Unconditional Means or Intercept-only MLM model

Level 1:

$$\boldsymbol{y_{ij} = \beta_{0j} + e_{ij}}$$

where β_{0j} is the mean* for group *j and* e_{ij} (others use r_{ij}) represents residual individual differences from the mean of group *j*.

Level 2:

$$\boldsymbol{\beta_{0j} = \gamma_{00} + \mu_{0j}}$$

where γ_{00} is the grand mean and μ_{0j} is the deviation between the group means and the grand mean.

With substitution:

$$\boldsymbol{y_{ij} = \gamma_{00} + \mu_{0j} + e_{ij}}$$

Level 2 Model Addresses 3 Questions (Kahn, 2011)

1. What are the average intercepts and slopes across groups? (fixed coefficients)

2. How much do the intercepts and slopes vary across groups? (random coefficients)

3. How useful are group-level variables for predicting group intercepts and slopes?

Peugh (2010) Example

Table 1
Model summaries: cross-sectional examples.

Parameters	Unconditional	Level-1: fixed	Level-1: random	Interaction
Regression coefficients (fixed effects)				
Intercept (γ_{00})	18.90 (.07)**	18.89 (.07)**	18.89 (.07)**	18.90 (.07)**
Student SES (γ_{10})	–	2.00 (.06)**	2.00 (.07)**	2.00 (.07)**
Student-to-Teacher Ratio (γ_{01})	–	–	–	−.10 (.01)**
Interaction (γ_{11})	–	–	–	−.04 (.02)**
Variance components (random effects)				
Residual (σ^2)	18.67 (.25)**	17.16 (.23)**	16.97 (.24)**	16.97 (.24)**
Intercept (τ_{00})	4.18 (.28)**	4.41 (.28)**	4.45 (.28)**	4.15 (.27)**
Slope (τ_{11})	–	–	.54 (.21)**	.49 (.21)**
Covariance (τ_{01})	–	–	.68 (.19)**	.59 (.18)**
Model summary				
Deviance statistic	71,308.01	70,394.40	70,374.06	70,310.45
Number of estimated parameters	3	4	6	8

Parameter estimate standard errors listed in parentheses.
** $p<.01$.

From: Peugh, J. L. (2010). A practical guide to multilevel modeling. Journal of School Psychology, 48, 85-112.

Peugh (2010) Example
Random Coefficients (intercepts only in this example)

Level-1

In the example below we add a level-1 predictor SES that has been group-mean centered

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_j\right) + e_{ij}$$

Level-2

$$\beta_{0j} = \gamma_{00} + \mu_{0j}$$

(same as in intercept model; the first term is the grand mean, and the second term is a residual that represents the variation in achievement means across schools).

$$\beta_{1j} = \gamma_{10}$$

Fixed coefficient – the impact of student SES on Ach does not vary across schools (lacks a μ_{1j} term)

Peugh (2010) Example

Combined

$$Y_{ij} = \gamma_{00} + \gamma_{10}\left(SES_{ij} - \overline{SES}_j\right) + \mu_{0j} + e_{ij}$$

Regression coefficient (fixed coefficients)

γ_{00} = 18.89 (grand mean or the mean of the school Achievement means)
γ_{10} = 2.00 (impact of SES on Achievement)

Variance components (random coefficients)

$var\ of\ \mu_{0j} = \tau_{00}$ = 4.41 (variance of group/school mean Ach scores)
$var\ of\ e_{ij} = \sigma^2$ = 17.16 (residual variance in Ach scores across students)

Peugh (2010) Example
Adding a Random Slope Coefficient

Level 1

$$Y_{ij} = \beta_{0j} + \beta_{1j}(SES_{ij} - \overline{SES}_j) + e_{ij}$$

Level 2

$$\beta_{0j} = \gamma_{00} + \mu_{0j}$$
$$\beta_{1j} = \gamma_{10} + \mu_{1j}$$

Combined

$$Y_{ij} = \gamma_{00} + \gamma_{10}(SES_{ij} - \overline{SES}_j) + \mu_{0j} + \mu_{1j}(SES_{ij} - \overline{SES}_j) + e_{ij}$$

The variance estimate for μ_{1j} is τ_{11}

Although not specified in the equation there is also a covariance term between the intercepts and the slopes (when both of these are random). This term is labeled τ_{01}. (In the example, the positive covariance estimate indicates that schools with higher SES-Ach slopes tend to have higher Ach means.)

Peugh (2010) Example
Adding a Level 2 Predictor
(Intercepts and Slopes as Outcomes Model)

Predictors at level-2 can be added to explain the variation in intercept and slope variance. In the Ach example, the level-2 predictor is ST_Ratio. I will label it STR. The level-2 equations are:

Level 1

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_j\right) + e_{ij}$$

Level 2

$$\beta_{0j} = \gamma_{00} + \gamma_{01}\left(STR_j - \overline{STR}\right) + \mu_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}\left(STR_j - \overline{STR}\right) + \mu_{1j}$$

Combined

$$\begin{aligned} Y_{ij} = \gamma_{00} &+ \gamma_{01}\left(STR_j - \overline{STR}\right) + \gamma_{10}\left(SES_{ij} - \overline{SES}_j\right) \\ &+ \gamma_{11}\left(SES_{ij} - \overline{SES}_j\right)\left(STR_j - \overline{STR}\right) + \mu_{0j} \\ &+ \mu_{1j}\left(SES_{ij} - \overline{SES}_j\right) + e_{ij} \end{aligned}$$

Peugh (2010) Example
Adding a Level 2 Predictor
(Intercepts and Slopes as Outcomes Model)

$$Y_{ij} = \gamma_{00} + \gamma_{01}(STR_j - \overline{STR}) + \gamma_{10}(SES_{ij} - \overline{SES}_j) + \gamma_{11}(SES_{ij} - \overline{SES}_j)(STR_j - \overline{STR}) + \mu_{0j} + \mu_{1j}(SES_{ij} - \overline{SES}_j) + e_{ij}$$

Table 1
Model summaries: cross-sectional examples.

Parameters	Unconditional	Level-1: fixed	Level-1: random	Interaction
Regression coefficients (fixed effects)				
Intercept (γ_{00})	18.90 (.07) **	18.89 (.07) **	18.89 (.07) **	18.90 (.07) **
Student SES (γ_{10})	–	2.00 (.06) **	2.00 (.07) **	2.00 (.07) **
Student-to-Teacher Ratio (γ_{01})	–	–	–	−.10 (.01) **
Interaction (γ_{11})	–	–	–	−.04 (.02) **
Variance components (random effects)				
Residual (σ^2)	18.67 (.25) **	17.16 (.23) **	16.97 (.24) **	16.97 (.24) **
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Model summary				
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Number of estimated parameters	3	4	6	8

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Example (from Joop Hox book, *Multilevel Analysis*, 2010)

DATA

Level 1: pupils (2000 in total)

Level 2: Classes (100 classes of approx 20 pupils each)

Outcome variable:
pupil popularity (scale 0-10)

Predictor variables at pupil level (level 1)
pupil gender (1=girl 0=boy)
pupil extraversion (scale 1-10)

Predictor variable(s) at class level (level 2)
teacher experience (scale 2-25)

Mplus Syntax: Intercept Only Model

```
TITLE:  Random Effects ANOVA using Mplus;
DATA:
  FILE IS "popular2.dat";
VARIABLE:
NAMES ARE class pupil cons extrav sex texp popular
popteach zextrav zsex ztexp zpopular zpoptch;
usevariables are popular;
CLUSTER IS class;
ANALYSIS:
TYPE IS TWOLEVEL;
ESTIMATOR IS ML; !default is MLR
MODEL:
  %within%
  popular;
 %between%
  popular;
OUTPUT: SAMPSTAT;
```

Mplus output: Intercept Only Model

```
SUMMARY OF DATA

     Estimated Intraclass Correlations for the Y Variables

                Intraclass
     Variable   Correlation

     POPULAR       0.362
```

0.695/(0.695 + 1.222) = .362

```
MODEL RESULTS
                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value
Within Level

 Variances
    POPULAR            1.222      0.047     26.199      0.000

Between Level

 Means
    POPULAR            5.078      0.087     58.394      0.000

 Variances
    POPULAR            0.695      0.108      6.421      0.000
```

Level-1 Predictor Fixed

Mplus - [popular2modified2]

```
TITLE:  Random Effects ANOVA using Mplus;
DATA:
  FILE IS "popular2.dat";
VARIABLE:
NAMES ARE class pupil cons extrav sex texp popular
popteach zextrav zsex ztexp zpopular zpoptch;
usevariables are popular extrav;
CLUSTER IS class;
within are extrav;
define: ! this statement applies to Mplus version 7 only
center extrav (groupmean); ! this statement applies to Mplus version 7 only
!centering = groupmean (extrav); use this statement in Mplus Version 6 or below
ANALYSIS:
TYPE IS TWOLEVEL;
ESTIMATOR IS ML; !default is MLR
MODEL:
  %within%
  popular on extrav;
  %between%
  popular;
OUTPUT: SAMPSTAT;
```

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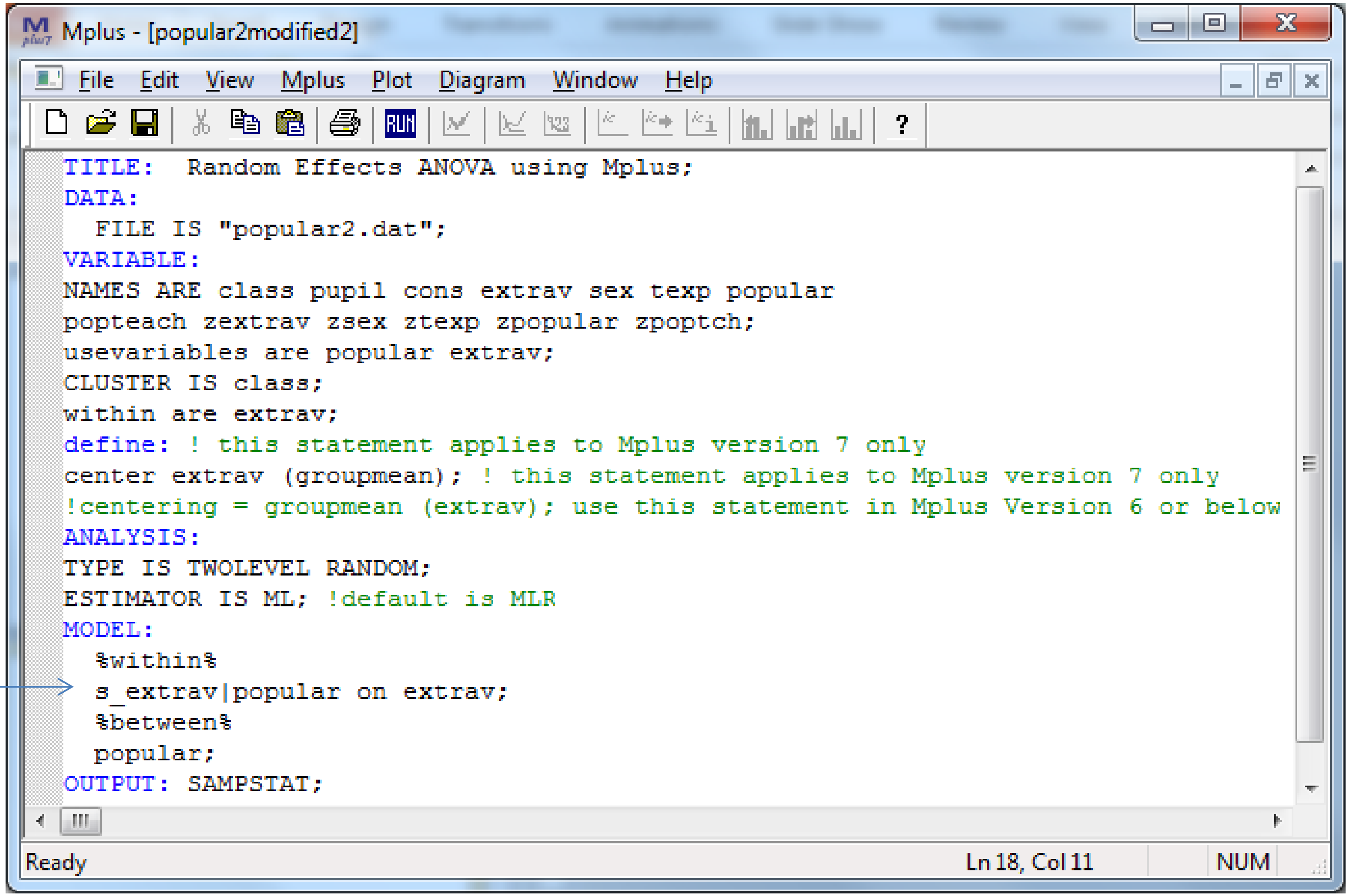
Level-1 Predictor Fixed

MODEL RESULTS

		Estimate	S.E.	Est./S.E.	Two-Tailed P-Value
	Within Level				
	POPULAR ON				
γ_{10}	EXTRAV	0.498	0.020	24.430	0.000
	Residual Variances				
e_{ij}	POPULAR	0.930	0.030	30.822	0.000
	Between Level				
	Means				
γ_{00}	POPULAR	5.078	0.087	58.378	0.000
	Variances				
μ_{0j}	POPULAR	0.710	0.107	6.627	0.000

Level-1 Predictor Random



Level-1 Predictor Random

MODEL RESULTS

		Estimate	S.E.	Est./S.E.	Two-Tailed P-Value
	Within Level				
	Residual Variances				
e_{ij}	POPULAR	0.892	0.030	30.018	0.000
	Between Level				
	Means				
γ_{00}	POPULAR	5.078	0.087	58.389	0.000
γ_{10}	S_EXTRAV	0.497	0.027	18.236	0.000
	Variances				
μ_{0j}	POPULAR	0.711	0.107	6.648	0.000
μ_{1j}	S_EXTRAV	0.033	0.010	3.103	0.002

Adding a Level-2 Predictor
(Intercept as Outcome Variable Model)

Mplus - [popular2modified3]

File Edit View Mplus Plot Diagram Window Help

```
TITLE:  Random Effects ANOVA using Mplus;
DATA:
  FILE IS "popular2.dat";
VARIABLE:
NAMES ARE class pupil cons extrav sex texp popular
popteach zextrav zsex ztexp zpopular zpoptch;
usevariables are popular extrav texp;
CLUSTER IS class;
within are extrav;
between are texp;
define: ! this statement applies to Mplus version 7 only
center extrav (groupmean);! this statement applies to Mplus version 7 only
center texp (grandmean); ! this statement applies to Mplus version 7 only
!centering = groupmean (extrav); use this statement in Mplus Version 6 or below
!centering = grandmean (texp);
ANALYSIS:
TYPE IS TWOLEVEL RANDOM;
ESTIMATOR IS ML; !default is MLR
MODEL:
  %within%
  s_extrav|popular on extrav;
  %between%
  popular on texp;
OUTPUT: SAMPSTAT;
```

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Adding a Level-2 Predictor (Intercept as Outcome Variable Model)

		Estimate	S.E.	Est./S.E.	Two-Tailed P-Value
	Within Level				
	Residual Variances				
e_{ij}	POPULAR	0.892	0.030	30.019	0.000
	Between Level				
	POPULAR ON				
γ_{01}	TEXP	0.062	0.012	5.266	0.000
	Means				
γ_{10}	S_EXTRAV	0.497	0.027	18.236	0.000
	Intercepts				
γ_{00}	POPULAR	5.076	0.077	65.956	0.000
	Variances				
μ_{1j}	S_EXTRAV	0.033	0.010	3.103	0.002
	Residual Variances				
μ_{0j}	POPULAR	0.547	0.084	6.533	0.000

Adding a Level-2 Predictor
(Intercept and Slope as Outcome Variables Model)

```
TITLE:  Random Effects ANOVA using Mplus;
DATA:
  FILE IS "popular2.dat";
VARIABLE:
NAMES ARE class pupil cons extrav sex texp popular
popteach zextrav zsex ztexp zpopular zpoptch;
usevariables are popular extrav texp;
CLUSTER IS class;
within are extrav;
between are texp;
define: ! this statement applies to Mplus version 7 only
center extrav (groupmean) texp (grandmean); ! this statement applies to Mplus version 7 only
!centering = groupmean (extrav); use this statement in Mplus Version 6 or below
!centering = grandmean (texp);
ANALYSIS:
TYPE IS TWOLEVEL RANDOM;
ESTIMATOR IS ML; !default is MLR
MODEL:
  %within%
  s_extrav|popular on extrav;
  %between%
  popular s_extrav on texp;
OUTPUT: SAMPSTAT;
```

Adding a Level-2 Predictor
(Intercept and Slope as Outcome Variables Model)

$$Y_{ij} = \gamma_{00} + \gamma_{01}(STR_j - \overline{STR}) + \gamma_{10}(SES_{ij} - \overline{SES}_j) + \gamma_{11}(SES_{ij} - \overline{SES}_j)(STR_j - \overline{STR}) + \mu_{0j} + \mu_{1j}(SES_{ij} - \overline{SES}_j) + e_{ij}$$

		Estimate	S.E.	Est./S.E.	Two-Tailed P-Value
	Within Level				
	Residual Variances				
e_{ij}	POPULAR	0.888	0.029	30.145	0.000
	Between Level				
	S_EXTRAV ON				
γ_{11}	TEXP	-0.027	0.003	-9.038	0.000
	POPULAR ON				
γ_{01}	TEXP	0.062	0.012	5.266	0.000
	Intercepts				
γ_{00}	POPULAR	5.076	0.077	65.956	0.000
γ_{10}	S_EXTRAV	0.498	0.020	24.438	0.000
	Residual Variances				
μ_{0j}	POPULAR	0.547	0.084	6.535	0.000
μ_{1j}	S_EXTRAV	0.002	0.007	0.246	0.806

(Intercept and Slope as Outcome Variables Model)
Allowing residuals to correlate

```
MODEL:
  %within%
  s_extrav|popular on extrav;
  %between%
  popular s_extrav on texp;
  popular with s_extrav;
```

τ_{01}

```
Two-Tailed
                            Estimate       S.E.  Est./S.E.    P-Value

Within Level

 Residual Variances
    POPULAR                    0.888      0.029     30.153      0.000

Between Level

 S_EXTRAV   ON
    TEXP                      -0.027      0.003     -8.914      0.000

 POPULAR    ON
    TEXP                       0.062      0.012      5.267      0.000

 POPULAR  WITH
    S_EXTRAV                  -0.017      0.017     -0.999      0.318

 Intercepts
    POPULAR                    5.076      0.077     65.956      0.000
    S_EXTRAV                   0.499      0.020     24.360      0.000

 Residual Variances
    POPULAR                    0.547      0.084      6.535      0.000
    S_EXTRAV                   0.002      0.007      0.289      0.773
```

Intercepts and Slopes as Outcome Variables Model
Mplus Diagram