditional interaction I(AB|C) = #21 + #30



$$I(AB \mid C) = \frac{1}{(\alpha - 1) f_{...}} \sum_{k=1}^{c} f_{..k} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{p_{ij \mid k}^{\alpha}}{q_{ij \mid k}^{\alpha-1}}; \quad p_{ij \mid k} = \frac{f_{ijk}}{f_{..k}},$$

$$q_{ij|k} = \frac{f_{i.k} f_{.jk}}{f_{..k}^{2}}; \qquad \text{DF} = c(ab - a - b + 1)$$

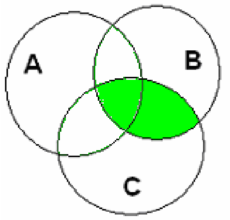
#25: Conditional interaction I(AC|B) = #22 + #30



$$I(AC \mid B) = \frac{1}{(\alpha - 1) f_{...}} \sum_{j=1}^{b} f_{.j.} \ln \sum_{i=1}^{a} \sum_{k=1}^{c} \frac{p_{ik \mid j}^{\alpha}}{q_{ik \mid j}^{\alpha-1}}; \quad p_{ik \mid j} = \frac{f_{ijk}}{f_{.j.}},$$

$$q_{ik \mid j} = \frac{f_{ij.} f_{.jk}}{f_{.j.}^{2}}; \qquad \text{DF} = b(ac - a - c + 1)$$

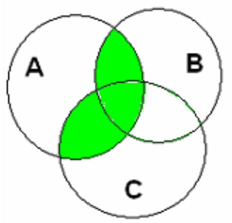
S#26: Conditional interactions I(BC | A) = #23 + #30



$$I(BC \mid A) = \frac{1}{(\alpha-1) f_{...}} \sum_{i=1}^{a} f_{i..} \ln \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{jk|i}^{\alpha}}{q_{jk|i}^{\alpha}}; \quad p_{jk \mid i} = \frac{f_{ijk}}{f_{i..}},$$

$$q_{jk \mid i} = \frac{f_{ij.} f_{i.k}}{f_{i..}^{2}}; \quad \text{DF} = a(bc - b - c + 1)$$

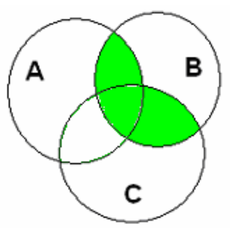
#27: Joint interaction I(A{B,C}) = #21 + #25



$$I(A\{B,C\}) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}}; \quad p_{ijk} = \frac{f_{ijk}}{f_{...}},$$

$$q_{ijk} = \frac{f_{i..} f_{.jk}}{f_{...}^{2}}; \quad \text{DF} = abc - b - c + 1$$

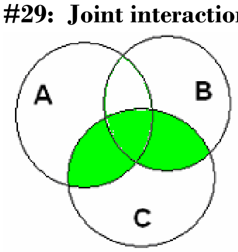
#28: Joint interaction I(B{A,C}) = #21 + #26



$$I(B\{A,C\}) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}}; \quad p_{ijk} = \frac{f_{ijk}}{f_{...}},$$

$$q_{ijk} = \frac{f_{.j.} f_{i.k}}{f_{...}^{2}}; \quad \text{DF} = abc - a - c + 1$$

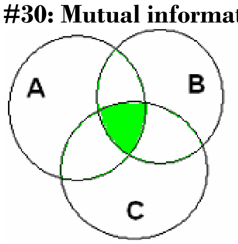
#29: Joint interaction I(C{A,B}) = #22 + #26



$$I(C\{A,B\}) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}}; \quad p_{ijk} = \frac{f_{ijk}}{f_{...}},$$

$$q_{ijk} = \frac{f_{..k} f_{ij.}}{f_{...}^{2}}; \quad \text{DF} = abc - b - c + 1$$

#30: Mutual information I(ABC) = #24 - #21

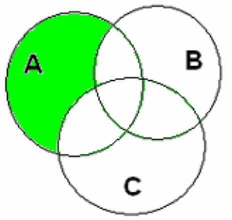


$$I(ABC) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}}; \quad p_{ijk} = \frac{f_{ijk}}{f_{...}},$$

$$q_{ijk} = \frac{f_{ij.} f_{i.k} f_{.jk}}{f_{i..} f_{.j.} f_{..k}}$$

DF = (c – 1)(ab – a – b + 1)= (b – 1)(ac – a – c +1) = (a – 1)(bc – b – c + 1)

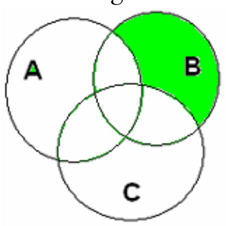
#31: Marginal effect I(A)



$$I(A) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{a} \frac{p_i^{\alpha}}{q_i^{\alpha-1}}; \quad p_i = \frac{f_{i..}}{f_{...}}, \; q_i = \frac{1}{a}$$

DF = a-1

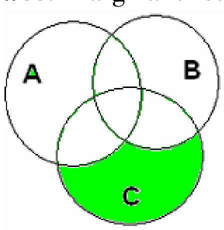
#32: Marginal effect I(B)



$$I(B) = \frac{1}{\alpha - 1} \ln \sum_{j=1}^{b} \frac{p_j^{\alpha}}{q_j^{\alpha-1}}; \quad p_j = \frac{f_{.j.}}{f_{...}}, \quad q_j = \frac{1}{b}$$

DF = b-1

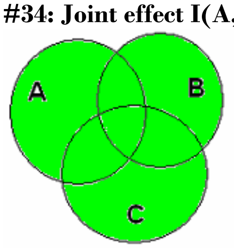
#33: Marginal effect I(C)



$$I(C) = \frac{1}{\alpha - 1} \ln \sum_{k=1}^{c} \frac{p_k^{\alpha}}{q_k^{\alpha-1}}; \quad p_k = \frac{f_{..k}}{f_{...}}, \; q_k = \frac{1}{c}$$

DF = c-1

#34: Joint effect I(A,B,C) = #20 + #31 + #32 + #33



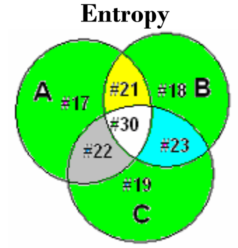
$$I(A,B,C) = \frac{1}{\alpha - 1} \ln \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \frac{p_{ijk}^{\alpha}}{q_{ijk}^{\alpha-1}}; \; p_{ijk} = \frac{f_{ijk}}{f_{...}}; \; q_{ijk} = \frac{1}{abc}$$

DF = abc-1

3. Elementary partitions

The graphs below identify partitions by type of function for quick reference to assist the construction of additive sequences of the order one *H* and *I* quantities.

Entropy



Information