Mandelbrot Polynomials and Matrices

Piers Lawrence and Rob Corless

ORCCA Ontario Research Centre for Computer Algebra

Western University · Canada

Mandelbrot Polynomials

Mandelbrot polynomials are defined by $p_0(\zeta) = 0$ and for $k \geq 0$ by

$$p_{k+1}(\zeta) = \zeta \left(p_k(\zeta)\right)^2 + 1\,.$$

Properties of $p_k(\zeta)$

- The degree of $p_k(\zeta)$ for $k > 0$ is $2^{k-1} - 1$.
- The roots of $p_k(\zeta)$ are periodic points of the Mandelbrot set with period k.
- The coefficients of $p_k(\zeta)$ when expressed in the monomial basis $1, \zeta, \zeta^2$, and so on, are nonnegative integers.
- Derivatives are can be computed from the recurrence relation by $p_0'(\zeta) = 0$ and for all $k \geq 0$ by

$$p_{k+1}'(\zeta) = p_k(\zeta)\left(p_k(\zeta) + 2\zeta p_k'(\zeta)\right)\,.$$

- $p_k(\zeta)$ and $p_k'(\zeta)$ can be simultaneously evaluated via their recurrence relations at a cost of $O(k-1)$ operations.

Results

The aim of this work was to compute the roots of the polynomials $p_k(\zeta)$, with the ultimate goal of computing all $2^{20} - 1 = 1{,}048{,}575$ roots of $p_{21}(\zeta)$ shown in Figure 1. Parallel eigenvalue computations were run on SHARCNET taking approximately 31 serial computing years.

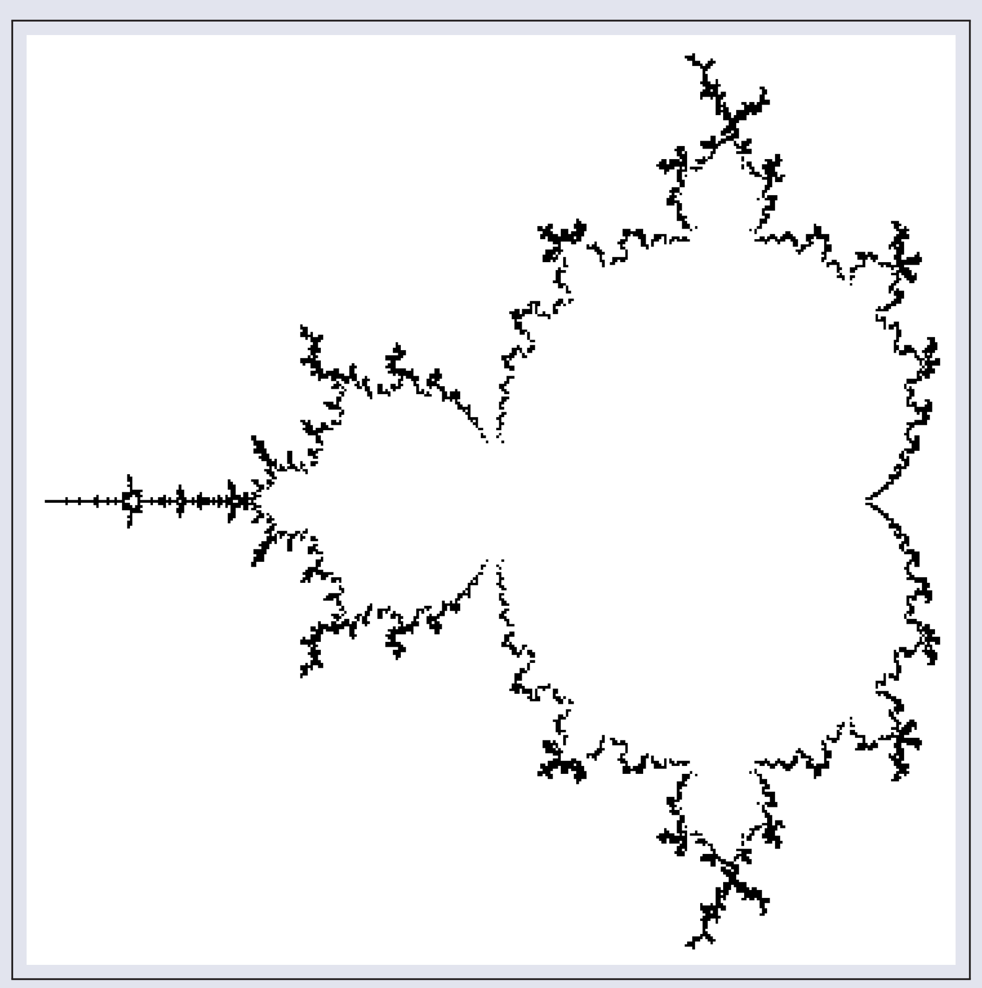

Figure 1: Roots of $p_{21}(\zeta)$

Monomial Basis Coefficients

Theorem 1. *The coefficients of $p_k(\zeta)$ expanded in the monomial basis contain both 1 and a number larger than $2^{2^{k-3}}$, for $k \geq 3$. That is, the coefficients of $p_k(\zeta)$ in the monomial basis grow doubly exponentially fast with k.*

Thus the condition number for evaluation of $p_k(\zeta)$ in the monomial basis

$$B_k(\zeta) = \sum_{i=0}^{2^{k-1}-1} |c_i||\zeta^i|\,,$$

where the c_i's are the monomial basis coefficients, must also grow doubly exponentially fast with k. This shows the ill-conditioning of the monomial basis in this case.

Location of Roots

By considering the perturbed polynomial

$$p_{k,\varepsilon}(\zeta) = \zeta\left(p_{k-1}(\zeta)\right)^2 + \varepsilon^2$$

we can see the relationship between the location of the roots of $p_k(\zeta)$ and $p_{k-1}(\zeta)$. Figure 2 shows the level curve $|p_5(\zeta)| = 1$, the roots of $p_{5,0}(\zeta)$ as circles, the roots of $p_5(\zeta)$ as squares and the small black dots are the roots of $p_{5,\varepsilon}(\zeta)$ as ε varies from 0 to 1.

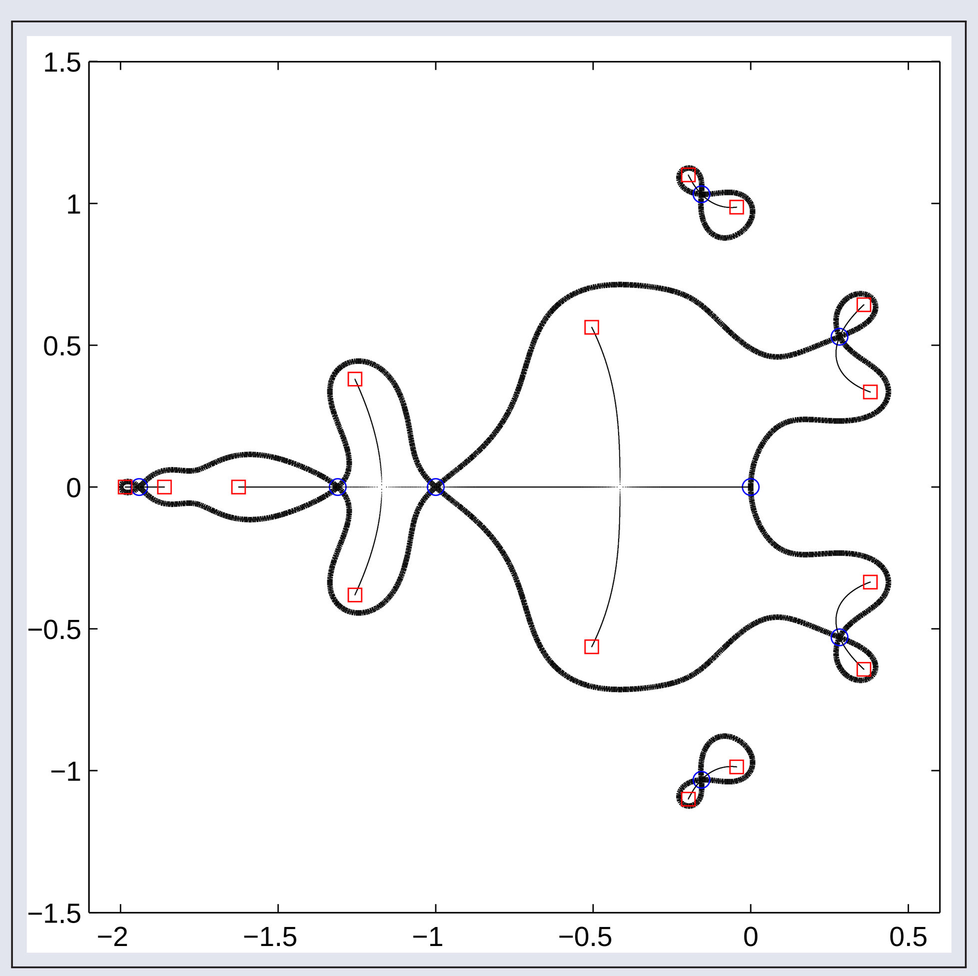

Figure 2: Level curve $|p_5(\zeta)| = 1$ and locations of roots of $p_{5,0}(\zeta)$, $p_5(\zeta)$ and $p_{5,\varepsilon}(\zeta)$ with $\varepsilon \in [0, 1]$

The complex roots of $p_k(\zeta)$ can be seen to be very close to the roots of $p_{k-1}(\zeta)$, this is not surprising as the roots tend to cluster near the boundary of the Mandelbrot set.

Mandelbrot Matrices

The Mandelbrot polynomials can also be generated as the characteristic polynomials of a family of recursively constructed upper Hessenberg matrices, $\mathbf{M}_k$, defined as follows: let $\mathbf{r}_k = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ be row and column vectors of length $2^{k-1} - 1$. Put

$$\mathbf{M}_2 = \begin{bmatrix} -1 \end{bmatrix}$$

and

$$\mathbf{M}_{k+1} = \begin{bmatrix} \mathbf{M}_k & & -\mathbf{c}_k\mathbf{r}_k \\ -\mathbf{r}_k & 0 & \\ & -\mathbf{c}_k & \mathbf{M}_k \end{bmatrix}$$

for all $k > 1$. It can be shown by repeatedly applying Laplace expansion on the $2^{k-1}th$ column of $(\zeta\mathbf{I} - \mathbf{M}_k)$ and the resulting submatrices that

$$\det\left(\zeta\mathbf{I} - \mathbf{M}_k\right) = p_k(\zeta)\,.$$

Thus the eigenvalues of $\mathbf{M}_k$ are exactly the zeros of $p_k(\zeta)$ that we wish to compute.

Krylov Based Solvers

- For values of $k > 13$ dense eigenvalue methods become computationally intractable.
- $\mathbf{M}_k$ and $\zeta\mathbf{I} - \mathbf{M}_k$ are both very sparse, with $2n - 1$ and $5/2n - 3/2$ nonzero entries respectively.
- We can use Krylov based eigenvalue techniques to locate the eigenvalues of $\mathbf{M}_k$ near a complex shift σ if we are able to solve $(\sigma\mathbf{I} - \mathbf{M}_k)\mathbf{x} = \mathbf{b}$ efficiently.
- The roots $\xi_{k-1,j}$ $1 \leq j \leq 2^{k-2} - 1$ of $p_{k-1}(\zeta)$ are close to the roots of $p_k(\zeta)$, and thus we can use these as shifts σ for a Krylov based eigenvalue solver.

LU Decomposition

Here we develop the LU decomposition of the resolvent matrix $(\sigma\mathbf{I} - \mathbf{M}_k)$. Firstly let:

$$\mathbf{P}_k = \begin{bmatrix} & \mathbf{I} \\ 1 & \end{bmatrix}$$

then factor the permuted matrix

$$\mathbf{L}_k\mathbf{U}_k = \mathbf{P}_k(\sigma\mathbf{I} - \mathbf{M}_k)$$

where $\mathbf{U}_k$ is unit-upper triangular and $\mathbf{L}_k$ is lower triangular.

LU Decomposition (continued)

$\mathbf{U}_k$ can be defined as follows: let $\hat{\mathbf{r}}_k = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\hat{\mathbf{c}}_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ be row and column vectors of length 2^{k-1}, and $\hat{\mathbf{U}}_i \in \mathbb{C}^{2^{i-1}\times 2^{i-1}}$ by

$$\hat{\mathbf{U}}_{i+1} = \begin{bmatrix} \hat{\mathbf{U}}_i & \sigma\hat{\mathbf{r}}_i^T\hat{\mathbf{c}}_i^T + \hat{\mathbf{c}}_i\hat{\mathbf{r}}_i \\ & \hat{\mathbf{U}}_i \end{bmatrix}, \quad \hat{\mathbf{U}}_1 = 1\,.$$

The *LU* factors are then

$$\mathbf{U}_k = \begin{bmatrix} \hat{\mathbf{U}}_1 & \sigma\hat{\mathbf{r}}_1^T\hat{\mathbf{c}}_2^T & & & \\ & \hat{\mathbf{U}}_2 & \sigma\hat{\mathbf{r}}_2^T\hat{\mathbf{c}}_3^T & & \\ & & \hat{\mathbf{U}}_3 & \ddots & \\ & & & \ddots & \sigma\hat{\mathbf{r}}_{k-2}^T\hat{\mathbf{c}}_{k-1}^T \\ & & & & \hat{\mathbf{U}}_{k-1} \end{bmatrix}$$

$$\mathbf{L}_k = \begin{bmatrix} \mathbf{I} & & \\ & \mathbf{I} & \\ \boldsymbol{\ell}_1 & \cdots & \boldsymbol{\ell}_{k-1} \end{bmatrix}$$

where the $\boldsymbol{\ell}_i$'s are blocked conformally with the $\hat{\mathbf{U}}_i$'s. The $\boldsymbol{\ell}_i$'s are defined as follows: let $\boldsymbol{\ell}_1 = \sigma + 1$ and for $2 \leq i \leq k-1$

$$\boldsymbol{\ell}_i\hat{\mathbf{U}}_i = \left(\hat{\mathbf{r}}_i - \boldsymbol{\ell}_{i-1}\sigma\hat{\mathbf{r}}_{i-1}^T\hat{\mathbf{c}}_i^T\right)\,.$$

The cost of one linear solve using this LU decomposition is $5 \cdot 2^{k-2} - k - 2$ operations.

Largest Roots of $p_k(\zeta)$

It was shown in [1] that

$$p_k\left(-2 + \frac{3}{2}\theta^2 \cdot 4^{-k}\right) = -\cos\theta + \left(-\frac{1}{8}\theta^3\sin\theta k + \theta^2 b(\theta)\right)4^{-k} + O(4^{-2k})$$

where

$$b(\theta) = -\frac{1}{2} + \frac{3}{8}\theta^2 - \frac{1}{18}\theta^4 + \frac{127}{43200}\theta^6 + \cdots$$

satisfies

$$b(\theta) = 4\cos\frac{1}{2}\theta \cdot b\left(\frac{1}{2}\theta\right) + \frac{1}{8}\theta\sin\theta + \frac{3}{2}\cos^2\frac{1}{2}\theta\,.$$

This approximately locates several zeros of $p_k(\zeta)$, near $\theta = (2\ell+1)\pi/2$, so long as $(2\ell+1)\pi/2 < O(2^{k/4})$.
Explicitly the zeros are near

$$z_{k,\ell} = -2 + \frac{3}{2}\left(\frac{(2\ell+1)\pi}{2}\right)^2 4^{-k} + O(4^{-2k})\,.$$

Interlacing

The largest roots have a curious interlacing property: between every root of $p_k(\zeta)$ there are two roots of $p_{k+1}(\zeta)$, as can be seen in Figure 3.

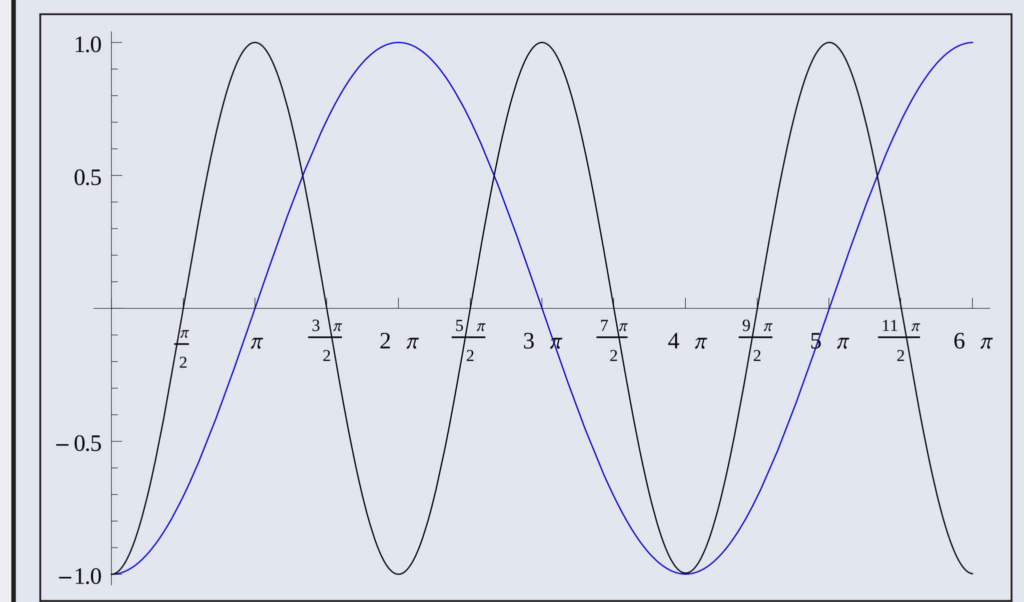

Figure 3: Plot of $p_{10}\left(-2 + \frac{3}{2}\theta^2 \cdot 4^{-9}\right)$ in black and $p_9\left(-2 + \frac{3}{2}\theta^2 \cdot 4^{-9}\right)$ in blue

Mandelbrot map iterations

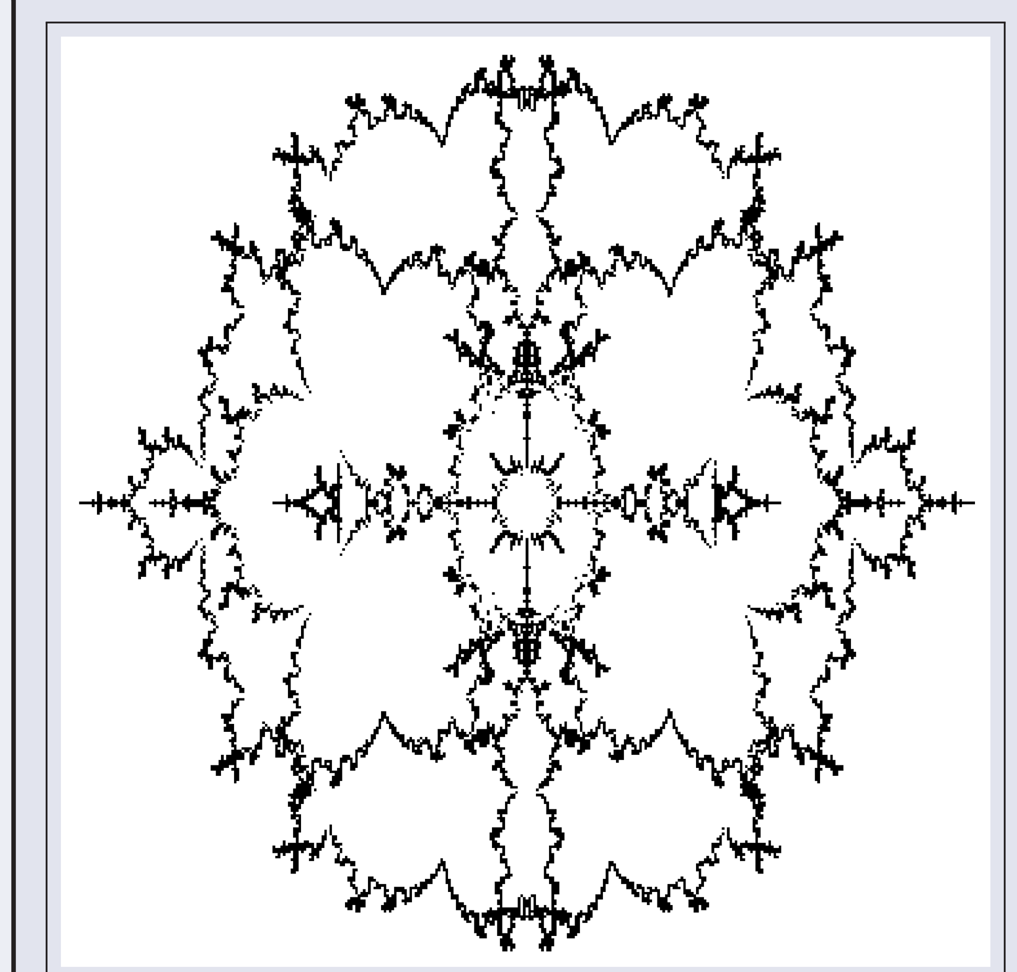

Figure 4: Iterating the Mandelbrot map $z \mapsto z^2 + c$ starting from the critical point $z = 0$ with the parameter c being the roots $\xi_{21,j}$ of $p_{21}(\zeta)$

References

[1] Robert M. Corless and Piers W. Lawrence. The largest roots of the Mandelbrot polynomials. In D. Bailey, H.H. Bauschke, P. Borwein, F. Garvan, M. Thera, J. Vanderwerff and H. Wolkowicz, editors, *Computational and Analytical Mathematics*, Springer Proceedings in Mathematics & Statistics, 2012. to appear.

[2] Robert M. Corless and Piers W. Lawrence. Mandelbrot Polynomials and Matrices, in preparation