I. Introduction

Despite Wittgenstein's own prolific writings on the topic, his ideas on the philosophy of mathematics are under-explored. A great deal of secondary literature concerns itself with exegetical controversies surrounding specific passages found in his later work that involve 'rule-following', 'meaning-as-use', 'family-resemblance' and other topics that have entered the philosophical lexicon.[1] In what follows I will argue that Wittgenstein's return to philosophy in the 1930s was motivated by mathematical and logical concerns about the *Tractatus* raised by constructivism, specifically surrounding quantification and the law of excluded middle.

It is estimated that half of Wittgenstein's writings from 1929-1944 were devoted to the philosophy of mathematics and logic (Monk 1990: 466). The neglect of these writings is partially due to the difficult and somewhat esoteric nature of both the topic and his discussion; familiarity with the *Principia* is assumed, as are the works of Frege, and it is widely supposed that Wittgenstein had read papers by Weyl, Skolem, Hilbert and other mathematicians of the time (Frascolla 1994: 43). But this is not the whole of the story—the posthumous publication of 'books' such as *On Certainty* or the *Investigations* gives the misleading appearance of unified, near-complete works. This is not the case—they are selections from the vast *Nachlass* :

> None of the material in the *Nachlass* has the finished form a publisher would expect ... Nearly every book that has been published under Wittgenstein's name has called for extensive and far-reaching decisions about how to select and arrange the source material in order to produce a conventional text. (Stern 1996: 446).

[1] Always prescient and ever pessimistic, Wittgenstein once concluded a year's lectures saying "the only seed I am likely to sow is a certain jargon". (Malcolm 1996: 63).

In the years immediately following Wittgenstein's death, the early choices made by his literary executors greatly influenced the direction of subsequent scholarship. The *Investigations*, the closest thing to a complete book Wittgenstein came to before his death, thus became the *de facto* reference for philosophers interested in Wittgenstein's thought post-*Tractatus*, and the selections which comprise that work are only peripherally concerned with logic and mathematics.

Moreover, the earliest reviews of Wittgenstein's philosophy of mathematics, following the publication of *Remarks on the Foundations of Mathematics* in 1956, were dismissive. This led many to feel that Wittgenstein was simply out of his depth in commenting on recent logical results. Wittgenstein's critics took this to impugn his philosophy altogether; and his partisans too often shied often from grappling with this part of his work (Floyd 2002: 67-68). The relative neglect, however, has recently been countered by thorough work from Marion, Frascolla and others, in which Wittgenstein's philosophy of mathematics is presented within a developmental perspective—from the quasi-formalism of the *Tractatus* to the 'finististic constructivism' of his middle period and subsequent refinements that came later on. I will draw on this work to make my case in what follows.

II. The Lectures of 1928

Wittgenstein 'retired' from philosophy after completing the *Tractatus*, becoming a school-teacher in Austria and then designing his sister's home in Vienna (von Wright 1966: 11-12). While involved in this architectural project, he was approached by members of the Vienna circle, who invited him to their meetings. To their dismay, he did not appear interesting in answering questions about the *Tractatus* (Feigl, quoted in Marion 2003: 104). This disdain for philosophical activity continued until 1929. It is supposed that his time in Vienna, and particularly his conversations with Ramsey, led him to return to philosophical speculations. What happened, exactly? Marion points

to Wittgenstein's attendance of a lecture by the intuitionist Brouwer as a decisive event. Both Feigl and von Wright mention the lecture, the latter commenting briefly that "it is rumoured to have been this which stirred him to take up philosophy again" (von Wright 1966: 13) and the former writing that

> When ... Brouwer was scheduled to lecture on intuitionism in mathematics in Vienna, Waismann and I managed to coax Wittgenstein, after much resistance, to join us in attending the lecture. When, afterwards, Wittgenstein went to a café with us, a great event took place. Suddenly and very volubly Wittgenstein began talking philosophy—at great length. Perhaps this was the turning point, for ever since that time, 1929, when he moved to Cambridge, Wittgenstein was a philosopher again (Feigl as, quoted in Marion 2003: 105).

The general atmosphere of excitement surrounding work in the foundations of mathematics could not have escaped Wittgenstein; after all, his original impetus towards philosophy was a result of curiosity into the foundations of mathematics while an engineering student (von Wright 1966: 4). Brouwer's Vienna lecture did not present any new or original material, but promoted his intuitionistic conception of mathematics.

While it is true that the distinguishing feature of constructive and intuitionistic mathematics is a rejection of the law of excluded middle (LEM), the original impetus was to interpret mathematical existence more strictly: in terms of 'constructivity,' that is, to only deal with what can be computed. The rejection of the LEM follows. This framework requires re-interpretation of standard connectives and quantifiers, of which disjunction and the universal quantifier are of particular interest here. In the case of the former, $P \vee Q$ is read as claiming that we either have a proof of P or a proof of Q. For the universal quantifier, we are to understand by $(x) . \in A\ P(x)$ that there is an algorithm which, when applied to (x) and together with a proof that it is an element of A, shows that property $P(x)$ holds (ibid., 96).

For the full re-interpretation we are indebted to the logician Heyting; and to return to classical mathematics, we need only add the LEM, $P \vee \sim P$, which leads to difficulties under the interpretation of disjunction given above: we must have either a construction of *P* or a of not-*P*.

Brouwer opened his lecture speaking of the temporal and causal grounding of mathematical abstraction, the so-called "ur-phenomenon of the falling apart of a life moment into two qualitatively distinct things" from which numerical multiplicity and the natural numbers are called forth (Brouwer, quoted in Marion 2003: 108). Mathematics, Brouwer claimed, is "an essentially languageless activity of the mind having its origin in the perception of a move of time" (ibid., 108). Marion suggests that while many elements of the *Tractatus* are similar to Brouwer, this was not one of them: Wittgenstein reacted strongly against the idea that a 'basal intuition' was needed in mathematics, rejecting it as "a form of psychologism" (ibid., 114). The essential difference here is that while Wittgenstein, in the *Tractatus*, took a strict operationalist stance—that is, mathematical propositions are purely syntactical constructions without truth-value—he would not see the need to affirm that what he views as rule-based symbolic manipulation is ultimately based on some languageless ur-phenomenon: for he already has an account of number.

This account rests on repeated iterations of the 'general form' of a natural number ($[0, \xi, \xi +1]$ (*Tractatus* 6.03)) which is given, in turn, in the context of his general conception of an 'operation' in the early sections of 6. It should be noted that for Wittgenstein the construction of propositions that follow from the application of an operation hinges on the formal properties of the propositions in question, and nothing else (viz. *Tractatus* 5.23; Frascolla 1994: 8-9). By this, Wittgenstein turns "numerical arithmetic into a sort of general theory of operations" (Frascolla 1998: 135). These operations require an agent to carry them out: hence they are a form of "phenomenological entity," contra Logicism (Marion 2003: 110). In

this sense, as Marion notes, Brouwer and Wittgenstein *are* close, despite the disagreement over basal intuitions. But Wittgenstein's insistence that mathematical equations are not truth-functional propositions—and hence neither true or false—also creates a certain tension with Brouwer's own position. Wittgenstein insists that the correctness of an equation or proof can be directly *seen*, but not compared with 'facts' regarding this correctness. This emphasis on perception and the manipulation of symbols is contrary to Brouwer's 'languageless' view (ibid., 113; 114).

By the late 1920s however Wittgenstein had begun to express doubts concerning the *Tractatus*. Indeed, by the time of his Cambridge lectures in the early 1930s, he became convinced of serious logical errors present in the *Tractatus*. Here we have the benefit of G.E. Moore's notes from the period:

> He [Wittgenstein] said that that there was a temptation, to which he had yielded in the *Tracatus*, to say that *(x) . fx* is identical with the logical product "*fa . fb . fc* ..." and (Ǝx) . *fx* identical with the logical sum "*fa v fb v fc* ..." but that this was in both cases a mistake [...] in supposing this he was committing a fallacy, which is common in the case of Mathematics, *e.g.* the fallacy of supposing that 1+1+1... is a sum, where it is only a *limit*. ... It is only a logical product if the dots are what he called the 'dots of laziness', as where we represent the alphabet by 'A, B, C... " and therefore the whole expression can be replaced by an enumeration (Moore 1993: 89).

Wittgenstein is making a typically constructivist point, namely that to treat infinite sequences as if they were merely arbitrarily long sequences of numbers is a mistake, doubly so because cases where universal and existential quantification do result in proper products and sums is relatively rare. It is an illegitimate extension of the

applicable logical notions to the infinite domain. Yet the assumption that this is not so is central to the *Tractatus*.[2]

In fact Wittgenstein's critiques of quantification theory during this period seem to demonstrate a commitment not only to constructivism, but finitism. While this critique is similar to that given by the intuitionists, Wittgenstein went much farther "by dispensing with quantification theory altogether" (Marion 1995: 156). Despite the admission of these errors, he is still committed to the same basic thesis concerning the essentially 'syntactical' nature of mathematics, against any form of 'Platonism' where our symbols merely *stand in* for numbers: "arithmetic doesn't talk about numbers, it works with numbers" (*Philosophical Remarks* §109; quoted in Rodych 2007: 2.1). Here we have another source of tension between intuitionism and Wittgenstein, as he criticized Brouwer for giving the misleading impression of having 'discovered' some "special facts about a certain class of propositions" where the law of excluded middle does not apply (Marion 1995: 159; 2003: 119).[3]

Wittgenstein's self-critique on quantification here echoes Weyl's, [4]who in 1921 argued that propositions with universal or existential quantifiers ranging over the natural numbers were not proper statements nor "abbreviations for an infinite disjunction" (Marion 1995: 146). It is rather a 'rule for judging', as Weyl wrote,

> a *hypothetical* proposition, saying something only in case that: "in case you come across a certain number (whatever number that may be), you may be sure if has the property [F] (Weyl, 'The Ghost of Modality', quoted ibid., 146).

[2] "The most important aspect of Wittgernstein's treatment of quantification [in the *Tractatus*] is that he assumed that there is no problem with quantification over an infinite domain" (Marion 1995: 143).

[3] It seems to be the case that Wittgenstein had to reject core theses of the *Tractatus* in order to maintain his anti-Platonism. More on this shortly.

[4] Weyl's views on the foundations of mathematics shift significantly over his lifetime; his enthusiasm, in the 1920s, for intuitionism waned considerably later on in his life, despite his very influential and widely-read articles in support of Brouwer.

Weyl's influence is likely, for Wittgenstein soon adopted a stance to infinite propositions at odds with the *Tractatus*, and closer to Weyl's, though expressed in his own terms: "the 'linguistic expression' of an infinite possibility is an infinite possibility in *language*" (Moore 1993: 90-91). This is because an infinite sequence cannot be understood extensionally—"infinity is the property of a law, not of its extension" (quoted in Marion 2003: 116).

Wittgenstein's interest in these questions can be gauged by the work he did on elaborating a constructive version of Euler's proof of the infinity of primes. The story begins with a conjecture by Felix Kaufmann in 1930 according to which "all existence theories of classical mathematics which do not rely on non-constructive existential axioms ... are in effect constructive," which an enterprising student named Heinrich Behmann set out to prove (Mancosu & Marion 2003: 172). Though unsuccessful in his general proof, he did provide an example of this method with Euler's proof and in a footnote claims that

> I simultaneously learned about a transformation of this proof into a constructive form by Mr. Wittgenstein and about Kaufmann's conjecture (ibid., 174).

Unearthing this comment prompted a search for this lost proof, which eventually turned up in a series of manuscripts dated c. 1930 that served as preparation for the *Big Typescript*, from which the book known as the *Philosophical Grammar* was eventually formed—where much of middle-period Wittgenstein's remarks on mathematics are culled. In this manuscripts one finds a series of calculations determining a bound below which a new prime greater than any given set of primes can be found (ibid., 175). Though the result is, according to commentators, of no great mathematical importance, it sheds light on Wittgenstein's conception of proof in mathematics. He is critical of Euler's original proof, commenting "no bridge has been

built, but we rest content on *seeing* the other bank ... in mathematics nothing *must* except what *is* - The bridge has to be built" (quoted ibid., 180). Wittgenstein's elaboration of a bound for the proof is perfectly in line with the type of constructive rigour demanded by Brouwer and Weyl. In the same section of the manuscript where the constructive proof is found, the authors report finding the modern-sounding remark that "in mathematics *everything* is algorithm and *nothing* is meaning" (ibid., 181). In a relatively recent paper Martin-Löf and Lozinski argued that

> it is the contention of the intuitionists (or constructivists, I shall use the terms synonymously) that the basic mathematical notions, above all the notion of function, ought to be interpreted in such a way that the cleave between mathematics (classical mathematics, that is) and programming that we are witnessing at present dissapears ... the true source of the uncomputable functions of classical mathematics is not the axiom of choice ... but the law of excluded middle and the law of indirect proof. (Martin-Löf & Lozinski 1984: 33).

Wittgenstein's strictures on mathematical proof should therefore not be seen as marginal or misdirected, but rather as both contemporary to the works of other leading mathematicians of the time and precursor to subsequent work in computability.[5] Much confusion surrounds his position, however, because without careful analysis his writing on the topic of the law of excluded middle seem ambiguous, if not downright hostile to the constructivist stance.

III. The Law of Excluded Middle

[5] To add another example: "The desire for algorithmic interpretability forces us to use intuitionistic logic, and that restriction of our logic seems to result, inevitably, in arguments that are entirely algorithmic in character." (Bridges 1999: 97).

Different commentators have advanced different theses concerning Wittgenstein's stance on the law of excluded middle. Peter Hacker, as Marion notes, argued in *Insight and Illusion* that Wittgenstein had no qualms whatsoever with the LEM (Marion 1995: 156). This is most likely an erroneous interpretation; still, Wittgenstein's position requires some spadework to discover, for often he will agree with a particular position, but sharply disagree with the underlying motivations given for its adoption. This disagreement often is mistaken for rejection of the thesis. So it is with the constructivist rejection of the LEM. It should also be noted that Wittgenstein attributes himself incorrect positions to his targets—Moore's notes from the Cambridge lectures of Lent 1930 reveal the following:

> What we are to say of the apparent question "Are there anywhere in the development of *pi* three consecutive 7's?" ... He first dealt with this apparent question in connexion with Brouwer's view that the Law of Excluded Middle does not apply to some mathematical propositions; i.e that some mathematical propositions are neither true nor false; that there is an alternative to being either true or false, *viz.* being 'undecidable' (Moore 1993: 94).

This, of course, is a distortion of Brouwer's position and of intuitionism in general. It is not as simple as the introduction of a 'third value', as shown by the validity of the double-negation of the LEM, for double-negation creates a weaker claim (on this, see Frascolla 1998: 106)[6]. Nevertheless Wittgenstein's position on the matter is far more sympathetic to the intuitionists than is commonly supposed. The passage above continues in a fashion once again strikingly similar to constructivist arguments:

[6] *P* requires a proof of *P*, while ~ ~*P* requires merely a proof that *P* does not lead to a contradiction. Wittgenstein repudiates this in the *Tractatus* specifically, viz. *TLP* **5.44**: "And if there were an object called '~', then '~ ~P' would have to say something other than P. For the one proposition would then treat of ~, the other would not." This becomes another fundamental shift away from the *Tractatus* largely due to constructivist concerns and arguments.

> [Wittgenstein] said we seem to be able to define *pi** as the number which, if there are three consecutive 7's in the development of *pi*, differs from *pi* in that, in the place in which three consecutive 7's occur in *pi*, there occur in it three consecutive 1's instead, but which, if there are not, does not differ from *pi* at all; and that we seem to be able to say that *pi**, so defined, either is identical with *pi* or not. But he said here that, since we have no way of finding out whether *pi** is identical with *pi* or not, the question whether it is or not 'has no meaning' (Moore 1993: 95).

Wittgenstein here thus presents his own example of a question to which the rule $p \vee \sim p$ is simply inapplicable, for we have no way of answering it. And this is where many commentators stumble—Wittgenstein often says such questions are 'meaningless', or more strongly even

> I need hardly say that where the law of excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer.) (Quoted in Marion 2003: 120).

But one must be careful here. As Marion and Frascolla both correctly note, what is being asserted here is not that the existence of meaningful mathematics propositions shows us that the LEM must apply, but that propositions such as the one above are not mathematical. This is more far-reaching than what the intuitionists claim: according to Wittgenstein, the LEM is valid for *no* genuine mathematical propositions. As I noted earlier, Wittgenstein objects to Brouwer's presentation of such cases as a sort of *discovery* made about infinite sets or peculiar numbers. In Marion's words, "Wittgenstein holds Brouwer to be right, but for the wrong reasons" (ibid., 120).[7]

[7] Marion again: "[For Wittgenstein] Brouwer incorrectly spoke as if he had just discovered some special fact about a certain class of propositions (those about infinite sets) in the same way as a physicist would speak of discovering the laws of nature" (ibid., 121). Wittgenstein's vehemence on the topic should be understood in light of his ruffled ontological sensibilities and not interpreted as a *casus belli* against Brouwer's programme.

This much should be clear already just from Wittgenstein's constructivization of Euler's proof above.

We may still legitimately inquire further as to what Wittgenstein intends to show us by insisting on the 'meaninglessness' of certain types of propositions here. In the case of *pi* and *pi** quoted above, the occurrence of three 7's—or any other string—is often mistakenly understood within the conception of the expansion of *pi* as a particularly long finite string, a totality which could be completed, in principle. Because of the possibility of applying $p \vee \sim p$ in relation to the occurrence of a string in the finite case, we are misled by this picture to suppose that the logical principle holds in the infinite case as well. This is very close, obviously, to the quantificational error Wittgenstein identified as problematic in the *Tractatus*. Now Wittgenstein also identifies a profound disconnect between finite and infinite sequences: he holds, as I mentioned above, that infinite sequences are 'rules for judging' or again 'properties of a law' and not of the extension, extension which *cannot be given*. It is not a question of peculiarly 'human' limitations: infinite strings are simply *not* numbers. A constructive answer to the *pi* question would involve a formula capable of producing an arbitrary string in the expansion of *pi* and giving its location, or at least providing a lower bound where the string could be found. In such a result, of course, dependence on LEM is nowhere to be found—hence Wittgenstein's stance that no applications of it are valid. Instead we should view general proofs, or proofs by induction, as mere schemata or templates which have to be filled in: they do not, by themselves, assert anything: "A proof by induction, if it were a proof, would be a proof of generality, not a proof a certain property for all numbers" (quoted in Marion 1995: 152). It is illegitimate to speak of a property holding for 'all numbers', since the notion of 'all numbers' is a mistake, one caused by the misleading picture of infinite strings as essentially similar to finite ones, but somehow just 'longer'.

Wittgenstein returned to this issue in the *Remarks on the Foundation of Mathematics*. Written from 1937-1944, the comments there illustrate a certain continuity in his thought on this topic. Returning to the question of an arbitrary string in *pi*, Wittgenstein shifts the focus to interpretation and decision—the rules that have to be *created* in order for a question to become meaningful: "the question—I want to say—changes its status, when it becomes *decidable*. For a connexion is made then, which formerly *was not there*" (RFM V, §9). Here we see the motivations of the later Wittgenstein more clearly—namely, the pursuit of philosophical therapy, the exorcism of false pictures, the very ones which he worked through in the early 1930s:

> When someone hammers away at us with the law of excluded middle as something which cannot be gainsaid, it is clear that there is something wrong with his question ... and if you say that the infinite expansion must contain the pattern *ó* or not contain it, you are so to speak shewing us the picture of an unsurveyable series reaching into the distance. But what is the picture began to flicker in the far distance? (RFM V, §10).

Difficult to see now, in the light of these remarks, any possible ambiguity concerning Wittgenstein's position relative to the LEM. Many difficult passages in the *Remarks on the Foundations of Mathematics* become fairly straightforward when interpreted in light of Wittgenstein's particular brand of constructive, 'algorithmic' mathematics.' To add one final example concerning the LEM—this imagined dialogue:

> 'Well, doesn't the rule of expansion *determine* the series completely? And if it does so, if it allows of no ambiguity, then it must implicitly determine *all* questions about the structure of the series.' Here you are thinking of finite series ... That is correct if it is supposed to mean that it is not the case that e.g. the so-and-so many'th is *not* determined. But you can see that *that* gives you no information about whether a particular pattern is going to appear in the series (if it has not appeared so far). *And so we can see* that we are using a misleading *picture*. (RFM V, §11).

Wittgenstein's treatment of the occurrence of a given string in an infinite expansion parallels rather precisely the understanding of mathematical existence of the intuitionists, though it goes without saying that Wittgenstein had both different motivations and aims. This is most likely the cause of the continuing controversy surrounding his position on the matter.

IV. Conclusion

It is difficult to trace the exact lines of influence that converged on Wittgenstein, after the *Tractatus*, to spur him to take up philosophy again—and, in the first few years of this return, to devote much of his time to questions in mathematics. It is significant that Wittgenstein never sought to re-work or amend the *Tractatus* to reflect his changes of opinion, preferring to write in a dialectical, dialogical fashion. Here his concern no doubt was to avoid creating the 'misleading pictures' he so often took to task, for example in Brouwer where a latent 'descriptivism' insinuates itself in the way in which he presents his arguments, or so Wittgenstein thought. Again and again Wittgenstein warns against falling into this trap—

> The dangerous, deceptive thing about the idea: "The real numbers cannot be arranged in a series," or again "The set is not denumerable" is that is makes the determination of a concept—concept formation—look like a fact of nature (RFM II §18).

Striving to cleanse mathematics and indeed philosophy of Platonic remnants of this sort underlies and explains Wittgenstein's sometimes puzzling formulations. This is especially subtle in mathematics where the inevitability of result often translates, darkly, into a feeling of discovery, not creation, as one works. Yet the creative act lies at the bottom of all intellectual endeavours: "Why should we say: The irrational numbers cannot be ordered?—We have method by which to upset *any* order" (RFM II §28). It is this methodology with which

Wittgenstein's return to philosophy is fundamentally concerned; it so happens that constructivity lays it bare in the case of mathematics and pushed Wittgenstein backs towards philosophy.

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