Mandelbrot Polynomials

Mandelbrot polynomials are defined by
\[ p_k(z) = z^k + 1 \]
for \( k \geq 0 \).

Properties of \( p_k(z) \)
- The degree of \( p_k(z) \) for \( k > 0 \) is \( 2^{k-1} - 1 \).
- The roots of \( p_k(z) \) are periodic points of the Mandelbrot set with period \( k \).
- The coefficients of \( p_k(z) \) when expressed in the monomial basis \( 1, z, z^2, \ldots \) are nonnegative integers.
- Derivatives are can be computed from the recurrence relation \( p_{k+1}(z) = p_k(p_k(z)) \).
- \( p_k(z) \) and \( p_{k-1}(z) \) can be simultaneously evaluated via their recurrence relations at a cost of \( O(k) \) operations.

Location of Roots

By considering the permuted polynomial
\[ p_k(z) = z(p_{k-1}(z))^2 + 1 \]
we can see the relationship between the location of the roots of \( p_k(z) \) and \( p_{k-1}(z) \). Figure 2 shows the level curve \( |p_k(z)| = 1 \), the roots of \( p_k(z) \) as circles, the roots of \( p_{k-1}(z) \) as squares and the small black dots are the roots of \( p_p(z) \) as \( \gamma \) varies from 0 to 1.

Mandelbrot Matrices

The Mandelbrot polynomials can also be generated as the characteristic polynomials of a family of recursively constructed upper Hemingway matrices, \( M_k \), defined as follows: let \( r_k = [0 \ 0 \ 0 \ 0 \ 1] \) and \( c_k = [1 \ 0 \ 0 \ 0 \ 0] \) be row and column vectors of length \( 2^{k-1} \), and
\[ U_k \in \mathbb{C}^{2^{k-1} \times 2^{k-1}} \]

The LU factors are then
\[ U_k = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \theta_k \end{bmatrix} \]
\[ U_{k+1} = \begin{bmatrix} 
\sigma_k & \epsilon_k \hat{T}_k \\
\hat{r}_k & \hat{c}_k & \hat{T}_k \\
\end{bmatrix} \]
where the \( \hat{\epsilon}_k \)'s are blocked conformally with the \( U_k \)'s. The \( \theta_k \)'s are defined as follows: let \( \theta_k = 1 + \gamma \) and for \( 2 \leq l \leq k \)
\[ \epsilon_k \hat{U}_k = (\hat{I}_k - \epsilon_{k-1} \sigma_k \hat{r}_k) \hat{U}_{k-1} \]

Krylov Based Solvers

- For values of \( k > 13 \) dense eigenvalue methods become computationally intractable.
- \( M_k \) and \( 4 - M_k \) are both very sparse, with \( 2n-1 \) and \( 2n-3 \) nonzero entries respectively.
- We can use Krylov based eigenvalue techniques to locate the eigenvalues of \( M_k \) near a complex shift \( \sigma \) if we are able to solve \( (\sigma I - M_k)x = b \) efficiently.
- The roots \( \hat{\epsilon}_{k-1} \), \( 1 \leq j \leq 2^{k-1} - 1 \) of \( p_{k-1}(z) \) are close to the roots of \( p_k(z) \), and thus we can use these as shifts \( \sigma \) for a Krylov based eigenvalue solver.

LU Decomposition

Here we develop the LU decomposition of the resolvent matrix \( (\sigma I - M_k) \). Firstly let:
\[ P_k = \begin{bmatrix} 1 & 1 \\ \end{bmatrix} \]
then factor the permuted matrix
\[ L_k U_k = P_k \sigma (I - M_k) \]
where \( U_k \) is unit-upper triangular and \( L_k \) is lower triangular.

References