Localization Theory in an ∞-topos

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The setting

We work with an ∞-*topos* in the sense of [Lur09].

Roughly, this is a category $\mathscr{E}$ such that:

- for $X, Y \in \mathscr{E}$, there is a *space* of maps $\mathscr{E}(X, Y)$;
- for $X, Y \in \mathscr{E}$, there is an *internal hom* Y^X;
- a few other properties are satisfied (we will see some as we go on).

You can think $\mathscr{E} = \text{Spaces}$ or $\mathscr{E} = \text{sPre}(\mathscr{C})$.

Our work is inspired by results on localization in homotopy type theory — see [CORS18].

Reflective subcategories of $\mathscr{E}$...

Definition

A *reflective subcategory* of an ∞-topos $\mathscr{E}$ is a sub ∞-category $\mathscr{D}$ of $\mathscr{E}$ such that the inclusion $\mathscr{D} \hookrightarrow \mathscr{E}$ has a left adjoint $L\colon \mathscr{E} \to \mathscr{D}$ (the *reflector* of $\mathscr{E}$ into $\mathscr{D}$).

We can think of $\mathscr{D}$ as the localization of $\mathscr{E}$ at the maps inverted by L.

Example

Let $\mathscr{E} = \text{Spaces}$ and p a prime number. A space X is *p-local* if, for every $x \in X$ and every prime $q \neq p$, the q-power map

$$(\cdot)^q\colon \Omega(X, x) \to \Omega(X, x)$$

is a weak equivalence. Then $\mathscr{D} = \{p - \text{local spaces}\}$ is a reflective subcategory of Spaces.

...and of $\mathscr{E}_{/X}$.

For any $X \in \mathscr{E}$, there is an ∞-topos $\mathscr{E}_{/X}$ whose objects are *maps* in $\mathscr{E}$ with codomain X. It is often interesting to consider reflective subcategories of $\mathscr{E}_{/X}$ and not just of $\mathscr{E}$.

Example

For $\mathscr{E} = \text{Spaces}$, let p be a prime number and X a space. Say that a map $f \colon E \to X$ is *p-local* if, for every $x \in X$, $\text{hofib}_x(f)$ is a $p-$local space. Then

$$\mathscr{D}_X = \{\text{all } p\text{-local maps}\}$$

is a reflective subcategory of $\text{Spaces}_{/X}$.

We want to abstract these (and some other) features of p-localization to the context of an arbitrary ∞-topos.

Object of Study

Fix an ∞-topos $\mathscr{E}$.

Definition ([RSS17, Def. A.3])

A *reflective subfibration (RS) L of $\mathscr{E}$* consists of, for every $X \in \mathscr{E}$, a reflective subcategory $\mathscr{D}_X \subseteq \mathscr{E}_{/X}$ with reflector $L_X \colon \mathscr{E}_{/X} \to \mathscr{D}_X$.

This data has to satisfy:

1. Given $f \colon X \to Y$ in $\mathscr{E}$, $f^* \colon \mathscr{E}_{/Y} \to \mathscr{E}_{/X}$ restricts to $f^* \colon \mathscr{D}_Y \to \mathscr{D}_X$.
2. Given $f \colon X \to Y$ in $\mathscr{E}$ and $p \in \mathscr{E}_{/Y}$, the natural map $L_X(f^*(p)) \to f^*(L_Y(p))$ is an equivalence in $\mathscr{E}_{/X}$.

We call the objects of $\mathscr{D}_X$ *L-local maps*. If $A \to 1$ is an L-local map, we say that A is an *L-local object*.

A few examples

In general, a reflective *subcategory* of $\mathscr{E}$ does not give rise to a whole reflective *subfibration* of $\mathscr{E}$.

What *does* work:

1. If S is a *set* of maps in $\mathscr{E}$, there is an RS L^S of $\mathscr{E}$ such that $X \in \mathscr{E}$ is L^S-local iff $X^f : X^B \to X^A$ is an equivalence for each $f \in S$.

2. Given a stable factorization system $\mathcal{F} = (\mathcal{L}, \mathcal{R})$, there is an associated RS $L^{\mathcal{F}}$ of $\mathscr{E}$ for which

$$\mathscr{D}_X = \{f : E \to X : \ f \in \mathcal{R}\}$$

Local maps and descent

Fix an RS L of $\mathscr{E}$ throughout, with associated reflective subcategories $\mathscr{D}_X \subseteq \mathscr{E}_{/X}$, for $X \in \mathscr{E}$. Set

$$\mathcal{M}_L := \{\text{all } L\text{-local maps}\}$$

Proposition (V.)

$\mathcal{M}_L$ satisfies descent.

This means: given any pullback square with f epi

$$\begin{array}{ccc} E & \longrightarrow & M \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ X & \underset{f}{\twoheadrightarrow} & Y \end{array}, \qquad p \in \mathcal{M}_L \Leftrightarrow q \in \mathcal{M}_L$$

Corollary

If $\mathscr{E} = \text{Spaces}$, $p\colon E \to X$ is an L-local map if and only if, for all $x \in X$, $\text{hofib}_x(p)$ is L-local.

Corollary ([Lur09][Prop. 6.1.6.7])

$\mathcal{M}_L$ has a classifying map $u_L\colon \widetilde{\mathcal{U}}_L \to \mathcal{U}_L$.

This means that, for every $q\colon Z \to Y$ in $\mathcal{M}_L$, there is a pullback square

$$\begin{array}{ccc} Z & \longrightarrow & \widetilde{\mathcal{U}}_L \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle u_L} \\ Y & \xrightarrow{\chi_q} & \mathcal{U}_L \end{array}$$

where χ_q is the *characteristic map* of q.

Remark

The class of *all* maps in $\mathscr{E}$ also has a classifying map $u\colon \widetilde{\mathcal{U}} \to \mathcal{U}$

L-separated maps

Proposition (V.)

$\Delta(\mathcal{U}_L)\colon \mathcal{U}_L \to \mathcal{U}_L \times \mathcal{U}_L$ *is an L-local map.*

Definition

A map $p\colon E \to X$ in $\mathscr{E}$ is *L-separated* if $\Delta(p)\colon E \to E \times_X E$ is an L-local map.

Example

If $E \in \text{Spaces}$, $\Delta(E) \simeq (p_E\colon E^{\Delta^1} \twoheadrightarrow E \times E)$. Then E is L-separated iff, for $(e, e') \in E \times E$, $\text{Path}(e, e')$ is L-local.

Facts:

- Every L-local map is L-separated.
- L-separated maps satisfy descent.

Mixing local and separated

There is some nice interplay between L-local and L-separated maps.

Remark

If $p\colon E \to X$ is L-local and X is L-separated, then E is $L-$separated.

Here is something more serious. Say that a map $\eta'\colon X \to X'$ is an *L'-localization of X* if X' is L-separated and $Y^{\eta'}\colon Y^{X'} \to Y^{X}$ is an equivalence when Y is L-separated.

Proposition (V.)

Let $\eta'\colon X \to X'$ be an L'-localization of X. Then $p\colon E \to X$ is L-local if and only if

$$\begin{array}{ccc} E & \xrightarrow{\eta_{X'}(\eta' p)} & L_{X'}E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle L_{X'}(\eta' p)} \\ X & \xrightarrow[\eta']{} & X' \end{array}$$

is a pullback square.

Corollary (Folklore?)

For $n \geq -2$, a map $p\colon E \to X$ in $\mathscr{E}$ is n-truncated if and only if

$$\begin{array}{ccc} E & \xrightarrow{|\cdot|_{n+1}} & \|E\|_{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \|p\|_{n+1}} \\ X & \xrightarrow[|\cdot|_{n+1}]{} & \|X\|_{n+1} \end{array}$$

is a pullback square.

Theorem (V., Characterization of L'-localizations)

TFAE for a map $\eta'\colon X \to X'$.

1. *η' is an L'-localization of X.*
2. *η' is epi and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\Delta(\eta')} & X \times_{X'} X \\ & {\scriptstyle \Delta(X)}\searrow & \downarrow{\scriptstyle t} \\ & & X \times X \end{array}$$

exhibits t as the reflection of $\Delta(X)$ into $\mathscr{D}_{X \times X}$.

New reflective subfibration from old

In some cases, it's easy to see that L-separated maps are the local maps for another RS L' of $\mathscr{E}$.

Example

- If $f: A \to B$ is a map in $\mathscr{E}$ and $L = L^f$, the RS associated to the set $S = \{f\}$, then $L' = L^{\Sigma f}$.
- If $L = L_n$ is the RS associated to the factorization system

$$(\text{n-connected}, \text{ n-truncated}),$$

then $L' = L_{n+1}$.

Theorem (V., Work in Progress)

This is always the case: for any RS L of $\mathscr{E}$ there is an RS L' of $\mathscr{E}$ such that the L'-local maps are exactly the L-separated maps.

- M. Anel, G. Biedermann, E. Finster, and A. Joyal, *A Generalized Blakers-Massey Theorem*, ArXiv e-prints (2017), `arXiv:1703.09050`.
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Thank you for your attention!