Algorithms to Compute Chern-Schwartz-Macpherson and Segre Classes and the Euler Characteristic

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Overview

Let V be a closed subscheme of $\mathbb{P}^n = \mathrm{Proj}(k[x_0, \dots, x_n])$, with k an algebraical closed field of characteristic zero.

- We will discuss probabilistic algorithms to compute the Euler characteristic and the Chern-Schwartz-MacPherson and Segre class of V.
- The algorithms can be implemented symbolically using Gröbner bases calculations, or numerically using homotopy continuation.
- The algorithms are tested on several examples and found to perform favourably compared to other existing algorithms.

The topological Euler characteristic

- The Euler characteristic χ is an important topological invariant which allows for the categorization of topological spaces.
- Has numerous applications, for example it is applied to problems of maximum likelihood estimation in algebraic statistics by Huh [9] and to string theory in physics by Collinucci et al. [5] and by Aluffi and Esole [3].
- For projective schemes it can be computed several different ways, for example from Hodge numbers, or as we do here, from the Chern-Schwartz-Macperhson class.
- In particular from $c_{SM}(V)$ we may immediately obtain the Euler characteristic of V, $\chi(V)$ using the well-known relation which states that $\chi(V)$ is equal to the degree of the zero dimensional component of $c_{SM}(V)$.

Chern-Schwartz-MacPherson Classes

- The Chow ring of $\mathbb{P}^n$ is $\oplus_{j=0}^{n} A^j(\mathbb{P}^n)$ where $A^j(\mathbb{P}^n)$ is the group of codimension j-cycles modulo rational equivalence.
- We have that $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$ where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is the class of a hyperplane in $\mathbb{P}^n$, so that a hypersurface W of degree d will be represented by $[W] = d \cdot h$ in $A^*(\mathbb{P}^n)$.
- We consider the c_{SM} class (and other characteristic classes) as elements of $A^*(\mathbb{P}^n)$.
- The c_{SM} class generalizes the Chern class of the tangent bundle to singular varieties/schemes, i.e. $c(TV) \cap [V] = c_{SM}(V)$ when V is a smooth subscheme of $\mathbb{P}^n$.

Chern-Schwartz-MacPherson Classes

- The c_{SM} class has important functorial properties, in particular its relation to the Euler characteristic.
- When V is a subscheme of $\mathbb{P}^n$ the class $c_{SM}(V)$ can, in a sense, be thought of as a more refined version of the Euler characteristic since it in fact contains the Euler characteristics of V and those of general linear sections of V for each codimension.
- Specifically, if $\dim(V) = m$, starting from $c_{SM}(V)$ we may directly obtain the list of invariants

$$\chi(V), \chi(V \cap L_1), \chi(V \cap L_1 \cap L_2), \dots, \chi(V \cap L_1 \cap \cdots \cap L_m)$$

where $L_1, \dots, L_m$ are general hyperplanes.

Example: c_{SM} Class and Euler Characteristics

- Consider the variety $V = V(x_0x_3 - x_1x_2)$ in $\mathbb{P}^3 = \mathrm{Proj}(k[x_0, \ldots, x_3])$ which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.
- We may compute $c_{SM}(V) = 4h^3 + 4h^2 + 2h$ and obtain the Euler characteristics of the general linear sections using an involution formula given by Aluffi in [2], specifically:
  - First consider the polynomial $p(t) = 4 + 4t + 2t^2 \in \mathbb{Z}[t]/(t^4)$ given by the coefficients of the c_{SM} class above.
  - Next apply Aluffi's involution

$$p(t) \mapsto \mathcal{I}(p) := \frac{t \cdot p(-t-1) + p(0)}{t+1} = 2t^2 - 2t + 4.$$

- This gives $\chi(V) = 4$, $\chi(V \cap L_1) = 2$, and $\chi(V \cap L_1 \cap L_2) = 2$.

Previous Work

Let V be an arbitrary subscheme of $\mathbb{P}^n$ defined by a homogeneous ideal $I = (f_0, \ldots, f_r)$.

- The first algorithm to compute $c_{SM}(V)$ was that of Aluffi [1].
  - For a hypersurface V, this algorithm requires the computation of the blowup of $\mathbb{P}^n$ along the singularity subscheme of V.
  - The computation of such blowups can be an expensive operation, making this algorithm impractical for many examples.
- Another algorithm to compute the c_{SM} class of a hypersurface was given by Jost in [10].
  - This method works by computing the degrees of certain residual sets via a particular saturation.

Previous Work

- Our algorithm described here uses a result originally proved by the author of this note in Theorem 3.1 of [8], which is given below as Theorem 1, combined with a result of Aluffi [1], given in (3) below.
    - This result is used to construct an algorithm which computes the projective degrees of a rational map and can be implemented in most computer algebra systems either symbolically with standard Gröbner bases algorithms, or numerically using homotopy continuation.
    - This method of computing the c_{SM} class using projective degrees provides a performance improvement in many cases, see Table 2.

Projective Degrees

- Consider a rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$.
- We may define the *projective degrees* of the map ϕ as a list of integers $(g_0, \dots, g_n)$ where

$$g_i = \mathrm{card}\left(\phi^{-1}\left(\mathbb{P}^{m-i}\right) \cap \mathbb{P}^i\right).$$

  - Here $\mathbb{P}^{m-i} \subset \mathbb{P}^m$ and $\mathbb{P}^i \subset \mathbb{P}^n$ are general hyperplanes of dimension $m - i$ and i respectively.
  - Additionally card is the cardinality of a zero dimensional set.
- Let $I = (f_0, \dots, f_m)$ be a homogeneous ideal in $R = k[x_0, \dots, x_n]$ defining a ϱ-dimensional scheme $V = V(I)$, and assume, without loss of generality, that all the polynomials f_i generating I have the same degree.

Projective Degrees

Theorem 1

The projective degrees $(g_0, \ldots, g_n)$ *of* $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$*,*
$\phi : p \mapsto (f_0(p) : \cdots : f_m(p))$, *are given by*

$$g_i = \dim_k \left(R[T]/(P_1 + \cdots + P_i + L_1 + \cdots + L_{n-i} + L_A + S) \right).$$

Here P_ℓ, L_ℓ, L_A *and* S *are ideals in* $R[T] = k[x_0, \ldots, x_n, T]$ *with*
$P_\ell = \left(\sum_{j=0}^m \lambda_{\ell,j} f_j \right)$ *for* $\lambda_{\ell,j}$ *a general scalar in* k*,*
$S = \left(1 - T \cdot \sum_{j=0}^m \vartheta_j f_j \right)$*, for* ϑ_j *a general scalar in* k*,* L_ℓ *a general homogeneous linear form for* $\ell = 1, \ldots, n$ *and* L_A *a general affine linear form.*

Segre Classes and the Projective Degrees

- Consider a subscheme $Y = V(J) \subset \mathbb{P}^n$ defined by a homogeneous ideal $J = (w_0, \dots, w_m) \subset R = k[x_0, \dots, x_n]$.
  - Assume, without loss of generality, that $\deg(w_i) = d$ for all i.
  - Let $(g_0, \dots, g_n)$ be the projective degrees of the rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (w_0(p) : \cdots : w_m(p))$.

By Proposition 3.1 of Aluffi [1] we have

$$s(Y, \mathbb{P}^n) = 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1 + dh)^{i+1}} \in A^*(\mathbb{P}^n). \tag{1}$$

This together with Theorem 1 can be used to construct an algorithm to compute Segre classes.

An Algorithm to Compute the Segre Class

- **Input:** A homogeneous ideal $J = (w_0, \dots, w_m)$ in $k[x_0, \dots, x_n]$ defining a scheme $Y = V(J)$ in $\mathbb{P}^n$.
- **Output:** The Segre class $s(Y, \mathbb{P}^n)$.
  - Let ϕ be the rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (w_0(p) : \cdots : w_m(p))$.
  - Compute the projective degrees $(g_0, \dots, g_n)$ of the rational map ϕ using Theorem 1.
  - Compute $s(Y, \mathbb{P}^n) = 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1+dh)^{i+1}}$.

Segre Class Algorithm Running Times

Input	Segre(Aluffi [1])	segreClass(E.J.P. [6])	**segre_proj_deg(Theorem 1)**
Rational normal curve in $\mathbb{P}^7$	-	7s (9s)	8s (15s)
Smooth surface in $\mathbb{P}^8$ defined by minors	-	59s	18s
Degree 48 surface in $\mathbb{P}^6$	-	172s	6s
Singular var. in $\mathbb{P}^9$	0.5 s	33s	10s
Singular var. in $\mathbb{P}^6$	-	173s	6s
Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^3$ in $\mathbb{P}^{11}$	2s	-	52s

Table : All algorithms are implemented in Macaulay2 [7]. Timings in () are for the numerical versions, using either Bertini [4] (via M2). Computations that do not finish within 600s are denoted -. Numerical timings which did not finish in 600s are omitted.

Segre Class Algorithm Discussion

- We note that in all cases considered here our algorithm using Theorem 1 to compute the projective degree performs favourably in comparison to other known algorithms which compute these Segre classes.
- Test computations for the symbolic version were performed over $\mathbb{GF}(32749)$, yielding the same results found by working over $\mathbb{Q}$ for all examples considered.
- The symbolic methods are somewhat slower when performed over $\mathbb{Q}$, but still much faster than the numeric ones.

Chern-Schwartz-MacPherson Class of a Hyper-surface

- Let $V = V(f) \subset \mathbb{P}^n$ be a hypersurface and let $(g_0, \ldots, g_n)$ be the projective degrees of the polar map

$$\varphi : p \mapsto \left(\frac{\partial f}{\partial x_0}(p) : \cdots : \frac{\partial f}{\partial x_n}(p) \right). \tag{2}$$

- Combining the expression for the Segre class in terms on the projective degree (1) with Theorem 2.1 of Aluffi [1] we have:

$$c_{SM}(V) = (1+h)^{n+1} - \sum_{j=0}^{n} g_j (-h)^j (1+h)^{n-j} \text{ in } A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}). \tag{3}$$

- This allows us to compute the c_{SM} class of any projective hypersurface using the projective degrees.

Inclusion/Exclusion for c_{SM} Classes

- For V_1, V_2 subschemes of $\mathbb{P}^n$ the inclusion-exclusion property for c_{SM} classes states

$$c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2).$$

- Inclusion/Exclusion allows for the computation of $c_{SM}(V)$ for V of any codimension.
- This requires exponentially many c_{SM} computations relative to the number of generators of I.
- Must consider c_{SM} classes of products of many or all of the generators of I, which may have significantly higher degree than the original scheme V.

Algorithm to Compute the Chern-Schwartz-MacPherson Class

- **Input:** A homogeneous ideal $(f_0, \ldots, f_r)$ in $k[x_0, \ldots, x_n]$ defining a scheme $V = V(I)$ in $\mathbb{P}^n$.
- **Output:** $c_{SM}(V)$ and/or $\chi(V)$.
  - Make a list $\mathcal{L}_I$ of all generators and all products of generators of the ideal I.
  - For each polynomial $f \in \mathcal{L}_I$ compute $c_{SM}(V(f))$.
    - Compute the projective degrees $(g_0, \ldots, g_n)$ of the polar map of f (2) using Theorem 1.
    - Compute $c_{SM}(V(f))$ from the projective degree $(g_0, \ldots, g_n)$ using (3).
  - Apply the inclusion/exclusion property of c_{SM} classes to obtain $c_{SM}(V)$.

Running times for c_{SM} Algorithm

Input	CSM (Aluffi)	CSM (Jost)	**csm_polar (M2)**	**csm_polar (Sage)**	euler
Twisted cubic	0.3s	0.1s (35s)	0.1s (37s)	0.1s (0.6s)	0.2s
Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$	0.4s	0.8s (148s)	0.2s (152s)	0.2s (57s)	0.2s
Smooth degree 3 surface in $\mathbb{P}^4$	-	1.2s (-)	0.6s (-)	0.2s (28s)	20.1s
Smooth degree 7 surface in $\mathbb{P}^4$	-	50s	42s	7.9s	-
Smooth degree 3 surface in $\mathbb{P}^8$	-	85.2s	2.2s	1.0s	-
Smooth degree 3 surface in $\mathbb{P}^{10}$	-	-	9.8s	2.0s	-
Sing. degree 3 surface in $\mathbb{P}^{10}$	-	-	10s	2.3s	n/a
Deg. 12 hypersurface in $\mathbb{P}^3$	25.3s	1.0s	0.2s	0.1s	n/a
Sing. Var. in $\mathbb{P}^5$	-	-	1.3s	0.3s	n/a

Table : Timings in () are for the numerical versions, using either Bertini [4] (via M2) or PHCpack [11] (via Sage). Computations that do not finish within 600s are denoted -, or are omitted for numerical versions.

c_{SM} Algorithm Discussion

- We note that in all cases considered here our algorithm using Theorem 1 to compute the projective degree performs favourably in comparison to other known algorithms which compute the c_{SM} class and Euler Characteristic.
- Test computations for the symbolic version were performed over $\mathbb{GF}(32749)$, yielding the same results found by working over $\mathbb{Q}$ for all examples considered.
- The symbolic methods are somewhat slower when performed over $\mathbb{Q}$, but still much faster than the numeric ones.

Future/Current Work

- Obtain running time bounds for the algorithms presented here.
- Try using the algorithms on examples arising from applications and see how they preform there.
- Continue work on an algorithm which computes the c_{SM} class without using inclusion/exclusion.

Thank you for listening!

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