

# Homotopy (Co)limits: a brief survey.

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## Abstract

We present here a short overview of possible different definitions for the notions of *homotopy limits* and *homotopy colimits* in the context of model categories, also providing suitable (and hopefully enough) context to compare them and show their equivalence when this comparison makes sense. We shall focus in particular on three distinct approaches, namely the one involving *derived functor*, the *homotopical approach* and the *simplicial version*.

## 1 Introduction

Given a model category  $\mathcal{M}$  and a small category  $\mathcal{C}$ , (co)completeness of  $\mathcal{M}$  ensures the existence of all (co)limits of functors  $\mathcal{C} \rightarrow \mathcal{M}$ . Hence, upon choosing a (co)limit for any object of  $\mathcal{M}^{\mathcal{C}}$ , one can always consider the colimit and limit functors, together with their adjoints as in

$$\operatorname{colim}: \mathcal{M}^{\mathcal{C}} \rightleftarrows \mathcal{M} : c \quad \text{and} \quad c: \mathcal{C} \rightleftarrows \mathcal{M}^{\mathcal{C}} : \operatorname{lim}, \quad (1)$$

where  $c$  is the diagonal (constant) functor and left adjoints are on the left. There are though a couple of remarks that should be done and may serve to give a (somewhat naive) motivation to the theory of homotopy (co)limits.

- (i) The (co)limit functor is not, in general, *homotopy invariant*, i.e. given a natural transformation  $\tau: F \Rightarrow G: \mathcal{C} \rightarrow \mathcal{M}$  which is a pointwise weak equivalence (we will call such natural transformations *natural weak equivalences*), it is not true that the induced arrow

$$\operatorname{colim} \tau: \operatorname{colim} F \rightarrow \operatorname{colim} G \quad (\text{or} \quad \operatorname{lim} \tau: \operatorname{lim} F \rightarrow \operatorname{lim} G)$$

is a weak equivalence in  $\mathcal{M}$ . For example, if  $\mathcal{C}$  is the pushout category

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

and  $\mathcal{M}$  is **Top** (with the Quillen model structure), then we can consider the following diagrams in  $\mathcal{M}$

$$F = * \longleftarrow S^n \longrightarrow D^{n+1} \quad \text{and} \quad G = * \longleftarrow S^n \longrightarrow *$$

(here  $S^n \rightarrow D^{n+1}$  is the boundary inclusion). We have an obvious natural weak equivalence  $\tau: F \rightarrow G$  which is the identity on  $*$  and on  $S^n$  and collapses the whole

$D^{n+1}$  to a point. However,  $\operatorname{colim} F \simeq S^{n+1}$  and  $\operatorname{colim} G \simeq *$ , so that the induced map  $\operatorname{colim} F \rightarrow \operatorname{colim} G$  is not a weak equivalence.

Thus, this lack of preserving weak equivalences is not a pathological behaviour, it is not a phenomenon arising when considering some weird model structure on some awkward category. On the contrary, it is somehow an unavoidable consequence of the very definition of (co)limits. One could then think of homotopy (co)limits as a homotopy "correction" to the ordinary (co)limits so that they become homotopy invariant.

- (ii) (Co)limits are unique up to within isomorphisms. This is a fundamental property which we are very pleased to have and which is the kind of uniqueness one seeks for in ordinary category theory. Still, when dealing with model categories, the isomorphism condition appears too strong (somehow redundant) and the right notion of "equality" (or uniqueness) seems more likely to be that of weak (or perhaps homotopy) equivalences (which become an isomorphism in the homotopy category). In other words and in light of 1., we may look for a notion of (co)limit which is *not* isomorphically unique, but rather *homotopically unique*. We will see that, following in particular one of the possible approaches to homotopy (co)limits (see Section 3 below), we will be able to give a precise meaning to this kind of uniqueness so as that such homotopy (co)limits match it.

There are several ways to approach the problem of defining homotopy (co)limits satisfactorily. Here, we have decided to treat and compare (briefly) three of them, namely:

1. the *derived functors approach*, which consists in defining homotopy (co)limits as derived functors of the ordinary ones, provided that the functor category  $\mathcal{M}^{\mathcal{C}}$  can be given a model structure for which  $\operatorname{colim}$  or  $\operatorname{lim}$  are (left or right) Quillen functors;
2. the *homotopical approach*, where homotopy (co)limits are defined using the notion of (*homotopical*) *approximation* of functors, without invoking any sort of model structure on  $\mathcal{M}^{\mathcal{C}}$  but utilizing only the homotopical informations retained by the model structure of  $\mathcal{M}$ ;
3. the *simplicial approach*, which defines homotopy (co)limits when  $\mathcal{M}$  is a simplicial model category, using explicitly the additional enriched structure.

## 2 Deriving $\operatorname{colim}$ and $\operatorname{lim}$

In this section we approach the problem of finding homotopy corrections to ordinary limits and colimits through the theory of Quillen adjunctions, thus looking for model structures on functor categories which allow us to derive such limit and colimit functors. Our main reference is [?].

We start with an easy observation, which actually is the motivating core for what follows.

**Remark 1.** Suppose that  $\mathcal{M}$  is a model category and  $\mathcal{C}$  is a small category such that  $\mathcal{M}^{\mathcal{C}}$  has a model structure where weak equivalences are natural transformations which are

pointwise weak equivalences. Assume also that, for this model structure,  $\text{colim}: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}$  is a left Quillen functor. Let  $Q: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{C}}$  be the cofibrant replacement functor given by the functorial factorization in  $\mathcal{M}^{\mathcal{C}}$ . Then, for every weak equivalence  $\tau: F \xrightarrow{\sim} G$  in  $\mathcal{M}^{\mathcal{C}}$ , the induced natural transformation  $QF \rightarrow QG$  is again a weak equivalence with both  $QF$  and  $QG$  cofibrant. Therefore,  $\text{colim } \tau: \text{colim } QF \rightarrow \text{colim } QG$  is a weak equivalence in  $\mathcal{M}$ , as  $\text{colim}$  is a left Quillen functor by hypothesis. Thus,  $\text{colim} \circ Q$  is homotopy invariant. Of course, also a dual statement involving  $\text{lim}$  as a right Quillen functor and the fibrant replacement  $R$  holds true.

Hence, if we could actually find model structures on  $\mathcal{M}^{\mathcal{C}}$  such that the requests of the Remark above were satisfied, we would have a natural candidate for our homotopy colimit (or limit), as we might just take (the total derived functor of)  $\text{colim} \circ Q$  (or  $\text{lim} \circ R$ ). Asking some specific properties to be satisfied either by the base model category  $\mathcal{M}$  or by the indexing category  $\mathcal{C}$  it is indeed possible to get such nice model structures on  $\mathcal{M}^{\mathcal{C}}$ .

**Definition 1.** Let  $\mathcal{M}$  be a model category and let  $\mathcal{C}$  be a small category. We say that a natural transformation  $\tau: F \Rightarrow G$  is

- a *natural weak equivalence* (or a *pointwise weak equivalence*) if, for all  $i \in \mathcal{C}$ ,  $\tau_i: F(i) \rightarrow G(i)$  is a weak equivalence in  $\mathcal{M}$ ;
- a *pointwise fibration* if, for all  $i \in \mathcal{C}$ ,  $\tau_i: F(i) \rightarrow G(i)$  is a fibration in  $\mathcal{M}$ ;
- a *pointwise cofibration* if, for all  $i \in \mathcal{C}$ ,  $\tau_i: F(i) \rightarrow G(i)$  is a cofibration in  $\mathcal{M}$ .

If there exists a model structure on  $\mathcal{M}^{\mathcal{C}}$  such that the weak equivalences and the fibrations are the natural weak equivalences and the pointwise fibrations respectively, then we call such a (uniquely determined) model structure the *projective model structure* on  $\mathcal{M}^{\mathcal{C}}$ . Dually, if there exists a model structure on  $\mathcal{M}^{\mathcal{C}}$  such that the weak equivalences and the cofibrations are the natural weak equivalences and the pointwise cofibrations respectively, then we call such a (uniquely determined) model structure the *injective model structure* on  $\mathcal{M}^{\mathcal{C}}$ .

The following result gives us sufficient conditions under which the projective or the injective model structures exist.

**Theorem 1.** *Let  $\mathcal{M}$  be a model category and let  $\mathcal{C}$  be a small category.*

1. *If  $\mathcal{M}$  is a cofibrantly generated model category, then  $\mathcal{M}^{\mathcal{C}}$  admits the projective model structure.*
2. *If  $\mathcal{M}$  is a combinatorial model category, then  $\mathcal{M}^{\mathcal{C}}$  admits the injective model structure.*

Colimits (respectively limits) behave as well as we hoped for with respect to the projective (respectively injective) model structure:

**Proposition 1.** *Let  $\mathcal{M}$  be a model category and  $\mathcal{C}$  be a small category such that the projective (resp. the injective) model structure on  $\mathcal{M}^{\mathcal{C}}$  is defined. Then the colimit (resp. the limit) functor*

$$\text{colim}: (\mathcal{M}^{\mathcal{C}})_{\text{proj}} \rightarrow \mathcal{M} \quad (\text{resp. } \text{lim}: (\mathcal{M}^{\mathcal{C}})_{\text{inj}} \rightarrow \mathcal{M})$$

*is a left (resp. right) Quillen functor.*

*Proof.* For the colimit case (the limit one being dual), one just notes that the constant functor  $c: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}$  (which is a left adjoint to the colimit functor) preserves clearly all (trivial) fibrations, by definition of the projective model structure.  $\square$

As announced, we can therefore give the following

**Definition 2.** Let  $\mathcal{M}$  be a model category and  $\mathcal{C}$  be a small category.

1. Assume that  $\mathcal{M}^{\mathcal{C}}$  has the projective model structure and let  $Q_{\text{Proj}}$  be the cofibrant replacement functor given by the functorial factorization in  $(\mathcal{M}^{\mathcal{C}})_{\text{Proj}}$ . Then, the *homotopy colimit* is the total left derived functor of  $\text{colim}$ , i.e. it is (up to isomorphism) the composite

$$\text{hcolim}^1: \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Proj}}) \xrightarrow{\text{Ho}(Q_{\text{Proj}})} \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Proj}}) \xrightarrow{\text{Ho}(\text{colim})} \text{Ho}(\mathcal{M}) \quad (2)$$

2. Assume that  $\mathcal{M}^{\mathcal{C}}$  has the injective model structure and let  $R_{\text{Inj}}$  be the fibrant replacement functor given by the functorial factorization in  $(\mathcal{M}^{\mathcal{C}})_{\text{Inj}}$ . Then, the *homotopy limit* is the total right derived functor of  $\text{lim}$ , i.e. it is (up to isomorphism) the composite

$$\text{hlim}^1: \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Inj}}) \xrightarrow{\text{Ho}(R_{\text{Inj}})} \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Inj}}) \xrightarrow{\text{Ho}(\text{lim})} \text{Ho}(\mathcal{M}) \quad (3)$$

In Theorem ?? we gave some sufficient condition on the model category  $\mathcal{M}$  so that, for all small categories  $\mathcal{C}$ , the functor category  $\mathcal{M}^{\mathcal{C}}$  admits a model structure with pointwise weak equivalences, which turns  $\text{colim}$  or  $\text{lim}$  into Quillen functors. One may also consider the specular point of view, that is to say one might look for some additional conditions on the small category  $\mathcal{C}$  so that, for all model categories  $\mathcal{M}$ ,  $\mathcal{M}^{\mathcal{C}}$  has a model category structure with the properties we are interested in. In this spirit, we recall the following result

**Theorem 2.** Let  $\mathcal{C}$  be a small Reedy category and  $\mathcal{M}$  a model category. For all objects  $n$  of  $\mathcal{C}$ , let  $L_n$  and  $M_n$  be the latching and the matching functor respectively. Then  $\mathcal{M}^{\mathcal{C}}$  has a model structure, called the Reedy model structure where, for all arrows  $\tau: F \rightarrow G$  in  $\mathcal{M}^{\mathcal{C}}$

(i)  $\tau$  is a weak equivalence if and only if it is a pointwise weak equivalence;

(ii)  $\tau$  is a cofibration if and only if, for all  $n \in \mathcal{C}$ , the induced morphism

$$F_n \amalg_{L_n F} L_n G \rightarrow G_n$$

is a cofibration in  $\mathcal{M}$  (here  $F_n := F(n)$ ).

(iii)  $\tau$  is a fibration if and only if, for all  $n \in \mathcal{C}$ , the induced morphism

$$F_n \rightarrow G_n \times_{M_n G} M_n F$$

is a fibration in  $\mathcal{M}$ .

Unluckily, in general, the colimit functor  $(\mathcal{M}^{\mathcal{C}})_{\text{Reedy}} \rightarrow \mathcal{M}$  is not a left Quillen functor (nor  $\text{lim}$  is a right Quillen functor). However, we can fix this problem as follows.

**Definition 3.** Let  $\mathcal{C}$  be a Reedy category.

1. We say that  $\mathcal{C}$  *has cofibrant constants* if, for all model categories  $\mathcal{M}$  and each cofibrant object  $A \in \mathcal{M}$ , the constant functor  $cA: \mathcal{C} \rightarrow \mathcal{M}$  is cofibrant in  $(\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}$ .
2. We say that  $\mathcal{C}$  *has fibrant constants* if, for all model categories  $\mathcal{M}$  and each fibrant object  $B \in \mathcal{M}$ , the constant functor  $cB: \mathcal{C} \rightarrow \mathcal{M}$  is fibrant in  $(\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}$ .

**Remark 2.** A Reedy category  $(\mathcal{C}, \mathcal{C}_+, \mathcal{C}_-)$  has cofibrant constants if and only if all the latching categories  $\partial(C_+ \downarrow n)$  are either empty or connected (and dually,  $(\mathcal{C}, \mathcal{C}_+, \mathcal{C}_-)$  has fibrant constants if and only if all the matching categories  $\partial(n \downarrow C_-)$  are either empty or connected)

**Example 1.** (i) The simplicial category  $\Delta$  is a Reedy category with fibrant constants.  
(ii) Every direct (resp. inverse) category is a Reedy category with fibrant (resp. cofibrant) constants.

Using Remark ??, one can prove the following

**Theorem 3.** *Let  $\mathcal{C}$  be a small Reedy category.*

1.  $\mathcal{C}$  has fibrant constants if and only if, for all model categories  $\mathcal{M}$ ,

$$\text{colim}: (\mathcal{M}^{\mathcal{C}})_{\text{Reedy}} \rightarrow \mathcal{M}$$

*is a left Quillen functor.*

2.  $\mathcal{C}$  has cofibrant constants if and only if, for all model categories  $\mathcal{M}$ ,

$$\text{lim}: (\mathcal{M}^{\mathcal{C}})_{\text{Reedy}} \rightarrow \mathcal{M}$$

*is a right Quillen functor.*

Thus, as prescribed by our underlying philosophy, we can give the

**Definition 4.** Let  $\mathcal{M}$  be a model category and  $\mathcal{C}$  a small Reedy category. Let also  $Q_{\text{Reedy}}$  and  $R_{\text{Reedy}}$  be the cofibrant and the fibrant replacement functors given by the functorial factorization in  $(\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}$ .

1. If  $\mathcal{C}$  has fibrant constants, the *homotopy colimit* is the total left derived functor of  $\text{colim}$ , i.e. it is (up to isomorphism) the composite

$$\text{hcolim}^2: \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}) \xrightarrow{\text{Ho}(Q_{\text{Reedy}})} \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}) \xrightarrow{\text{Ho}(\text{colim})} \text{Ho}(\mathcal{M}) \quad (4)$$

2. If  $\mathcal{C}$  has cofibrant constants, the *homotopy limit* is the total right derived functor of  $\text{lim}$ , i.e. it is (up to isomorphism) the composite

$$\text{hlim}^2: \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}) \xrightarrow{\text{Ho}(R_{\text{Reedy}})} \text{Ho}((\mathcal{M}^{\mathcal{C}})_{\text{Reedy}}) \xrightarrow{\text{Ho}(\text{lim})} \text{Ho}(\mathcal{M}) \quad (5)$$

We could now wonder what happens when we are in a situation where we could apply both Definition (??) and Definition (??) to get our homotopy colimits. The (predictable) answer comes from the following

**Theorem 4.** *Let  $\mathcal{C}$  be a small Reedy category and let  $\mathcal{M}$  be a model category such that  $\mathcal{M}^{\mathcal{C}}$  can be endowed with the projective model structure. Then the adjoint pair*

$$\text{id}: (\mathcal{M}^{\mathcal{C}})_{\text{proj}} \rightleftarrows (\mathcal{M}^{\mathcal{C}})_{\text{Reedy}} : \text{id}$$

*is a Quillen equivalence.*

*Proof.* Since the weak equivalences are the same both in the Reedy and in the projective model structure, we only need to prove that a Reedy fibration is a pointwise fibration and that a cofibration in the projective model structure is also a Reedy cofibration.

Let then  $\tau: F \implies G$  be a fibration in the Reedy model structure and  $n$  an object of  $\mathcal{C}$ , so that the induced morphism  $t_n: F_n \longrightarrow G_n \times_{M_n G} M_n F$  is a fibration in  $\mathcal{M}$ . Let us consider the pullback square

$$\begin{array}{ccc} G_n \times_{M_n G} M_n F & \xrightarrow{p_n} & G_n \\ \downarrow & & \downarrow \\ M_n F & \xrightarrow{M_n \tau} & M_n G \end{array}$$

Now, the matching category  $\partial(n \downarrow \mathcal{C}_-)$  is an inverse category by letting  $(\partial(n \downarrow \mathcal{C}_-))_- = \partial(n \downarrow \mathcal{C}_-)$  and then it has cofibrant constants. Thus, by Theorem ??,  $M_n \tau$  is a fibration in  $\mathcal{M}$  and then so is  $p_n$ . Since  $\tau_n = p_n \circ t_n$ ,  $\tau_n$  is a fibration in  $\mathcal{M}$ , hence  $\tau$  is a fibration in the projective model structure.

If now  $\sigma: F \implies G$  is a cofibration in the projective model structure, what we have just seen implies that  $\sigma$  has the left lifting property with respect to all Reedy trivial fibration, so it is a Reedy cofibration itself.  $\square$

This theorem tells us in particular that the cofibrant replacement given by the functorial factorization in the projective model structure (which we denoted by  $Q_{\text{Proj}}$  above) is a cofibrant replacement also for the Reedy model structure. Since, given any object  $X$  in a model category  $\mathcal{M}$ , if  $Q$  is the cofibrant replacement functor given by the functorial factorization in  $\mathcal{M}$ , there is a weak equivalence between any cofibrant replacement of  $X$  and  $QX$ , we immediately get the

**Corollary 1.** *Let  $\mathcal{C}$  be a small Reedy category having fibrant constants and let  $\mathcal{M}$  be a model category such that  $\mathcal{M}^{\mathcal{C}}$  can be endowed with the projective model structure. Then, for all  $F \in \mathcal{M}^{\mathcal{C}}$ , there is an isomorphism*

$$\text{hcolim}^1(F) \simeq \text{hcolim}^2(F).$$

Of course, also the dual results involving injective model structures, Reedy categories with cofibrant constants and homotopy limits hold true.

### 3 The Homotopical approach

In the former section we defined homotopy (co)limits under the hypotheses of the existence of a model structure on the functor category  $\mathcal{M}^{\mathcal{C}}$ , where  $\mathcal{M}$  is a model category

and  $\mathcal{C}$  is a small category. We will show that, actually, one can define homotopically unique homotopy (co)limits *without any further assumption* on the basis model category  $\mathcal{M}$  nor on the indexing small category  $\mathcal{C}$ . This goal will be achieved adopting a purely homotopical point of view which allows us to see any model category  $\mathcal{M}$  and all functor categories  $\mathcal{M}^{\mathcal{C}}$  simply as nice *categories with weak equivalences*, the so called *homotopical categories*. In particular, the definition of homotopy (co)limits will make no use of cofibrations or fibrations at all, stressing the preeminent importance of weak equivalences in the theory of homotopy (co)limits. We shall follow [?] in our discussion.

Before giving the basic definition of this section, a remark is needed. In what follows, we shall also consider functor categories of the form  $\mathcal{N}^{\mathcal{M}}$  where  $\mathcal{M}$  is not necessarily small, even if, in general, these categories are not locally small. This is mainly because we will be interested in the case where  $\mathcal{M}$  and  $\mathcal{N}$  are model categories (see Corollary ?? and the discussion preceding Theorem ?? below), and the smallness condition for a model category implies that its underlying category is a poset (admitting arbitrary infima and suprema), as a model category is complete and cocomplete by hypothesis. Therefore, the smallness request would trivialize our discussion too much when we deal with model categories. We then ask the reader to forgive our sloppiness about size problems for functor categories, reassured by the fact that a proper, formal treatment of this issue is possible, enlarging the working universe (see, for example, the discussion in §8.1 of [?]).

**Definition 5.** A *homotopical category* is a pair  $(\mathcal{H}, \mathcal{W})$ , where  $\mathcal{H}$  is a category and  $\mathcal{W}$  is a subclass of the class of all morphisms in  $\mathcal{H}$  having the following properties:

1. for all objects  $A$  of  $\mathcal{H}$ ,  $id_A \in \mathcal{W}$ ;
2.  $\mathcal{W}$  satisfies the *2-out-of-6* property, i.e. if  $f, g, h$  are morphisms in  $\mathcal{H}$  such that the two compositions  $gf$  and  $hg$  exist and are in  $\mathcal{W}$ , then also  $f, g, h$  and  $hgf$  belong to  $\mathcal{W}$ .

The elements of the distinguished class  $\mathcal{W}$  are called *weak equivalences*.

**Remark 3.** Assuming that both  $gf$  and  $hg$  are identity morphisms or that at least one among  $f, g$  and  $h$  is an identity morphism, one readily proves that, if  $(\mathcal{H}, \mathcal{W})$  is a homotopical category, then  $\mathcal{W}$  *contains all isomorphisms* in  $\mathcal{H}$  and has the *2-out-of-3* property. In particular,  $\mathcal{W}$  is a *subcategory* of  $\mathcal{H}$ .

**Definition 6.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be homotopical categories.

1. A functor  $F: \mathcal{H} \rightarrow \mathcal{L}$  is called a *homotopical functor* if it *preserves weak equivalences*, i.e., if  $f: A \rightarrow B$  is a weak equivalence in  $\mathcal{H}$ , then  $F(f): F(A) \rightarrow F(B)$  is a weak equivalence in  $\mathcal{L}$ .
2. Given (not necessarily homotopical) functors,  $F, G: \mathcal{H} \rightarrow \mathcal{L}$ , a *natural weak equivalence* from  $F$  to  $G$  is a natural transformation  $\tau: F \rightrightarrows G$  such that, for all  $A \in \mathcal{H}$ , the  $A$ -th component  $\tau_A$  of  $\tau$  is a weak equivalence in  $\mathcal{L}$ .
3. Two homotopical functors  $F, G: \mathcal{H} \rightarrow \mathcal{L}$  are *naturally weakly equivalent* if there is a *zigzag* of natural weak equivalences connecting them.
4. We will denote by  $(\mathcal{L}^{\mathcal{H}})_w$  the full subcategory of the functor category  $\mathcal{L}^{\mathcal{H}}$  given by homotopical functors. (So, in particular, morphisms between two homotopical functors are given by *all* natural transformations between them).

We have the following, unsurprising result

**Proposition 2.** *Let  $\mathcal{L}$  be a homotopical category.*

- (i) *If  $\mathcal{C}$  is an ordinary category, then the functor category  $\mathcal{L}^{\mathcal{C}}$  is a homotopical category with weak equivalences given by natural weak equivalences.*
- (ii) *If  $\mathcal{H}$  is a homotopical category, then the category  $(\mathcal{L}^{\mathcal{H}})_w$  is a homotopical category with weak equivalences given by natural weak equivalences.*

Model categories fits perfectly into the theory of homotopical categories, since

**Proposition 3.** *Let  $(\mathcal{M}, \mathcal{W}, \text{Cof}(\mathcal{M}), \text{Fib}(\mathcal{M}))$  be a model category. Then  $(\mathcal{M}, \mathcal{W})$  is a homotopical category.*

*Proof.* We have to prove that weak equivalences in a model category satisfy the 2-out-of-6 property. Let then

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

be composable morphisms in  $\mathcal{M}$  with  $gf$  and  $hg$  weak equivalences. If  $\gamma: \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$  is the localization functor, we have that  $\gamma(gf)$  and  $\gamma(hg)$  are isomorphism in  $\text{Ho}(\mathcal{M})$ . Since  $\gamma(f)(\gamma(gf))^{-1}$  and  $(\gamma(hg))^{-1}\gamma(h)$  are a right and a left inverse for  $\gamma(g)$  respectively,  $g$  is a weak equivalence in  $\mathcal{M}$ . Using the two-out-of-three property, we then get that also  $f, h$  and  $hgf$  are weak equivalences, as required.  $\square$

Now, we are interested in studying functors between homotopical categories which are not necessarily globally homotopical, but become such when restricted to suitable subcategories of their domain. More precisely,

**Definition 7.** Let  $\mathcal{H}$  be a homotopical category and  $\mathcal{H}_0$  a subcategory of  $\mathcal{H}$ . We say that  $\mathcal{H}_0$  is a *left* (resp. *right*) *deformation retract* of  $\mathcal{H}$  if there is a pair  $(R, \tau)$  (called *left* (resp. *right*) *deformation*), where  $R: \mathcal{H} \rightarrow \mathcal{H}$  is a homotopical functor sending  $A \in \mathcal{H}$  to  $R(A) \in \mathcal{H}_0$  and

$$\tau: R \implies id_{\mathcal{H}} \quad (\text{resp. } \tau: id_{\mathcal{H}} \implies R)$$

is a natural weak equivalence.

**Definition 8.** Let  $F: \mathcal{H} \rightarrow \mathcal{L}$  be a functor between homotopical categories. We say that  $F$  is *left deformable* (resp. *right deformable*) if  $F$  is homotopical when restricted to a left (resp. right) deformation retract of  $\mathcal{H}$ .

Using the cofibrant and fibrant replacement functors and Ken Brown's Lemma we get immediately the following key result

**Theorem 5.** *Let  $\mathcal{M}$  be a model category and let  $\mathcal{M}_c$  and  $\mathcal{M}_f$  denote the full subcategory of  $\mathcal{M}$  consisting of cofibrant and fibrant objects respectively. Then the following properties hold.*

1.  *$\mathcal{M}_c$  is a left deformation retract of  $\mathcal{M}$ , whereas  $\mathcal{M}_f$  is a right deformation retract of  $\mathcal{M}$ .*
2. *Every functor between model categories preserving weak equivalences between cofibrant (resp. fibrant) objects is left (resp. right) deformable. In particular, every left (resp. right) Quillen functor is left (resp. right) deformable.*

Before defining homotopy (co)limits within this new perspective, we still need a couple of concepts which will automatically give to homotopy (co)limits the properties of homotopy invariance and homotopy uniqueness we seek for.

Let us first remark a somewhat unorthodox characterization of initial and terminal objects in an ordinary category.

**Remark 4.** Let  $\mathcal{C}$  be a category. Then an object  $Z \in \mathcal{C}$  is initial (resp. terminal) if and only if there is a natural transformation  $\sigma: cZ \Longrightarrow id_{\mathcal{C}}$  (resp.  $\sigma: id_{\mathcal{C}} \Longrightarrow cZ$ ), where  $cZ$  is the constant functor at  $Z$ , such that  $\sigma_Z: Z \rightarrow Z$  is an isomorphism.

We are then lead to give the following

**Definition 9.** Let  $\mathcal{H}$  be a homotopical category. An object  $Z \in \mathcal{H}$  is called *homotopically initial* (resp. *homotopically terminal*) if there exist functors  $F_0, F_1: \mathcal{H} \rightarrow \mathcal{H}$  and a natural transformation  $\tau: F_0 \Longrightarrow F_1$  (resp.  $\tau: F_1 \Longrightarrow F_0$ ) such that:

- (i)  $F_0$  is naturally weak equivalent (see Definition ??) to  $cZ$ ;
- (ii)  $F_1$  is naturally weak equivalent to  $id_{\mathcal{H}}$ ;
- (iii)  $\tau_Z$  is a weak equivalence in  $\mathcal{H}$ .

The following proposition explains the terminology just introduced

**Proposition 4.** *Let  $\mathcal{H}$  be a homotopical category and let  $Z$  be a homotopically initial (resp. homotopically terminal) object in  $\mathcal{H}$ . Then an object  $A \in \mathcal{H}$  is a homotopically initial (resp. homotopically terminal) object in  $\mathcal{H}$  if and only if it is weakly equivalent to  $Z$ , i.e. if and only if there is a zigzag of weak equivalences in  $\mathcal{H}$  connecting  $A$  and  $Z$ . In other words, a homotopically initial (resp. homotopically terminal) object is homotopically unique, i.e. it is unique up to within (zigzags of) weak equivalences.*

Now, given a (not necessarily homotopical) functor  $F: \mathcal{H} \rightarrow \mathcal{L}$ , where  $\mathcal{H}$  and  $\mathcal{L}$  are homotopical categories, we can consider the full subcategory of the over category  $(\mathcal{L}^{\mathcal{H}} \downarrow F)$  consisting of all those natural transformations  $\sigma: G \Longrightarrow F$  where  $G$  is a homotopical functor. We will denote such a full subcategory by  $((\mathcal{L}^{\mathcal{H}})_w \downarrow F)$ . Note that it naturally inherits a structure of homotopical category from  $(\mathcal{L}^{\mathcal{H}})_w$ . Similarly, we can define  $(F \downarrow (\mathcal{L}^{\mathcal{H}})_w)$ . Initial and terminal objects in these homotopical categories are so fundamental for us that they deserve special names.

**Definition 10.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be homotopical categories and let  $F: \mathcal{H} \rightarrow \mathcal{L}$  be a (not necessarily homotopical) functor between them. A *left approximation* (resp. *right approximation*) is a homotopically terminal (resp. initial) object in  $((\mathcal{L}^{\mathcal{H}})_w \downarrow F)$  (resp. in  $(F \downarrow (\mathcal{L}^{\mathcal{H}})_w)$ ).

A sufficient condition for the existence of left and right approximations is provided by the following

**Theorem 6.** *Let  $\mathcal{H}$  and  $\mathcal{L}$  be homotopical categories and let  $F: \mathcal{H} \rightarrow \mathcal{L}$  be a (not necessarily homotopical) functor between them. If  $F$  is left deformable (resp. right deformable), then there exists a left approximation (resp. a right approximation) of  $F$ .*

*Proof.* We prove only the left version, the other being dual. Let  $\mathcal{H}_0$  be a left deformation retract with left deformation  $(R, \tau)$  such that  $F$  is homotopical when restricted to  $\mathcal{H}_0$ . Note that  $FR$  is a homotopical functor, so  $(FR, F\tau)$  is an object in  $\mathcal{C} := ((\mathcal{L}^{\mathcal{H}})_w \downarrow F)$ . We claim that actually  $(FR, F\tau)$  is a left approximation for  $F$ . Indeed, let  $(H, \sigma)$  be an object in  $\mathcal{C}$  and consider the diagram

$$\begin{array}{ccc} HR & \xrightarrow{\sigma R} & FR \\ \downarrow H\tau & & \downarrow F\tau \\ H & \xrightarrow{\sigma} & F \end{array}$$

which commutes by naturality of  $\sigma$ . We can then define a functor  $F_1: \mathcal{C} \rightarrow \mathcal{C}$  which sends  $(H, \sigma)$  into  $(HR, \sigma H\tau)$ . Since  $\tau$  is a natural weak equivalence and  $H$  is a homotopical functor,  $H\tau$  is a natural weak equivalence  $HR \Rightarrow H$ , thus it is a weak equivalence in the homotopical category  $\mathcal{C}$  between the object  $(HR, \sigma H\tau)$  and the object  $(H, \sigma)$ . Therefore,  $F_1$  is naturally weak equivalent to  $id_{\mathcal{C}}$ . If we define  $F_0$  as the constant functor with value  $(FR, F\tau)$ , we actually have that the assignment  $(H, \sigma) \mapsto \sigma R$  defines (the  $(H, \sigma)$ -th component of) a natural transformation  $F_1 \Rightarrow F_0$  such that

$$F_1(FR, F\tau) = (FRR, F\tau FR\tau) \xrightarrow{F\tau R} (FR, F\tau) = F_0(FR, F\tau)$$

is a weak equivalence in  $\mathcal{C}$ , as  $FR$  is a homotopical functor. Thus,  $(FR, F\tau)$  is a left approximation for  $F$ .  $\square$

In view of Theorem ??, we immediately get the following

**Corollary 2.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be model categories. Then any left Quillen functor (resp. right Quillen functor)  $\mathcal{M} \rightarrow \mathcal{N}$  has a left approximation (resp. right approximation).*

Finally, we can give the promised

**Definition 11.** Let  $\mathcal{M}$  be a model category and let  $\mathcal{C}$  be a small category.

1. A *homotopy colimit*

$$\mathrm{hcolim}^3: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M} \tag{6}$$

is a left approximation of  $\mathrm{colim}: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}$ .

2. A *homotopy limit*

$$\mathrm{hlim}^3: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M} \tag{7}$$

is a right approximation of  $\mathrm{lim}: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}$ .

Note that, by the above discussion, when they exist,  $\mathrm{hcolim}^3$  and  $\mathrm{hlim}^3$  are automatically *homotopically unique* (they are naturally weakly equivalent to any other left or right approximation of  $\mathrm{colim}$  and or  $\mathrm{lim}$  respectively) and *homotopy invariant*, as they are homotopical functors by definition.

Actually, one can show, through a lengthy and rather technical proof, involving also Reedy model structures on particular functor categories, not only the existence of homotopy limits and colimits for *all* model categories  $\mathcal{M}$  and *all* small categories  $\mathcal{C}$ , but also the following, more general

**Theorem 7.** *Let  $\mathcal{M}$  be a model category and  $\mathcal{C}, \mathcal{D}$  small categories. For each functor  $U: \mathcal{C} \rightarrow \mathcal{D}$ , let*

$$\operatorname{colim}^U: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{D}} \quad \text{and} \quad \operatorname{lim}^U: \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{D}}$$

*be arbitrary but fixed left and right adjoints to the functor  $- \circ U: \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{C}}$ .<sup>1</sup> Then there exist a left approximation of  $\operatorname{colim}^U$  and a right approximation of  $\operatorname{lim}^U$ .*

We conclude this section by establishing the link between the above version of homotopy (co)limits and the one given in Definitions ?? and ??.

Recall that if  $\mathcal{M}$  is a model category, then the full subcategory  $\mathcal{M}_c$  of  $\mathcal{M}$  given by cofibrant objects is a left deformation retract of  $\mathcal{M}$  (see Theorem ??) via the left deformation  $(Q, \tau)$ , where  $Q$  is the cofibrant replacement functor given by the functorial factorization in  $\mathcal{M}$  and  $\tau_A$  is, for all  $A \in \mathcal{M}$ , the weak equivalence  $QA \xrightarrow{\sim} A$ . By the proof of Theorem ??, we know that a left approximation for a Quillen functor  $F: \mathcal{M} \rightarrow \mathcal{N}$  between model categories is given precisely by  $(FQ, F\tau)$ . Therefore, we immediately get the following

**Theorem 8.** *Let  $\mathcal{M}$  be a model category and  $\mathcal{C}$  a small category.*

1. *Assume that  $\mathcal{M}^{\mathcal{C}}$  has the projective model structure (see Definition ??). Then  $\operatorname{hcolim}^3$  and  $\operatorname{colim} \circ Q_{\operatorname{Proj}}$  are naturally weakly equivalent.*
2. *Assume that  $\mathcal{C}$  is a Reedy model category, so that  $\mathcal{M}^{\mathcal{C}}$  has the Reedy model structure (see Theorem ??). Then  $\operatorname{hcolim}^3$  and  $\operatorname{colim} \circ Q_{\operatorname{Reedy}}$  are naturally weakly equivalent.*

Of course, dual results involving homotopy limits and fibrant replacement functors hold true, thus providing (passing to the derived functors) the equivalence between Definitions ??, ?? and ??.

## 4 Simplicial version

We will develop here a notion of homotopy colimits for simplicial model categories and see how they can fit into the context of Quillen adjunctions and derived functors, so as to provide a link with section 2 above. Our main references are given by [?] and [?].

Before getting started, let us recall/introduce some notions.

**Definition 12.** Let  $\mathbb{C}$  be a simplicial category (i.e. a **SSet**-enriched category) and let  $\mathbb{C}(-, \bullet)$  be the **SSet**-enriched Hom-bifunctor of  $\mathbb{C}$ . We say that  $\mathbb{C}$  is *tensored* if, for all  $A \in \mathbb{C}$ , there is a **SSet**-adjunction

$$(-) \otimes A: \mathbf{SSet} \rightleftarrows \mathbb{C} : \mathbb{C}(A, -).$$

Dually, we say that  $\mathbb{C}$  is *cotensored* if, for all  $B \in \mathbb{C}$ , there is a **SSet**-adjunction

$$[-, B]: \mathbf{SSet} \rightleftarrows \mathbb{C}^{\operatorname{op}} : \mathbb{C}(-, B).$$

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<sup>1</sup>Since  $\mathcal{M}$  is complete and cocomplete, such left and right adjoint always exist and can be constructed as *Left* and *Right Kan extensions* respectively.

Thus, the existence of tensors and cotensors for a simplicial category  $\mathbb{C}$  is equivalent to the existence, for every  $K \in \mathbf{SSet}$  and for all  $A, B \in \mathbb{C}$ , of natural  $\mathbf{SSet}$ -enriched isomorphisms

$$\mathbb{C}(K \otimes A, B) \simeq \underline{\mathbf{Hom}}(K, \mathbb{C}(A, B)) \quad \text{and} \quad \mathbb{C}(A, [K, B]) \simeq \underline{\mathbf{Hom}}(K, \mathbb{C}(A, B)) \quad (8)$$

respectively. Here (and in what follows)  $\underline{\mathbf{Hom}}(-, \bullet)$  is the internal Hom in  $\mathbf{SSet}$  (which gives the closed monoidal structure to simplicial sets). Therefore, when  $\mathbb{C}$  is both tensored and cotensored, there is a  $\mathbf{SSet}$ -adjunction in two variables

$$((-) \otimes (\bullet), \mathbb{C}(-, \bullet), [-, \bullet])$$

Contemplating the definition of simplicial adjunctions in two variables (as a  $\mathbf{SSet}$ -enriched adjunctions in two variables), simplicial left and right Quillen functors in two variables and simplicial model categories leads to the following

**Proposition 5.** *Let  $\mathbb{C}$  be a tensored and cotensored simplicial category and assume that its underlying ordinary category is endowed with a model structure. The following conditions are then equivalent:*

- (i) *the functor  $(-) \otimes (\bullet): \mathbf{SSet} \times \mathbb{C} \rightarrow \mathbb{C}$  is a simplicial left Quillen functor in two variables;*
- (ii) *the functor  $\mathbb{C}(-, \bullet): \mathbb{C}^{op} \times \mathbb{C} \rightarrow \mathbf{SSet}$  is a simplicial right Quillen functor in two variables;*
- (iii) *the functor  $[-, \bullet]: \mathbf{SSet}^{op} \times \mathbb{C} \rightarrow \mathbb{C}$  is a simplicial right Quillen functor in two variables.*

Moreover, these conditions hold if and only if  $\mathbb{C}$  is a simplicial model category.

From now on we will then work with simplicial model categories  $\mathbb{M}$  for which the simplicial model structure is given by the bifunctors  $(-) \otimes (\bullet): \mathbf{SSet} \times \mathbb{M} \rightarrow \mathbb{M}$  and  $[-, \bullet]: \mathbf{SSet}^{op} \times \mathbb{M} \rightarrow \mathbb{M}$ .

We can immediately give the following

**Definition 13.** Let  $\mathcal{C}$  be a small category and let  $\mathbb{M}$  be a simplicial model category. Let also a functor  $F: \mathcal{C} \rightarrow \mathbb{M}$  be given. If  $Q$  and  $R$  are the functorial cofibrant and fibrant replacement of (the underlying model category of)  $\mathbb{M}$  respectively, we denote by  $Q^\bullet F$  and  $R_\bullet F$  the composite functors

$$\mathcal{C} \xrightarrow{F} \mathbb{M} \xrightarrow{Q} \mathbb{M} \quad \text{and} \quad \mathcal{C} \xrightarrow{F} \mathbb{M} \xrightarrow{R} \mathbb{M}$$

respectively.

1. The *homotopy colimit* of  $F$  is the (ordinary) coend

$$\mathrm{hcolim}^4(F) = \int^{n \in \mathcal{C}} N(n \downarrow \mathcal{C})^{op} \otimes (Q^\bullet F)(n), \quad (9)$$

where  $N(n \downarrow \mathcal{C})^{op}$  is the nerve of the category  $(n \downarrow \mathcal{C})^{op}$ .

2. The *homotopy limit* of  $F$  is the (ordinary) end

$$\mathrm{hlim}^4(F) = \int_{n \in \mathcal{C}} [N(\mathcal{C} \downarrow n), (R_\bullet F)(n)], \quad (10)$$

where  $N(\mathcal{C} \downarrow n)$  is the nerve of the category  $(\mathcal{C} \downarrow n)$ .

**Example 2.** Let us consider the category **Top** of topological spaces, endowed with the Quillen model structure. We know that **Top** becomes a simplicial model category **Top** if one sets, for all simplicial set  $K$  and all topological spaces  $X, Y$ ,

$$K \otimes X := |K| \times X, \quad [K, X] := X^{|K|}, \quad \mathrm{Top}(X, Y) := \mathbf{Top}(X \times |\Delta[-]|, Y).$$

Here  $|K|$  is the geometric realization of the simplicial set  $K$ ,  $X^{|K|}$  has the compact open topology ( $|K|$  is a  $CW$ -complex, so, in particular, it is compactly generated) and  $\Delta[-]$  is the functor  $\Delta \rightarrow \mathbf{SSet}$  sending each  $[n] \in \Delta$  to the Hom functor  $\Delta[n] := \Delta(-, [n])$  (whose geometric realization is the standard  $n$ -simplex  $\Delta^n$ ).

Let us now take the pushout category

$$\mathcal{C} = 1 \longleftarrow 0 \longrightarrow 2$$

Note that the over-categories  $(1 \downarrow \mathcal{C})$  and  $(2 \downarrow \mathcal{C})$  are both the terminal category with one object, whereas  $(0 \downarrow \mathcal{C})$  is isomorphic to  $\mathcal{C}$ . Taking then the geometric realizations of the nerve of (the opposite of) these categories, one can see that, given a pushout diagram

$$X \longleftarrow A \longrightarrow Y$$

where all objects are cofibrant (for example,  $CW$ -complexes) in **Top**, the formula (??) gives a topological space which is homeomorphic to

$$(X \coprod (A \times [0, 1]) \coprod Y) / \sim, \quad (11)$$

where  $\sim$  is the equivalence relation generated by

$$(a, 0) \sim f(a) \quad \text{and} \quad (a, 1) \sim g(a) \quad \text{for all } a \in A.$$

As announced at the beginning of this section, we would like to show how this new definition of homotopy (co)limits fits into the context of Quillen adjunction, i.e. into the working assumptions of section 2. We will actually restrict to the study of homotopy colimits, the case of homotopy limits being dual.

To this end, we will assume that the small indexing category  $\mathcal{C}$  is actually itself a simplicial category: this does not result into a loss of generality, as any small category  $\mathcal{C}$  can be seen as a **SSet**-enriched category in a trivial way (given two object  $k, n$  in  $\mathcal{C}$ , we can regard  $\mathcal{C}(k, n)$  as a discrete simplicial set). If  $\mathbb{C}$  and  $\mathbb{D}$  are simplicial categories with  $\mathbb{C}$  small, we denote with  $[\mathbb{C}, \mathbb{D}]$  the simplicial category of simplicial functors from  $\mathbb{C}$  to  $\mathbb{D}$ , for which, given  $T, S: \mathbb{C} \rightarrow \mathbb{D}$ , the **SSet**-enriching hom object  $[\mathbb{C}, \mathbb{D}](T, S)$  is the **SSet**-enriched end  $\int_{n \in \mathbb{C}} \mathbb{D}(T(n), S(n))$ .<sup>2</sup> In other words,  $[\mathbb{C}, \mathbb{D}]$  is the simplicial category

<sup>2</sup>Given a **SSet**-enriched functor  $T: \mathbb{C}^{op} \times \mathbb{C} \rightarrow \mathbf{SSet}$ ,  $\int_{n \in \mathbb{C}} T(n, n)$  is a simplicial set such that there exists a universal **SSet**-natural transformation  $(\lambda_n: \int_{n \in \mathbb{C}} T(n, n) \rightarrow T(n, n))_{n \in \mathbb{C}}$ .

whose underlying category has simplicial functors  $\mathbb{C} \rightarrow \mathbb{D}$  as objects and simplicial natural transformations among them as arrows. In particular, when dealing with a simplicial model category  $\mathbb{M}$ , it makes still sense to consider the projective and the injective model structure on the underlying category of  $[\mathbb{C}, \mathbb{D}]$ , as in Definition ??.

Now, the linking bridge for our desired comparison is given by the notion of *weighted colimit*.

**Definition 14.** Let  $\mathbb{C}$  and  $\mathbb{D}$  be simplicial categories with  $\mathbb{C}$  small. We say that  $\mathbb{D}$  *has (or admits)  $\mathbb{C}$ -weighted colimits* if, for all  $A \in [\mathbb{C}, \mathbb{D}]$ , there is an adjoint pair

$$(-) \otimes_{\mathbb{C}} A: [\mathbb{C}^{op}, \mathbf{SSet}] \rightleftarrows \mathbb{D} : \mathbb{D}(A(\bullet), -), \quad (12)$$

where, for all  $P \in \mathbb{D}$ ,  $\mathbb{D}(A(\bullet), -)(P)$  is the simplicial functor  $\mathbb{D}(A(\bullet), P)$  sending  $n \in \mathbb{C}^{op}$  to  $\mathbb{D}(A(n), P)$ . For each  $X \in [\mathbb{C}^{op}, \mathbf{SSet}]$ , the object  $X \otimes_{\mathbb{C}} A$  is called the  *$X$ -weighted colimit* of  $A$ .

If  $\mathbb{D}$  has  $\mathbb{C}$ -weighted colimits for all small simplicial categories  $\mathbb{C}$ , we say that  $\mathbb{D}$  *has (all) weighted colimits*.

In other words,  $\mathbb{D}$  has  $\mathbb{C}$ -weighted colimits if and only if, for all  $X \in [\mathbb{C}^{op}, \mathbf{SSet}]$  and all  $A \in [\mathbb{C}, \mathbb{D}]$ , the simplicial functor

$$\mathbb{D} \ni P \mapsto [\mathbb{C}^{op}, \mathbf{SSet}](X(-), \mathbb{D}(A(-), P)) \in \mathbf{SSet}$$

is representable.

**Remark 5.** It can be shown that, given a small simplicial category  $\mathbb{C}$ , if  $1: \mathbb{C}^{op} \rightarrow \mathbf{SSet}$  is the constant functor at  $1 \in \mathbf{SSet}$  and  $K: \mathbb{C} \rightarrow \mathbf{SSet}$  is a simplicial functor, then the ordinary colimit of (the underlying functor of)  $K$  can be expressed as a weighted colimit via the isomorphism of simplicial sets

$$\operatorname{colim} K \simeq 1 \otimes_{\mathbb{C}} K. \quad (13)$$

**Definition 15.** Let  $\mathbb{D}$  and  $\mathbb{C}$  be simplicial categories with  $\mathbb{C}$  small. Given a simplicial bifunctor  $G: \mathbb{C}^{op} \times \mathbb{C} \rightarrow \mathbb{D}$ , its (*simplicial*) *coend*, if it exists, is

$$\int^{n \in \mathbb{C}} G(n, n) := \mathbb{C}(-, \bullet) \otimes_{\mathbb{C}^{op} \times \mathbb{C}} G \in \mathbb{D}$$

It can be seen that actually the simplicial coend of  $G$  is completely determined by the existence of simplicial natural isomorphisms

$$\mathbb{D} \left( \int^{n \in \mathbb{C}} G(n, n), P \right) \simeq \int_{n \in \mathbb{C}} \mathbb{D}(G(n, n), P) \quad (14)$$

for  $P \in \mathbb{D}$  and where the right hand side is the usual end for  $\mathbf{SSet}$ -enriched functors with values in  $\mathbf{SSet}$ . If  $\mathbb{D} = \mathbf{SSet}$ , the notions of simplicial coend just defined and the usual one involving universal  $\mathbf{SSet}$ -natural transformations coincide.

We have the following

**Proposition 6.** *Let  $\mathbb{D}$  and  $\mathbb{C}$  be simplicial categories with  $\mathbb{C}$  small. If  $\mathbb{D}$  is tensored, then, for all simplicial functors  $A: \mathbb{C} \rightarrow \mathbb{D}$  and  $X: \mathbb{C}^{op} \rightarrow \mathbf{SSet}$ ,*

$$X \otimes_{\mathbb{C}} A \simeq \int^{n \in \mathbb{C}} X(n) \otimes A(n) \quad (15)$$

where one side exists if the other does.

*Proof.* For all  $P \in \mathbb{D}$  we have

$$[\mathbb{C}^{op}, \mathbf{SSet}](X(-), \mathbb{D}(A(-), P)) \stackrel{\text{def}}{\simeq} \int_{n \in \mathbb{C}} \underline{\text{Hom}}(X(n), \mathbb{D}(A(n), P)) \stackrel{(\text{??})}{\simeq} \int_{n \in \mathbb{C}} \mathbb{D}(X(n) \otimes A(n), P)$$

and we can conclude comparing the definition of  $X \otimes_{\mathbb{C}} A$  with (??) and using (the enriched version of ) Yoneda's lemma.  $\square$

One can prove the following result, which gives sufficient conditions for the existence of all weighted colimits

**Theorem 9.** *Let  $\mathbb{D}$  be a tensored and cotensored simplicial category such that its underlying category is cocomplete. Then  $\mathbb{D}$  admits all weighted colimits. In particular, a simplicial model category has all weighted colimits.*

Finally, we can apply all this machinery to our simplicial model setting.

**Theorem 10.** *Let  $\mathbb{M}$  be a simplicial model category and  $\mathbb{C}$  a small simplicial category. Assume that  $[\mathbb{C}^{op}, \mathbf{SSet}]$  is equipped with the simplicial model structure given by the injective model structure and that  $[\mathbb{C}, \mathbb{M}]$  is a simplicial model category when equipped with the projective model structure (see Definition ??). Then the  $(\mathbb{C}-)$ weighted colimit functor*

$$(-) \otimes_{\mathbb{C}} (\bullet): [\mathbb{C}^{op}, \mathbf{SSet}] \times [\mathbb{C}, \mathbb{M}] \rightarrow \mathbb{M}, \quad (X, A) \mapsto X \otimes_{\mathbb{C}} A$$

is a (simplicial) left Quillen functor in two variables.

*Proof.* Consider the functors

$$\Gamma: [\mathbb{C}, \mathbb{M}]^{op} \times \mathbb{M} \rightarrow [\mathbb{C}^{op}, \mathbf{SSet}], \quad (A, B) \mapsto \mathbb{M}(A(-), B) \quad (16)$$

$$\Lambda: [\mathbb{C}^{op}, \mathbf{SSet}]^{op} \times \mathbb{M} \rightarrow [\mathbb{C}, \mathbb{M}], \quad (X, B) \mapsto [X(-), B]. \quad (17)$$

Using the fact that, for cotensored simplicial categories  $\mathbb{M}$ , the existence of  $X$ -weighted colimits (for  $X \in [\mathbb{C}^{op}, \mathbf{SSet}]$ ) is equivalent to the existence of an adjoint

$$X \otimes_{\mathbb{C}} (-): [\mathbb{C}, \mathbb{M}] \rightleftarrows \mathbb{M} : \Lambda(X, -),$$

the definition of  $\mathbb{C}$ -weighted colimits implies that

$$((-) \otimes_{\mathbb{C}} (\bullet), \Gamma, \Lambda)$$

is an adjunction in two variables. On the other hand, the assumption that  $\mathbb{M}$  is a simplicial model category and Proposition ?? say that the cotensor functor  $[-, \bullet]: \mathbf{SSet}^{op} \times \mathbb{M} \rightarrow \mathbb{M}$  is a right Quillen functor in two variables. By the definition of injective and projective model structure and of opposite model category, one notices that this implies that

$$\Lambda: [\mathbb{C}^{op}, \mathbf{SSet}]_{\text{Inj}}^{op} \times \mathbb{M} \rightarrow [\mathbb{C}, \mathbb{M}]_{\text{Proj}}$$

is a (simplicial) right Quillen functor into two variables. This is equivalent to our thesis.  $\square$

In a completely analogous manner, but looking at  $\Gamma$  instead of  $\Lambda$ , one can also show the following

**Theorem 11.** *Let  $\mathbb{M}$  be a simplicial model category and  $\mathbb{C}$  a small simplicial category. Assume that  $[\mathbb{C}^{op}, \mathbf{SSet}]$  is equipped with the simplicial model structure given by the projective model structure and that  $[\mathbb{C}, \mathbb{M}]$  is a simplicial model category when equipped with the injective model structure (see Definition ??). Then the  $(\mathbb{C}-)$ weighted colimit functor*

$$(-) \otimes_{\mathbb{C}} (\bullet): [\mathbb{C}^{op}, \mathbf{SSet}] \times [\mathbb{C}, \mathbb{M}] \longrightarrow \mathbb{M}, \quad (X, A) \mapsto X \otimes_{\mathbb{C}} A$$

is a (simplicial) left Quillen functor in two variables.

We may then conclude that, since there are two ways of turning the weighted colimit functor into a left Quillen functor in two variables, *there are two different, yet equivalent, ways to compute its total left derived functor* as well.

The first way is the one revealed by Theorem ??: under the same hypotheses as there, if

$$(-) \otimes_{\mathbb{C}}^L (\bullet): \mathrm{Ho}([\mathbb{C}^{op}, \mathbf{SSet}]) \times \mathrm{Ho}([\mathbb{C}, \mathbb{M}]) \longrightarrow \mathrm{Ho}(\mathbb{M})$$

is the total left derived functor of  $(-) \otimes_{\mathbb{C}} (\bullet)$ , then, for all  $X: \mathbb{C}^{op} \longrightarrow \mathbf{SSet}$  and all  $A: \mathbb{C} \longrightarrow \mathbb{M}$ , we get

$$X \otimes_{\mathbb{C}}^L A = X \otimes_{\mathbb{C}} Q_{\mathrm{Proj}}(A), \quad (18)$$

where  $Q_{\mathrm{Proj}}$  is the cofibrant replacement of the projective model structure given by the functorial factorization. Note that actually we should have written the right hand side of (??) as  $Q_{\mathrm{Inj}}(X) \otimes_{\mathbb{C}} Q_{\mathrm{Proj}}(A)$ , but the simplification is possible since each object is cofibrant in  $\mathbf{SSet}$  (for the Quillen model structure) and being cofibrant for the injective model structure on  $[\mathbb{C}^{op}, \mathbf{SSet}]$  means being cofibrant pointwise. Now, the homotopy category of a simplicial model category is enriched over  $\mathrm{Ho}(\mathbf{SSet})$  (which is a monoidal category), therefore, by Remark ??, the homotopy colimit

$$\mathrm{hcolim}: \mathrm{Ho}([\mathbb{C}, \mathbb{M}]) \longrightarrow \mathrm{Ho}(\mathbb{M})$$

as derived functor of the (enriched) colimit functor (as in Section 2) sends  $A: \mathbb{C} \longrightarrow \mathbb{M}$  into  $1 \otimes_{\mathbb{C}}^L A$  (where  $1: \mathbb{C}^{op} \longrightarrow \mathbf{SSet}$  is the functor with constant value the terminal object in  $\mathbf{SSet}$ ). Hence, we get the following formula for homotopy colimits, which is precisely the simplicially enriched version of the one given in Definition ??

$$\mathrm{hcolim}(A) \simeq \mathrm{colim} Q_{\mathrm{Proj}}(A), \quad \text{for } A: \mathbb{C} \longrightarrow \mathbb{M}. \quad (19)$$

The second way to compute  $(-) \otimes_{\mathbb{C}}^L (\bullet)$  is given under the assumptions of Theorem ???. Indeed, in such a situation we have, for  $X: \mathbb{C}^{op} \longrightarrow \mathbf{SSet}$  and  $A: \mathbb{C} \longrightarrow \mathbb{M}$ ,

$$X \otimes_{\mathbb{C}}^L A = Q_{\mathrm{Proj}}(X) \otimes_{\mathbb{C}} Q_{\mathrm{Inj}}(A). \quad (20)$$

Then, comparing with what we have just seen in (??) above, we could set

$$\mathrm{hcolim}(A) := Q_{\mathrm{Proj}}(1) \otimes_{\mathbb{C}} Q_{\mathrm{Inj}}(A), \quad \text{for } A: \mathbb{C} \longrightarrow \mathbb{M}. \quad (21)$$

Since the nerve functor  $N(- \downarrow \mathbb{C})^{op}: \mathbb{C}^{op} \rightarrow \mathbf{SSet}$  is a projective cofibrant replacement for  $1: \mathbb{C}^{op} \rightarrow \mathbf{SSet}$ , we can use (??) to get

$$\mathrm{hcolim}(A) \simeq \int^{n \in \mathbb{C}} N(n \downarrow \mathbb{C})^{op} \otimes (Q_{\mathrm{Inj}}(A))(n), \quad (22)$$

which is the enriched counterpart of (??). The picture of the link (actually, the equivalence) between the derived functor approach of (??) and the simplicial one (??) is then completed.

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