Nonlinear Analysis: Real World Applications 41 (2018) 82–106

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

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Bifurcation of limit cycles at infinity in piecewise polynomial systems

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A R T I C L E I N F O

Article history:
Received 20 April 2017
Received in revised form 2 October 2017
Accepted 7 October 2017

Keywords:
Piecewise polynomial system
Lyapunov constant
Center
Limit cycle

A B S T R A C T

In this paper, we study bifurcation of limit cycles from the equator of piecewise polynomial systems with no singular points at infinity. We develop a method for computing the Lyapunov constants at infinity of piecewise polynomial systems. In particular, we consider cubic piecewise polynomial systems and study limit cycle bifurcations in the neighborhood of the origin and infinity. Moreover, an example is presented to show 11 limit cycles bifurcating from infinity.

1. Introduction

One of the well-known mathematical problems is the second part of Hilbert's 16th problem, which considers the maximal number and relative positions of limit cycles bifurcating in polynomial vector fields of degree n, given by

$$\dot{x} = f_n(x, y), \quad \dot{y} = g_n(x, y), \tag{1}$$

where the dot denotes differentiation with respect to time t. Since Hilbert proposed the problem in 1900, a great deal of works has been done in studying this problem, for example see [1–8]. Let $H(n)$ denote the upper bound of the number of limit cycles that system (1) can have. Chen and Wang [1], and Shi [2] proved the existence of 4 limit cycles with $\{3, 1\}$ distribution, i.e., $H(2) \geq 4$. However, this problem is even not

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https://doi.org/10.1016/j.nonrwa.2017.10.003

completely solved for $n = 2$. For cubic systems, Yu and Han [4,5], Liu and Huang [6] proved $H(3) \geq 12$ by studying Hopf bifurcation. Later, Li et al. [7] constructed a Hamiltonian system and applied proper perturbations to prove $H(3) \geq 13$. On the other hand, Liu and Li [8] investigated the cyclicity problem for a Z_2-equivariant cubic system, and showed that this system can have 13 limit cycles, with a large-amplitude limit cycle at infinity, surrounding 12 small-amplitude limit cycles around two symmetric foci.

To completely study bifurcation of limit cycles in system (1), it is necessary to include studying the bifurcation of limit cycles at infinity. The bifurcation of limit cycles at infinity was studied by Shi [2] 30 years ago, and later the birth of a unique limit cycle at infinity is shown by Sotomayor [9]. In order to find maximal number of limit cycles bifurcating from infinity for cubic systems, Blows and Rousseau [10] computed the first five Lyapunov quantities at infinity for a class of cubic systems:

$$\begin{cases} \dot{x} = \lambda_1 x - \eta y + Ax^2 + (B+2D)xy + Cy^2 + \lambda_2 x(x^2+y^2) - y(x^2+y^2), \\ \dot{y} = \eta x - \lambda_1 y + Dx^2 + (E-2A)xy - Dy^2 + x(x^2+y^2) + \lambda_2 y(x^2+y^2), \end{cases} \tag{2}$$

and studied the limit cycles bifurcating from the origin and infinity. Liu and Chen [11] constructed an example of cubic system with 6 limit cycles bifurcating from infinity. Liu and Huang [12] proved that a cubic polynomial system can have 7 limit cycles near infinity. Actually, studying the bifurcation of limit cycles at infinity is quite similar to studying Hopf bifurcation at the origin, via a transformation based on Poincaré return map. However, a uniform upper bound of the number of limit cycles bifurcating at infinity for polynomial vector fields is still unknown.

Recently, increasing interest has been focused on bifurcation of limit cycles in discontinuous or non-differentiable, i.e., non-smooth dynamical systems. In this paper, we consider the piecewise polynomial system (or the so-called switching polynomial system) with a switching line on the x-axis, given in the form of

$$(\dot{x}, \dot{y}) = \begin{cases} \left(\sum_{k=1}^{+\infty} X_k^+(x,y,\lambda), \sum_{k=1}^{+\infty} Y_k^+(x,y,\lambda) \right), & \text{for} \quad y > 0, \\ \left(\sum_{k=1}^{+\infty} X_k^-(x,y,\lambda), \sum_{k=1}^{+\infty} Y_k^-(x,y,\lambda) \right), & \text{for} \quad y < 0, \end{cases} \tag{3}$$

where $X_k^{\pm}(x,y,\lambda)$ and $Y_k^{\pm}(x,y,\lambda)$ are homogeneous polynomials of degree k in x and y , $\lambda \in \Lambda \subset \mathbf{R}^s$ is a parameter vector. System (3) includes two systems: the first one is called the upper system, defined for $y > 0$, and the second one is called the lower system, defined for $y < 0$.

The investigation of the more general piecewise systems, described by

$$(\dot{x}, \dot{y}) = \begin{cases} (X^+(x,y), Y^+(x,y)), & \text{for} \quad y > 0, \\ (X^-(x,y), Y^-(x,y)), & \text{for} \quad y < 0, \end{cases} \tag{4}$$

started a half century ago [13–15]. Here, $X^{\pm}(x,y)$ and $Y^{\pm}(x,y)$ are real analytic functions in a neighborhood of the origin. Note that system (4) is usually considered as a differential system with discontinuous right sides, and simply called discontinuous system. Such systems can exhibit rich complex dynamical phenomena. Since the analytic functions $X^{\pm}(x,y)$ and $Y^{\pm}(x,y)$ in (4) can be expanded into the form of (3) with the coefficients treated as parameters, researchers generally consider them equivalent and use either one as they wish. Filippov established some basic qualitative theory in [15] for such discontinuous systems. In the study of analytic system (4), the cyclicity problem is fundamental in the qualitative analysis. Coll et al. [16] developed a method for computing the Lyapunov constants to study bifurcation of small-amplitude limit cycles. They derived the explicit formulas for computing the first three Lyapunov quantities. Let $P(n)$ denote the maximal number of limit cycles for system (3) of degree n. Gasull and Torregrosa [17] obtained

$P(2) \geq 5$, showing that quadratic piecewise polynomial systems have two more limit cycles than that of quadratic smooth polynomial systems. Moreover, center conditions have been obtained for piecewise Kukles system [17], piecewise Liénard system [18] and piecewise Bautin system [19]. Note that planar smooth linear systems cannot generate limit cycles, but piecewise smooth linear systems can. In fact, Han and Zhang [20] proved $P(1) \geq 2$. Further, Huan and Yang [21], and Freire et al. [22] respectively proved $P(1) \geq 3$. Buzzi et al. [23] studied the limit cycles that bifurcate from a linear center using a piecewise linear perturbation in two zones. They proved that the maximal numbers of limit cycles that can appear with up to a Nth order perturbation are 1, 1, 2, 3, 3, 3, 3 when $N = 1, 2, \ldots, 7$. Llibre et al. [24,25] studied the limit cycles that bifurcate from the quadratic and cubic isochronous centers when the systems are perturbed within the class of piecewise quadratic and cubic polynomial differential systems, respectively. Chen et al. [26] constructed a class of piecewise quadratic Bautin systems to show $P(2) \geq 8$. Recently, Tian and Yu [27] gave a complete classification for the quadratic Bautin system with a singular point being a center, and proved $P(2) \geq 10$. Li et al. [28] considered a piecewise cubic polynomial system to show that $P(3) \geq 15$.

So far, there are very few studies on bifurcation of limit cycles at infinity for piecewise polynomial systems. Llibre et al. [29] obtained one limit cycle bifurcating from infinity in planar piecewise linear vector fields. Li et al. [30] presented a piecewise cubic polynomial system which can have 7 limit cycles in the neighborhood of infinity. In this paper, we develop a recursive algorithm to compute the Lyapunov constants at infinity for piecewise polynomial systems, and apply it to study bifurcation of limit cycles in the following piecewise cubic polynomial system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} A_1x + A_2y + A_3x^2 + A_4xy + A_5y^2 + (\mu x - y)(x^2+y^2) \\ A_6x + A_7y + A_8x^2 + A_9xy + A_{10}y^2 + (x + \mu y)(x^2+y^2) \end{pmatrix}, & \text{for} \quad y > 0, \\ \begin{pmatrix} B_1x + B_2y + B_3x^2 + B_4xy + B_5y^2 + (\mu x - y)(x^2+y^2) \\ B_6x + B_7y + B_8x^2 + B_9xy + B_{10}y^2 + (x + \mu y)(x^2+y^2) \end{pmatrix}, & \text{for} \quad y < 0, \end{cases} \tag{5}$$

where $(A_1, \ldots, A_{10}, B_1, \ldots, B_{10}) \in \mathbf{R}^{20}$.

The rest of the paper is organized as follows. In the next section, we introduce a recursive procedure to compute the Lyapunov constants at the origin of system (6), which can be carried out by using a computer algebraic system such as Mathematica, Maple. Using this method, we study the center conditions and limit cycle bifurcation for one class of system (5) in Section 3. In Section 4, we develop a recursive procedure for computing the Lyapunov constants at infinity for piecewise polynomial systems. In Section 5, we give a complete calculation of the Lyapunov constants at infinity for a special case of system (5) and investigate possible simultaneous Hopf bifurcations at the origin and infinity. We will show that 13 limit cycles with either $\{9, 4\}$ or $\{4, 9\}$ distribution at the origin and infinity can exist in this system. In Section 6, we present another example of system (5) to show 11 limit cycles bifurcating from infinity, with a concrete numerical example to illustrate the existence of 11 limit cycles. This is a new lower bound on the number of large-amplitude limit cycles for such polynomial cubic systems near infinity.

2. Computation of Lyapunov quantities

In this section, we present a method for computing the Lyapunov constants at the origin of the piecewise polynomial system,

$$(\dot{x}, \dot{y}) = \begin{cases} \left(\delta x - \beta y + \sum_{k=2}^{n} X_k^+(x,y),\ \beta x + \delta y + \sum_{k=2}^{n} Y_k^+(x,y)\right), & \text{for} \quad y > 0, \\ \left(\delta x - \beta y + \sum_{k=2}^{n} X_k^-(x,y),\ \beta x + \delta y + \sum_{k=2}^{n} Y_k^-(x,y)\right), & \text{for} \quad y < 0, \end{cases} \tag{6}$$

with $\beta > 0$ and $\delta \in \mathbf{R}$. For analytic smooth systems, the computation of Lyapunov quantities is the classical method of determining the center type equilibria and weak foci. We present some basic formulas for computing the Lyapunov constants of the general differential system,

$$\begin{cases} \dot{x} = \delta x - \beta y + \sum_{k=2}^{n} X_k(x,y), \\ \dot{y} = \beta x + \delta y + \sum_{k=2}^{n} Y_k(x,y), \end{cases} \tag{7}$$

where $X_k(x,y)$, $Y_k(x,y)$ are homogeneous polynomials of degree k in x and y. Introducing the polar coordinate transformation, $x = r\cos\theta$ and $y = r\sin\theta$, into (7) yields

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\delta r + \sum_{k=2}^{n} \varUpsilon_k(\theta) r^k}{\beta + \sum_{k=2}^{n} \varTheta_k(\theta) r^{k-1}}, \tag{8}$$

where

$$\begin{aligned} \varUpsilon_k(\theta) &= \cos\theta X_k(\cos\theta, \sin\theta) + \sin\theta Y_k(\cos\theta, \sin\theta), \\ \varTheta_k(\theta) &= \cos\theta Y_k(\cos\theta, \sin\theta) - \sin\theta X_k(\cos\theta, \sin\theta), \end{aligned} \tag{9}$$

in which X_k and Y_k are polynomials in $\sin\theta$ and $\cos\theta$. Further, (8) can be expressed in the power series of r as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \sum_{k=1}^{\infty} R_k(\theta) r^k, \tag{10}$$

where $R_k(\theta)$ is a polynomial in $\sin\theta$ and $\cos\theta$. Note that

$$\begin{aligned} \frac{\delta r + \sum_{k=2}^{n} \varUpsilon_k(\theta) r^k}{\beta + \sum_{k=2}^{n} \varTheta_k(\theta) r^{k-1}} &= \frac{1}{\beta}\Big[\delta r + \sum_{k=2}^{n} \varUpsilon_k(\theta) r^k\Big]\Big[1 + \sum_{i=1}^{\infty}\Big(-\sum_{k=2}^{n} \frac{\varTheta_k(\theta)}{\beta} r^{k-1}\Big)^i\Big] \\ &= \frac{1}{\beta}\Big[\delta r + \sum_{k=2}^{n} \varUpsilon_k(\theta) r^k\Big]\Big[1 + \sum_{k=1}^{\infty} \tilde{\varTheta}_k(\theta) r^k\Big]. \end{aligned} \tag{11}$$

It follows from (10) and (11) that $R_1(\theta) = \frac{\delta}{\beta}$ and

$$R_k(\theta) = \frac{1}{\beta}\Big[\sum_{i=2}^{k-1}(\varUpsilon_i(\theta)\tilde{\varTheta}_{k-i}(\theta) + \delta\tilde{\varTheta}_{k-1}(\theta)) + \varUpsilon_k(\theta)\Big], \quad k \ge 2. \tag{12}$$

The general solution of (10) can be expressed as

$$r(\rho,\theta) = \sum_{i\ge 1} v_k(\theta)\rho^k, \quad |\rho| \ll 1, \tag{13}$$

where $v_1(0) = 1$, $v_k(0) = 0$, $\forall k \ge 2$. Substituting the above solution (13) into Eq. (10), we obtain ${v_1}'(\theta) = \frac{\delta}{\beta} v_1(\theta)$ and

$${v_k}'(\theta) = R_k(\theta)\varOmega_{k,k}(\theta) + R_{k-1}(\theta)\varOmega_{k-1,k}(\theta) + \cdots + R_2(\theta)\varOmega_{2,k}(\theta), \quad k \ge 2, \tag{14}$$

where $\Omega_{i,j}(\theta)$ are polynomials in $v_l(\theta)$, $2 \le l \le j$. Further, we have

$$\begin{aligned}
v_1(\theta) &= e^{\int_0^\theta \frac{\delta}{\beta} d\theta}, \\
v_2(\theta) &= \int_0^\theta R_2(\theta) v_1^2(\theta) d\theta, \\
v_3(\theta) &= \int_0^\theta (R_3(\theta) v_1^3(\theta) + 2R_2(\theta) v_2(\theta) v_1(\theta)) d\theta, \\
v_4(\theta) &= \int_0^\theta (R_4(\theta) v_1^4(\theta) + 3R_3(\theta) v_2(\theta) v_1^2(\theta) + R_2(\theta)(v_2^2(\theta) + 2v_3(\theta) v_1(\theta))) d\theta, \\
&\cdots \\
v_k(\theta) &= \int_0^\theta (R_k(\theta)\Omega_{k,k}(\theta) + R_{k-1}(\theta)\Omega_{k-1,k}(\theta) + \cdots + R_2(\theta)\Omega_{2,k}(\theta)) d\theta, \quad k \ge 2.
\end{aligned} \tag{15}$$

However, as k grows, computation of $v_k(\theta)$ becomes more and more involved by direct integration. For convenience, we present a method developed in [27] to simplify the computation, which only needs the use of multiplication in the sum formula of trigonometric functions, which can be easily implemented using a computer algebra system. Then, Eq. (14) can be rewritten as

$$v_k'(\theta) = \sum_{i=0}^{3k-3} T_i(\theta)\sin(i\theta) + D_i(\theta)\cos(i\theta), \tag{16}$$

where $T_i(\theta)$ and $D_i(\theta)$ are polynomials in θ. Thus, integrating the above equation results in

$$\begin{aligned}
v_k(\theta) &= \sum_{i=0}^{3k-3} \int_0^\theta \left[T_i(\theta)\sin(i\theta) + D_i(\theta)\cos(i\theta)\right] d\theta \\
&= \sum_{i=0}^{3k-3} A_i(\theta)\cos(i\theta) + B_i(\theta)\sin(i\theta),
\end{aligned} \tag{17}$$

where $A_i(\theta)$ and $B_i(\theta)$ are polynomials in θ.

Like for analytic systems, we also need something alike to deal with the piecewise polynomial system (6). Using the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, system (6) can be written as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \begin{cases} \dfrac{\delta r + \sum_{k=2}^n \Upsilon_k^+(\theta) r^k}{\beta + \sum_{k=2}^n \Theta_k^+(\theta) r^{k-1}}, & \text{for} \quad \theta \in (0, \pi), \\ \dfrac{\delta r + \sum_{k=2}^n \Upsilon_k^-(\theta) r^k}{\beta + \sum_{k=2}^n \Theta_k^-(\theta) r^{k-1}}, & \text{for} \quad \theta \in (\pi, 2\pi), \end{cases} \tag{18}$$

where $\Upsilon_k^\pm(\theta)$ and $\Theta_k^\pm(\theta)$ are polynomials in $\sin\theta$ and $\cos\theta$ of degrees $k+1$. Suppose $r^+(\rho,\theta) = \sum_{k\ge1} v_k^+(\theta)\rho^k$ and $r^-(\rho,\theta) = \sum_{k\ge1} v_k^-(\theta)\rho^k$ are respectively the solutions of the upper and lower systems of (18), satisfying $r^+(\rho, 0) = r^-(\rho, \pi) = \rho$. Although a return map cannot be simply defined for system (6) like that for smooth systems, we define the positive half-return map $\Pi^+(\rho) = r^+(\rho, \pi)$ and the negative half-return map $\Pi^-(\rho) = r^-(\rho, 2\pi)$. Then we can define the displacement function,

$$d(\rho) = \Pi(\rho) - \rho = \Pi^-(\Pi^+(\rho)) - \rho = \sum_{k\ge1} V_k \rho^k, \tag{19}$$

as illustrated in Fig. 1(a). Here, V_k is called the kth-order Lyapunov constant of the piecewise polynomial system (6). It is not difficult to get $V_1 = e^{\frac{2\delta\pi}{\beta}} - 1$ since $\Pi^+(\rho) = \Pi^-(\rho) = e^{\frac{\delta\pi}{\beta}}\rho + O^\pm(\rho^2)$. Thus, $V_1 = 0$ if and only if $\delta = 0$.

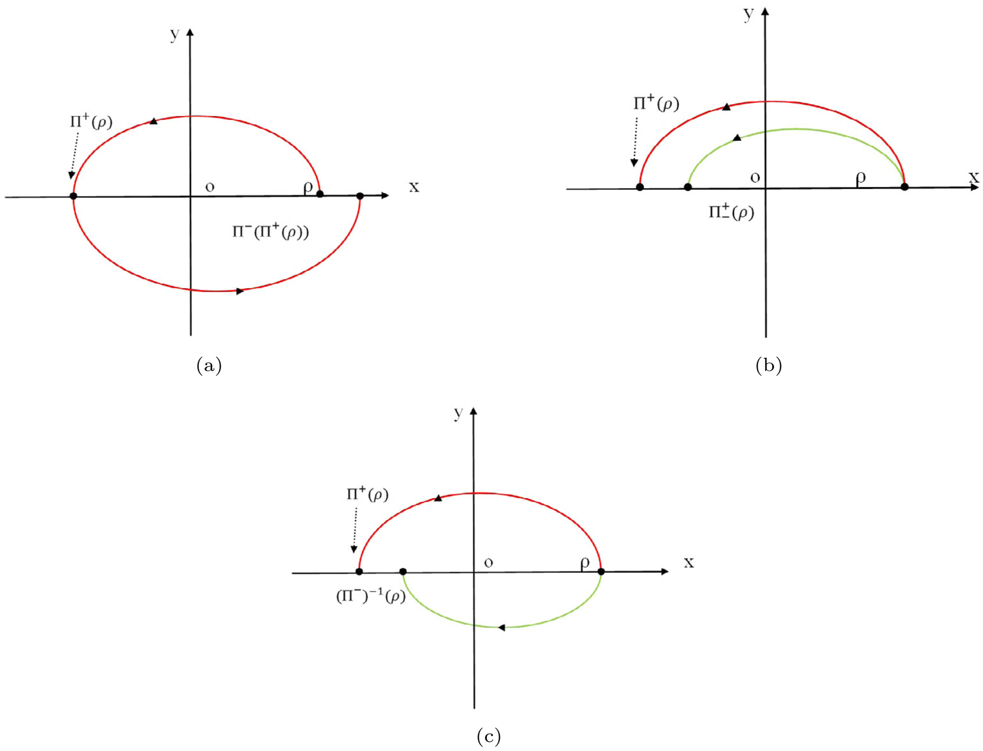

Fig. 1. (a) Map $\Pi(\rho)$, (b) Map $\Pi_-^+(\rho)$, (c) Map $(\Pi^-)^{-1}(\rho)$.

Another method to compute the Lyapunov constants can be found in [17]. To make the computation more convenient we substitute $(x, y, t) \to (x, -y, -t)$ into the lower system of (6) to obtain a new system,

$$\begin{cases} \dot{x} = -\delta x - \beta y - \displaystyle\sum_{k=2}^{n} X_k^-(x, -y), \\ \dot{y} = \beta x - \delta y + \displaystyle\sum_{k=2}^{n} Y_k^-(x, -y), \end{cases} \quad \text{for} \quad y > 0, \tag{20}$$

which defines a new positive half-return map $\Pi_-^+(\rho) = r_-^+(\rho, \pi) = \sum_{k\geq 1} v_{-k}^+(\pi)\rho^k$, as illustrated in Fig. 1(b). Let $(\Pi^-)^{-1}(\rho)$ denote the inverse of the negative half-return map $\Pi^-(\rho)$, as illustrated in Fig. 1(c). The map $(\Pi^-)^{-1}(\rho)$ of (6) is equivalent to the map $\Pi_-^+(\rho)$ of (20). Coll et al. [18] proved that the following expressions are equivalent:

$$g(f(\rho)) - \rho \quad \text{and} \quad f(\rho) - g^{-1}(\rho),$$

where f and g are analytic functions satisfying that $f(0) = g(0) = 0$ and $f'(0) = g'(0) = 1$. Gasull and Torregrosa [17] introduced a new function,

$$\Pi^-(\Pi^+(\rho)) - \rho = \Pi^+(\rho) - (\Pi^-)^{-1}(\rho) = \Pi^+(\rho) - \Pi_-^+(\rho) = \sum_{k\geq 1} W_k \rho^k. \tag{21}$$

Thus, originally computing one positive half-return map and one negative half-return map becomes computing two positive half-return maps. It has been proved [17] that the conditions $V_j = 0$, $1 \leq j \leq k-1$, $V_k \neq 0$ for (19) are equivalent to $W_j = 0$, $1 \leq j \leq k-1$, $W_k \neq 0$ for (21). Hence, in the following, we still use

V_k instead of W_k for simplicity. We can use the procedure for computing $v_k^+(\pi)$ and $v_{-k}^+(\pi)$ to compute the Lyapunov constants for the positive half-return maps $\Pi^+(\rho)$ and $\Pi_-^+(\rho)$, so that we obtain the Lyapunov constants V_k for piecewise polynomial system (6).

Note that the Lyapunov constant V_k is a polynomial in terms of the coefficients of the original piecewise polynomial system (6). It is well known that the origin of system (6) is a center if and only if $d(\rho) = 0$ for $0 < \rho \ll 1$, which means that for all integer k, $V_k = 0$. But the center problem for the piecewise polynomial system (6) is more complicated. The following lemma can be used for proving the center conditions at the origin of system (6).

Lemma 2.1 ([31]). *If the upper and lower systems in (6) have the first integrals $H^+(x, y)$ and $H^-(x, y)$ near the origin, respectively, and either both $H^+(x, 0)$ and $H^-(x, 0)$ are even functions in x, or $H^+(x, 0) \equiv H^-(x, 0)$, then the origin of system (6) is a center.*

Lemma 2.1 can be used to identify centers in the case that both the upper and lower systems are analytic and have a center at the origin. In addition, there is another useful result, as given in the following lemma.

Lemma 2.2 ([28]). *Assuming that $\delta = 0$, if system (6) is symmetric with respect to the x-axis, i.e., the functions on the right-hand side of system (6) satisfy*

$$X_k^+(x, y) = -X_k^-(x, -y), \quad Y_k^+(x, y) = Y_k^-(x, -y), \tag{22}$$

or system (6) is symmetric with respect to the y-axis, i.e., the functions on the right-hand side of system (6) satisfy

$$\begin{aligned} X_k^+(x, y) &= X_k^+(-x, y), \quad Y_k^+(x, y) = -Y_k^+(-x, y), \\ X_k^-(x, y) &= X_k^-(-x, y), \quad Y_k^-(x, y) = -Y_k^-(-x, y), \end{aligned} \tag{23}$$

then the origin of system (6) is a center.

Lemma 2.2 redefines symmetry of piecewise polynomial systems, which can be used to derive the center conditions for such systems. Moreover, the isolated zeros of $d(\rho) = 0$ near $\rho = 0$ correspond to the limit cycles around the origin. The origin of system (6) is called $\frac{k}{2}$-order ($k \in \mathbf{N}$) weak focus if there exists $\lambda_* \in \Lambda$ such that

$$V_1(\lambda_*) = V_2(\lambda_*) = \cdots = V_k(\lambda_*) = 0, \quad V_{k+1}(\lambda_*) \neq 0. \tag{24}$$

It is well known that for the nonzero Lyapunov constant V_k of smooth polynomial systems, k must be an odd number. However k can be any positive integer for piecewise polynomial systems. Based on Lemma 4 of [27], we have the following lemma, which gives the sufficient conditions for proving the existence of limit cycles.

Lemma 2.3. *If there exists a critical point $\lambda^* = (a_{1c}, a_{2c}, \ldots, a_{kc})$ such that $V_{i_1}(\lambda^*) = V_{i_2}(\lambda^*) = \cdots = V_{i_k}(\lambda^*) = 0$, $V_{i_{k+1}}(\lambda^*) \neq 0$, with $1 = i_1 < i_2 < \cdots < i_k$, and*

$$\det\left[\frac{\partial(V_{i_1}, V_{i_2}, \ldots, V_{i_k})}{\partial(a_{1c}, a_{2c}, \ldots, a_{kc})}(\lambda^*)\right] \neq 0, \tag{25}$$

then small appropriate perturbations about $\lambda = \lambda^*$ lead to that system (6) has exact k limit cycles bifurcating from the origin.

3. An example for limit cycle bifurcation at the origin

In this section, we apply the results presented in the previous section to consider an example of system (5). We study the center conditions and the number of bifurcating limit cycles for a family of piecewise cubic polynomial systems, obtained by setting $A_1 = A_7 = B_1 = B_7 = \delta$, $A_2 = B_2 = -1$, $A_6 = B_6 = 1$ and $A_4 = A_{10} = B_4 = B_{10} = 0$ in system (5), as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y + A_3x^2 + A_5y^2 + (\mu x - y)(x^2+y^2) \\ x + \delta y + A_8x^2 + A_9xy + (x+\mu y)(x^2+y^2)\end{pmatrix}, & \text{for} \quad y > 0, \\ \begin{pmatrix} \delta x - y + B_3x^2 + B_5y^2 + (\mu x - y)(x^2+y^2) \\ x + \delta y + B_8x^2 + B_9xy + (x+\mu y)(x^2+y^2)\end{pmatrix}, & \text{for} \quad y < 0. \end{cases} \tag{26}$$

With the aid of a computer algebra system, symbolic computations are carried out to find the Lyapunov constants associated with the origin of system (26), which is summarized in the following theorem.

Theorem 3.1. *Assume $\mu = 0$. System (26) has a center at the origin if and only if $\delta = 0$ and one of the following conditions holds:*

$$\begin{aligned} \mathrm{I}:&\quad A_8 = B_8 = 0,\\ \mathrm{II}:&\quad A_8 - B_8 = 2A_3 + A_9 = 2B_3 + B_9 = 0,\\ \mathrm{III}:&\quad A_8 - B_8 = A_3 + B_3 = A_5 + B_5 = A_9 + B_9 = 0. \end{aligned} \tag{27}$$

Proof. For system (26), the Lyapunov constants with $\mu = 0$ are obtained by using the algorithm described in the previous section:

$$\begin{aligned} V_1 &= e^{2\delta\pi} - 1,\\ V_2 &= \frac{2}{3}(A_8 - B_8),\\ V_3 &= -\frac{1}{8}A_8(2A_3 + A_9 + 2B_3 + B_9)\pi,\\ V_4 &= -\frac{2}{45}A_8(2A_3 + A_9)(15A_3 + 2A_5 + 15B_3 + 2B_5),\\ V_5 &= \frac{5}{384}A_8(2A_3 + A_9)(A_3 + B_3)(49A_3 + 12A_5 + A_9 + 43B_3)\pi. \end{aligned}$$

We compute the common zeros of V_k, $k = 1, \ldots, 5$, and consequently obtain the necessary conditions I, II and III. Letting $V_1 = V_2 = 0$ yields $\delta = A_8 - B_8 = 0$. Then, setting $V_3 = 0$ we have $A_8 = 0$ or $2A_3 + A_9 + 2B_3 + B_9 = 0$. If $A_8 = 0$, we obtain condition I, otherwise we have $B_9 = -2A_3 - A_9 - 2B_3$. Further, letting $V_4 = 0$ yields $(2A_3 + A_9)(15A_3 + 2A_5 + 15B_3 + 2B_5) = 0$. If $2A_3 + A_9 = 0$, we obtain condition II. Otherwise, we have $B_5 = -A_5 - \frac{15}{2}(A_3 + B_3)$. Taking $A_3 + B_3 = 0$ yields $V_5 = 0$, we obtain condition III.

Now, we assume $A_3 + B_3 \neq 0$ and let $A_5 = -\frac{49A_3 + A_9 + 43B_3}{12}$, for which $V_5 = 0$. Then we get

$$\begin{aligned} V_6 &= -\frac{1}{4725}A_8(2A_3 + A_9)(A_3 + B_3)F_1,\\ V_7 &= \frac{1}{12902400}A_8(2A_3 + A_9)(A_3 + B_3)F_2\pi,\\ V_8 &= -\frac{1}{65840947200}A_8(2A_3 + A_9)(A_3 + B_3)F_3,\\ V_9 &= -\frac{1}{12641461862400}A_8(2A_3 + A_9)(A_3 + B_3)F_4,\\ V_{10} &= -\frac{1}{131407996059648000}A_8(2A_3 + A_9)(A_3 + B_3)F_5, \end{aligned}$$

where

$$
\begin{aligned}
F_1 =\ & -10979A_3^2 + 3150A_8^2 + 275A_3A_9 + 76A_9^2 - 22042A_3B_3 + 29A_9B_3 - 11225B_3^2, \\
F_2 =\ & -1428000A_3 + 5079237A_3^3 - 714000A_9 + 6271596A_3^2A_9 - 90153A_3A_9^2 - 15024A_9^3 \\
& + 17850781A_3^2B_3 + 12728596A_3A_9B_3 + 9A_9^2B_3 + 20510215A_3B_3^2 + 6451920A_9B_3^2 \\
& + 7704375B_3^3, \\
F_3 =\ & 975884378112A_3^2 - 2350928519168A_3^4 - 277995110400A_8^2 - 578642190336A_3^2A_8^2 \\
& + 279620812800A_8^4 - 39467409408A_3A_9 + 107337293824A_3^3A_9 + 42830143488A_3A_8^2A_9 \\
& - 16046456832A_9^2 - 31414665216A_3^2A_9^2 + 54402121728A_8^2A_9^2 + 2283864064A_3A_9^3 \\
& + 757891072A_9^4 + 1900941557760A_3B_3 - 9409968816128A_3^3B_3 - 112059437875A_3A_8^2B_3 \\
& - 24718417920A_9B_3 - 13346193408A_3^2A_9B_3 + 174778343424A_8^2A_9B_3 \\
& - 49995276288A_3A_9^2B_3 + 3779264512A_9^3B_3 + 990633369600B_3^2 - 14454233186304A_3^2B_3^2 \\
& - 446693990400A_8^2B_3^2 - 339699941376A_3A_9B_3^2 - 26928463872A_9^2B_3^2 \\
& - 10085791268864A_3B_3^3 - 236149809152A_9B_3^3 - 2697406423040B_3^4 + 21861252000A_3A_8\pi \\
& + 156952867239A_3^3A_8\pi - 67341229200A_3A_8^3\pi + 10930626000A_8A_9\pi \\
& + 174951476532A3^2A_8A_9\pi - 79429215600A_8^3A_9\pi - 7178885091A_3A_8A_9^2\pi \\
& - 1686385008A_8A_9^3\pi + 516913738047A_3^2A_8B_3\pi - 91517202000A_8^3B_3\pi \\
& + 352331144172A_3A_8A_9B_3\pi - 2939424957A_8A_9^2B_3\pi + 566367043725A_3A_8B_3^2\pi \\
& + 183430396800A_8A_9B_3^2\pi + 208174546125A_8B_3^3\pi, \\
F_4 =\ & 722478248755200A_3^2A_8 - 1519825046208512A_3^4A_8 - 206009008128000A_8^3 \\
& - 148378343178240A_3^2A_8^3 + 116530348032000A_8^5 - 27711630213120A_3A_8A_9 \\
& + 7066276593664A_3^3A_8A_9 + 39753005137920A_3A_8^3A_9 - 10947510927360A_8A_9^2 \\
& - 45644557320192A_3^2A_8A_9^2 + \cdots, \\
F_5 =\ & -7401962187325440000A_3^2 + 46486215725497712640A_3^4 - 92550580964698357760A_3^6 \\
& + 2068016746070016000A_8^2 + 14697869051080212480A_3^2A_8^2 - 27162659432913960960A_3^4A_8^2 \\
& - 6378965932179456000A_8^4 - 21638785878510796800A_3^2A_8^4 + 25437992323645440000A_8^6 \\
& + 2901914991643852800A_3A_9 + \cdots.
\end{aligned}
$$

Since the five polynomial equations contain only four independent parameters, A_3, B_3, A_8 and A_9 with the restriction $A_3 + B_3 \neq 0$, there in general exist solutions such that $F_1 = F_2 = F_3 = F_4 = 0$ but $F_5 \neq 0$. We first consider the four polynomial equations: $F_1 = F_2 = F_3 = F_4 = 0$ to find all real solutions of these equations, and then verify if they satisfy the equation $F_5 = 0$. If none of the solutions does, then there do not exist real solutions satisfying $F_1 = F_2 = F_3 = F_4 = F_5 = 0$, and thus the best result obtained from $F_1 = F_2 = F_3 = F_4 = 0$ yields maximal number of limit cycles.

Note that F_2 does not contain A_8 and F_1 contains only one term A_8^2. So, we first solve A_8^2 from $F_1 = 0$ to obtain

$$
A_8^2 = \frac{1}{3150}\left[10979A_3^2 + 22042A_3B_3 + 11225B_3^2 - A_9(76A_9 + 275A_3 + 29B_3)\right],
$$

which is used to simplify F_3 and F_4. Then, we perform a symbolic computation on F_2, F_3 and F_4 to eliminate A_3 to obtain a solution,

$$
A_3 = A_3(B_3, A_9),
$$

and two resultants:

$$
F_{23} = F_{23}(B_3, A_9), \quad F_{24} = F_{24}(B_3, A_9).
$$

Next, we again perform symbolic computation on F_{23} and F_{24} to eliminate B_3 to obtain the final resultant:

$$
F_{2324} = F_{2324}(A_9^2),
$$

which is a single-variate, 24th-degree polynomial in A_9^2. Today the technique for solving single-variate polynomial is mature, and all real and complex solutions of such a polynomial can be found. Here, we are only interested in real solutions. In fact, with a computer algebra system such as Maple or Mathematica, we can perform *interval computation* to identify the real solutions in intervals with arbitrary accuracy as one wishes.

Four sets of solutions from $F_{2324} = 0$ have been obtained as follows:

$$\begin{aligned} A_3 &= \pm\, 136.2014111281\cdots, \quad A_8 = \pm 2.0625751157\cdots, \\ B_3 &= \mp\, 138.0681608497\cdots, \quad A_9 = \pm 79.3892644835\cdots, \end{aligned} \tag{28}$$

and

$$\begin{aligned} A_3 &= \pm\, 138.0681608497\cdots, \quad A_8 = \pm 2.0625751157\cdots, \\ B_3 &= \mp\, 136.2014111281\cdots, \quad A_9 = \pm 75.6557650402\cdots. \end{aligned} \tag{29}$$

For these four solutions, $F_1 = F_2 = F_3 = F_4 = 0$, but $F_5 \neq 0$. Hence, for the solutions given in (28) and (29), system (26) may generate maximal number of limit cycles around the origin.

Now we prove the sufficiency of conditions I, II and III. When the condition I is satisfied, system $(26)|_{\delta=0}$ can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + A_3x^2 + A_5y^2 - y(x^2+y^2) \\ x + A_9xy + x(x^2+y^2) \end{pmatrix}, & \text{for } \ y > 0, \\ \begin{pmatrix} -y + B_3x^2 + B_5y^2 - y(x^2+y^2) \\ x + B_9xy + x(x^2+y^2) \end{pmatrix}, & \text{for } \ y < 0. \end{cases} \tag{30}$$

Obviously, system (30) is symmetric with respect to the y-axis. Hence, by Lemma 2.2 the origin of system $(26)|_{\delta=0}$ is a center.

If the condition II holds, system $(26)|_{\delta=0}$ becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + A_3x^2 + A_5y^2 - y(x^2+y^2) \\ x + A_8x^2 - 2A_3xy + x(x^2+y^2) \end{pmatrix}, & \text{for } \ y > 0, \\ \begin{pmatrix} -y + B_3x^2 + B_5y^2 - y(x^2+y^2) \\ x + A_8x^2 - 2B_3xy + x(x^2+y^2) \end{pmatrix}, & \text{for } \ y < 0. \end{cases} \tag{31}$$

The upper and lower systems in (31) are Hamiltonian systems, having respectively the Hamiltonian functions:

$$\begin{aligned} H^+(x,y) &= -\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{3}A_8x^3 + A_3x^2y + \frac{1}{3}A_5y^3 - \frac{1}{4}x^4 - \frac{1}{2}x^2y^2 - \frac{1}{4}y^4, \\ H^-(x,y) &= -\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{3}A_8x^3 + B_3x^2y + \frac{1}{3}B_5y^3 - \frac{1}{4}x^4 - \frac{1}{2}x^2y^2 - \frac{1}{4}y^4. \end{aligned} \tag{32}$$

Thus, the condition $H^+(x,0) \equiv H^-(x,0)$ in Lemma 2.1 is satisfied, which implies that the origin of system $(26)|_{\delta=0}$ is a center.

If the condition III holds, system (26) can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + A_3x^2 + A_5y^2 - y(x^2+y^2) \\ x + A_8x^2 + A_9xy + x(x^2+y^2) \end{pmatrix}, & \text{for } \ y > 0, \\ \begin{pmatrix} -y - A_3x^2 - A_5y^2 - y(x^2+y^2) \\ x + A_8x^2 - A_9xy + x(x^2+y^2) \end{pmatrix}, & \text{for } \ y < 0. \end{cases} \tag{33}$$

It is easy to see that system (33) is symmetric with respect to the x-axis. Hence, the origin of system $(26)|_{\delta=0}$ is a center.

Therefore, the conditions I, II and III are sufficient for the origin of system $(26)|_{\delta=0}$ being a center. □

From the proof of Theorem 3.1, if the following conditions

$$\begin{aligned} &\delta = A_8 - B_8 = F_1 = F_2 = F_3 = F_4 = 0, A_8(2A_3 + A_9)(A_3 + B_3) \neq 0, \\ &B_9 = -2A_3 - A_9 - 2B_3, B_5 = -\frac{15A_3 + 2A_5 + 15B_3}{2}, A_5 = -\frac{49A_3 + A_9 + 43B_3}{12} \end{aligned} \tag{34}$$

are satisfied, we have $V_i = 0$, $i = 1, 2, \ldots, 9$, $V_{10} \neq 0$, indicating that the origin of system (26) is a 4.5-order weak focus. For our purpose, we choose one of the solutions in (28),

$$\begin{aligned} A_3 &= 2 - 136.2014111281\cdots, & A_8 &= -2.0625751157\cdots, \\ B_3 &= 138.0681608497\cdots, & A_9 &= -79.3892644835\cdots, \end{aligned} \tag{35}$$

to prove the existence of 9 limit cycles. Then, we have $V_{10} = -3.4990999739 \times 10^9$. A direct calculation shows that the determinant,

$$\det\left[\frac{\partial(V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9)}{\partial(A_3, A_5, A_8, A_9, B_3, B_5, B_8, B_9)}\right]_{(34),(35)} = -9.8120724589\cdots \times 10^{26} \neq 0. \tag{36}$$

Hence, we can take appropriate perturbations on δ, A_3, A_5, A_8, A_9, B_3, B_5, B_8 and B_9 such that

$$0 < V_1 \ll -V_2 \ll V_3 \ll -V_4 \ll V_5 \ll -V_6 \ll V_7 \ll -V_8 \ll V_9 \ll 1,$$

and so the polynomial equation $d(r) = 0$ has 9 simple zeros near $r = 0$. Thus, the following result follows directly from Lemma 2.3.

Theorem 3.2. *For system (26), there exist* 9 *small-amplitude limit cycles bifurcating from the origin.*

4. Lyapunov quantities at infinity

In this section, we consider bifurcation of limit cycles from the equator of the Poincaré sphere in the piecewise polynomial system (3) of degree $2n + 1$. With the ideas taken from [11,32], we suppose that there exists $\sigma > 0$ such that

$$xY^{\pm}_{2n+1}(x, y) - yX^{\pm}_{2n+1}(x, y) \geq \sigma(x^2 + y^2)^2, \tag{37}$$

which indicates that the equator Γ_∞ on the Poincaré closed sphere is a trajectory of system (3), having no real singular points. Let $\Gamma_\infty = \Gamma^+_\infty \cup \Gamma^-_\infty$ denote the equator cycle or infinity (on Gauss sphere) of system (3), where Γ^+_∞ and Γ^-_∞ represent the semi-equator cycle of the upper and lower systems of (3), respectively. We apply the polar coordinate transformation, $x = \cos\theta/r$ and $y = \sin\theta/r$, under which $r = 0$ corresponds to the equator. Then, system (3) can be transformed into

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \begin{cases} \dfrac{\sum_{k=1}^{2n+1} \varphi^+_{2n+2-k}(\theta) r^k}{\sum_{k=1}^{2n+1} \psi^+_{2n+2-k}(\theta) r^{k-1}}, & \text{for} \quad \theta \in (0, \pi), \\ \dfrac{\sum_{k=1}^{2n+1} \varphi^-_{2n+2-k}(\theta) r^k}{\sum_{k=1}^{2n+1} \psi^-_{2n+2-k}(\theta) r^{k-1}}, & \text{for} \quad \theta \in (\pi, 2\pi), \end{cases} \tag{38}$$

where

$$\begin{aligned} \varphi^{\pm}_{k+1}(\theta) &= \cos\theta X_k(\cos\theta, \sin\theta) + \sin\theta Y_k(\cos\theta, \sin\theta), \\ \psi^{\pm}_{k+1}(\theta) &= \sin\theta X_k(\cos\theta, \sin\theta) - \cos\theta Y_k(\cos\theta, \sin\theta), \end{aligned} \tag{39}$$

$k = 0, 1, 2\cdots$. Particularly, condition (37) implies $\psi^{\pm}_{2n+2} \geq \sigma > 0$.

Similar to Hopf bifurcation analysis associated with a singular point at the origin of a dynamical system, we construct the displacement map of (38). Let $\tilde{r}^+(\rho,\theta) = \sum_{k\geq 1} u_k^+(\theta)\rho^k$ and $\tilde{r}^-(\rho,\theta) = \sum_{k\geq 1} u_k^-(\theta)\rho^k$ denote the solutions of the upper and lower systems of (38) with the initial conditions $\tilde{r}^+(\rho,0) = \tilde{r}^-(\rho,\pi) = \rho$, respectively. Define respectively the positive half-return map $\Pi_\infty^+(\rho)$ and the negative half-return map $\Pi_\infty^-(\rho)$ by

$$\Pi_\infty^+(\rho) = \tilde{r}^+(\rho,\pi) = \sum_{k\geq 1} u_k^+\rho^k, \quad \Pi_\infty^-(\rho) = \tilde{r}^-(\rho,2\pi) = \sum_{k\geq 1} u_k^-\rho^k,$$

where $u_k^\pm$'s are the coefficients of Taylor expansion. The return map for (38) can now be defined as

$$\Pi_\infty(\rho) = \Pi_\infty^-(\Pi_\infty^+(\rho)) = \sum_{k\geq 1} u_k\rho^k, \tag{40}$$

where u_k's are the coefficients of Taylor expansion. Then, the displacement function of (38) is given by

$$d_\infty(\rho) = \Pi_\infty(\rho) - \rho = (u_1-1)\rho + \sum_{k\geq 2} u_k\rho^k = \sum_{k\geq 1} U_k\rho^k, \tag{41}$$

where U_k is called the kth-order Lyapunov quantity at infinity (also called kth-order focal values at infinity). We have the following definitions and results, which are similar to that of Hopf bifurcation analysis around a singular point at the origin.

Definition 4.1. For system (3), if there exists $\lambda = \lambda_*$ such that

$$U_1(\lambda_*) = \cdots = U_k(\lambda_*) = 0, \quad U_{k+1}(\lambda_*) \neq 0, \tag{42}$$

then infinity is called a weak focus of $\frac{k}{2}$-order ($k \in \mathbf{N}$); and if $U_k = 0$ for all integer k, infinity is a center.

It follows that we must consider the displacement function (41) of system (38), when we are interested in the limit cycles bifurcating from infinity. The number of fixed points of $\Pi_\infty(\rho)$ (or zeros of $d_\infty(\rho)$) corresponds to the maximal number of limit cycles of system (38). If the displacement function (41) satisfies $U_1 = \cdots = U_k = 0$, $U_{k+1} \neq 0$, then any perturbation of (3) has at most k limit cycles bifurcating at infinity.

In the following, we consider a particular case of system (3) with higher-order terms, given exactly by

$$X_{2n+1}^\pm(x,y) = (\mu x - y)(x^2+y^2)^n, \quad Y_{2n+1}^\pm(x,y) = (x+\mu y)(x^2+y^2)^n, \tag{43}$$

under which system (3) becomes

$$(\dot{x},\dot{y}) = \begin{cases} \begin{pmatrix} \sum_{k=1}^{2n} X_k^+(x,y) + (\mu x - y)(x^2+y^2)^n \\ \sum_{k=1}^{2n} Y_k^+(x,y) + (x+\mu y)(x^2+y^2)^n \end{pmatrix}, & \text{for} \quad y > 0, \\ \begin{pmatrix} \sum_{k=1}^{2n} X_k^-(x,y) + (\mu x - y)(x^2+y^2)^n \\ \sum_{k=1}^{2n} Y_k^-(x,y) + (x+\mu y)(x^2+y^2)^n \end{pmatrix}, & \text{for} \quad y < 0. \end{cases} \tag{44}$$

For system (44), we alternatively introduce a transformation to change infinity of the system into the origin, and then use the methods in studying limit cycle bifurcation at the origin to investigate bifurcation of limit cycles at infinity of system (44). Under the following transformation,

$$x = \frac{\xi}{(\xi^2+\eta^2)^{n+1}}, \quad y = \frac{\eta}{(\xi^2+\eta^2)^{n+1}}, \tag{45}$$

and time rescaling,

$$\mathrm{d}t = (x^2+y^2)^{-n}\mathrm{d}\tau, \tag{46}$$

we have

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= (\xi^2+\eta^2)^{-1-n}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + (-1-n)\xi(\xi^2+\eta^2)^{-2-n}\Big(2\xi\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + 2\eta\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\Big),\\
\frac{\mathrm{d}y}{\mathrm{d}\tau} &= (\xi^2+\eta^2)^{-1-n}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + (-1-n)\eta(\xi^2+\eta^2)^{-2-n}\Big(2\xi\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + 2\eta\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\Big),
\end{aligned} \tag{47}$$

and thus,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \Big[\sum_{k=0}^{2n} X_k^{\pm}(x,y) + (\mu x - y)(x^2+y^2)^n\Big](x^2+y^2)^{-n}\\
&= (\xi^2+\eta^2)^{-1-n}(-\eta+\mu\xi) + (\xi^2+\eta^2)^{n+2n^2}\sum_{k=0}^{2n}(\xi^2+\eta^2)^{-k-nk}X_k^{\pm}(\xi,\eta),\\
\frac{\mathrm{d}y}{\mathrm{d}\tau} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \Big[\sum_{k=0}^{2n} Y_k^{\pm}(x,y) + (x + \mu y)(x^2+y^2)^n\Big](x^2+y^2)^{-n}\\
&= (\xi^2+\eta^2)^{-1-n}(\xi+\mu\eta) + (\xi^2+\eta^2)^{n+2n^2}\sum_{k=0}^{2n}(\xi^2+\eta^2)^{-k-nk}Y_k^{\pm}(\xi,\eta).
\end{aligned} \tag{48}$$

Further, solving $\frac{\mathrm{d}\xi}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}\eta}{\mathrm{d}\tau}$ from (47) and using (48) yields

$$\begin{pmatrix}\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau}\end{pmatrix} = \begin{cases}\begin{pmatrix}\frac{-\mu}{2n+1}\xi - \eta + \sum_{k=0}^{2n} P^{+}_{2n+2+2nk+k}(\xi,\eta)\\ \xi + \frac{-\mu}{2n+1}\eta + \sum_{k=0}^{2n} Q^{+}_{2n+2+2nk+k}(\xi,\eta)\end{pmatrix}, & \text{for} \quad \eta > 0,\\ \begin{pmatrix}\frac{-\mu}{2n+1}\xi - \eta + \sum_{k=0}^{2n} P^{-}_{2n+2+2nk+k}(\xi,\eta)\\ \xi + \frac{-\mu}{2n+1}\eta + \sum_{k=0}^{2n} Q^{-}_{2n+2+2nk+k}(\xi,\eta)\end{pmatrix}, & \text{for} \quad \eta < 0,\end{cases} \tag{49}$$

where

$$\begin{aligned}
P^{\pm}_{2n+2+2nk+k}(\xi,\eta) &= \Big[(\eta^2 - \frac{\xi^2}{2n+1})X^{\pm}_{2n-k}(\xi,\eta) - \frac{2n+2}{2n+1}\xi\eta Y^{\pm}_{2n-k}(\xi,\eta)\Big](\xi^2+\eta^2)^{k(n+1)},\\
Q^{\pm}_{2n+2+2nk+k}(\xi,\eta) &= \Big[(\xi^2 - \frac{\eta^2}{2n+1})Y^{\pm}_{2n-k}(\xi,\eta) - \frac{2n+2}{2n+1}\xi\eta X^{\pm}_{2n-k}(\xi,\eta)\Big](\xi^2+\eta^2)^{k(n+1)}.
\end{aligned} \tag{50}$$

The origin of system (49) corresponds to infinity of system (44). Hence, the origin of system (49) being a center is equivalent to that infinity of system (44) being a center. We have the following theorem.

Theorem 4.2. *Infinity of system (44) is a center if and only if the origin of system (49) is a center.*

As discussed in Section 2, we know that the problem to determine the center conditions and bifurcation of limit cycles at infinity of system (44) can be studied by using the Lyapunov constants at the origin of system (49). Moreover, the results presented in this section allow us to consider limit cycle bifurcations around the origin, or near infinity or in both places.

Definition 4.3. The notation $\{k_1, k_2\}$ denotes the configuration of a vector field with k_1 (small-amplitude) limit cycles around the origin and k_2 (large-amplitude) limit cycles near infinity.

5. Simultaneous bifurcations of limit cycles at the origin and infinity

In this section, we study bifurcation of limit cycles at infinity of system (26). We find that the Lyapunov quantities at infinity of system (26) has a great similarity with that at the origin. Under the transformation,

$$x = \frac{\xi}{(\xi^2+\eta^2)^2}, \quad y = \frac{\eta}{(\xi^2+\eta^2)^2}, \tag{51}$$

and time rescaling,

$$\mathrm{d}t = (x^2+y^2)^{-1}\mathrm{d}\tau, \tag{52}$$

system (26) is transformed into

$$\begin{pmatrix} \frac{\mathrm{d}\xi}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\frac{\mu}{3}\xi - \eta + A_5\eta^4 + \frac{3A_3-A_5-4A_9}{3}\eta^2\xi^2 - \frac{4}{3}A_8\eta\xi^3 - \frac{1}{3}A_3\xi^4 \\ -\eta^7 - \frac{\delta}{3}\eta^6\xi - 3\eta^5\xi^2 - \delta\eta^4\xi^3 - 3\eta^3\xi^4 - \delta\eta^2\xi^5 - \eta\xi^6 - \frac{\delta}{3}\eta^7, \\ \xi - \frac{\mu}{3}\eta + A_8\xi^4 + \frac{3A_9-4A_3}{3}\eta\xi^3 - \frac{1}{3}A_8\eta^2\xi^2 - \frac{4A_5+A_9}{3}\eta^3\xi \\ +\xi^7 - \frac{\delta}{3}\xi^6\eta + 3\xi^5\eta^2 - \delta\xi^4\eta^3 + 3\xi^3\eta^4 - \delta\xi^2\eta^5 + \xi\eta^6 - \frac{\delta}{3}\eta^7, \end{pmatrix}, & \text{for } \eta > 0, \\ \begin{pmatrix} -\frac{\mu}{3}\xi - \eta + B_5\eta^4 + \frac{3B_3-B_5-4B_9}{3}\eta^2\xi^2 - \frac{4}{3}B_8\eta\xi^3 - \frac{1}{3}B_3\xi^4 \\ -\eta^7 - \frac{\delta}{3}\eta^6\xi - 3\eta^5\xi^2 - \delta\eta^4\xi^3 - 3\eta^3\xi^4 - \delta\eta^2\xi^5 - \eta\xi^6 - \frac{\delta}{3}\eta^7, \\ \xi - \frac{\mu}{3}\eta + B_8\xi^4 + \frac{3B_9-4B_3}{3}\eta\xi^3 - \frac{1}{3}B_8\eta^2\xi^2 - \frac{4B_5+B_9}{3}\eta^3\xi \\ +\xi^7 - \frac{\delta}{3}\xi^6\eta + 3\xi^5\eta^2 - \delta\xi^4\eta^3 + 3\xi^3\eta^4 - \delta\xi^2\eta^5 + \xi\eta^6 - \frac{\delta}{3}\eta^7, \end{pmatrix}, & \text{for } \eta < 0. \end{cases} \tag{53}$$

We are interested in identifying the center conditions at infinity of system (26). We will show that they can be classified as two types: Hamiltonian system or one having symmetry with respect to a line. These conditions are also the center conditions for the origin of (26). Moreover, we will show that system (26) can have limit cycles bifurcating simultaneously from the origin and the equator.

Theorem 5.1. *Assume $\delta = 0$. System (26) has a center at infinity (correspondingly system (53) has a center at the origin) if and only if $\mu = 0$ and one of the conditions* I, II *and* III *given in Theorem 3.1 holds.*

Proof. When $\delta = 0$, we compute the Lyapunov quantities at the origin of system (53) (corresponding to the Lyapunov quantities at infinity of system (26)), and find that the Lyapunov constants are identically equal to zero except U_{3k+1}. Actually, we have

$$\begin{aligned} U_1 &= e^{\frac{2\mu\pi}{3}} - 1, \\ U_4 &= \frac{2}{9}(A_8 - B_8), \\ U_7 &= -\frac{1}{24}A_8(2A_3 + A_9 + 2B_3 + B_9)\pi, \\ U_{10} &= \frac{2}{189}A_8(2A_3 + A_9)(15A_3 + 2A_5 + 15B_3 + 2B_5), \\ U_{13} &= -\frac{1}{1152}A_8(2A_3 + A_9)(A_3 + B_3)(72A_3 + 24A_5 - 11A_9 + 86B_3)\pi. \end{aligned}$$

The proof of the necessity of center conditions is similar to the proof for Theorem 3.1, and is thus omitted here for brevity. Assume that $A_3 + B_3 \neq 0$ and let $A_5 = -\frac{72A_3 - 11A_9 + 86B_3}{24}$, for which $U_{13} = 0$, then we obtain

$$
\begin{aligned}
U_{16} &= \frac{1}{2211300} A_8(2A_3 + A_9)(A_3 + B_3)F_6, \\
U_{19} &= \frac{1}{2080899072000} A_8(2A_3 + A_9)(A_3 + B_3)F_7\pi, \\
U_{22} &= \frac{1}{3572072595534643200} A_8(2A_3 + A_9)(A_3 + B_3)F_8, \\
U_{25} &= -\frac{1}{5426410061612187648000} A_8(2A_3 + A_9)(A_3 + B_3)F_9, \\
U_{28} &= \frac{1}{1436921724707171987122814976000} A_8(2A_3 + A_9)(A_3 + B_3)F_{10},
\end{aligned}
$$

where

$$
\begin{aligned}
F_6 \; = \; & -303912A_3^2 + 88200A_8^2 - 14472A_3A_9 - 9833A_9^2 - 668520A_3B_3 - 24860A_9B_3 - 314300B_3^2, \\
F_7 \; = \; & -69995520000A_3 + 113754063024A_3^3 - 34997760000A_9 + 48174235056A_3^2A_9 \\
& + 5874190738A_3A_9^2 + 1876956983A_9^3 + 216930158640A_3^2B_3 + 106461471640A_3A_9B_3 \\
& + 5387551160A_9^2B_3 + 68844491800A_3B_3^2 + 47200955900A_9B_3^2 - 25886700000B_3^3, \\
F_8 \; = \; & 18412553007079096\,32A_3^2 - 1144863919341305856A_3^4 - 394121451877171200A_8^2 \\
& - 1719043246824357888A_3^2A_8^2 + 527079896933990400A_8^4 + 773005929741287424A_3A_9 \\
& - 972617423970631680A_3^3A_9 - 780906842711654440A_3A_8^2A_9 + 277301306147733504A_9^2 \\
& - 424557724494200832A_3^2A_9^2 - 43170343811022848A_8^2A_9^2 - 53498864134520832A_3A_9^3 \\
& - 1439590578860851\,2A_9^4 + 343750464890103398\,4A_3B_3 - 4471175657023340544A_3^3B_3 \\
& - 3759486934506799104A_3A_8^2B_3 + 336199294849646592A_9B_3 \\
& - 2647999461030100992A_3^2A_9B_3 - 94590690972925952A_8^2A_9B_3 \\
& - 953419734762651648A_3A_9^2B_3 - 61668382174347264A_9^3B_3 + 1404448665816268800B_3^2 \\
& - 6251270020043636736A_3^2B_3^2 - 1735543253526118400A_8^2B_3^2 \\
& - 2221924006383058944A_3A_9B_3^2 - 449066278613680128A_9^2B_3^2 \\
& - 3584100795873755136A_3B_3^3 - 544296089627394048A_9B_3^3 - 700203548918415360B_3^4 \\
& + 307061120830464000A_3A_8\pi - 1059156131672214576A_3^3A_8\pi + 162559052645109120A_3A_8^3\pi \\
& + 153530560415232000A_8A_9\pi - 669882531931717104A_3^2A_8A_9\pi \\
& + 125337019129882560A_8^3A_9\pi - 64457741654788482A_3A_8A_9^2\pi - 2220719316344228\,7A_8A_9^3\pi \\
& - 248739434769289464\,0A_3^2A_8B_3\pi + 88114985614656000A_8^3B_3\pi \\
& - 1477312740939835800A_3A_8A_9B_3\pi - 68785417325865240A_8A_9^2B_3\pi \\
& - 1549165231431057240A_3A_8B_3^2\pi - 67853788327387062\,0A_8A_9B_3^2\pi \\
& - 200435510879604000A_8B_3^3\pi, \\
F_9 \; = \; & 1023496617800289484800\,0A_3^2A_8 - 58040357465873480417\,28A_3^4A_8 \\
& - 2354895499988828160000A_8^3 - 37181817497656177459\,20A_3^2A_8^3 \\
& + 1268445739638325248000A_8^5 + \cdots, \\
F_{10} \; = \; & -20760628159916599576262344704\,00A_3^2 + 30236961974478350976316499558\,40A_3^4 \\
& - 13182804209851927883308597248\,00A_3^6 + 19575779000154687004055961600\,0A_8^2 \\
& + 62764697253725651777688384307\,20A_3^2A_8^2 + \cdots.
\end{aligned}
$$

Here, similarly following the proof for Theorem 3.1, we can show that the four polynomial equations $F_6 = F_7 = F_8 = F_9 = 0$ have 8 sets of real solutions given below:

$$\begin{aligned}
&A_3 = \pm 1.1935491673\cdots, \ B_3 = \mp 1.9598517046\cdots\\
&A_8 = \pm 0.8583332974\cdots, \ A_9 = \pm 0.4170129068\cdots\\
&A_3 = \mp 1.0820295512\cdots, \ B_3 = \pm 0.3530483910\cdots\\
&A_8 = \pm 1.2580741376\cdots, \ A_9 = \pm 0.6980676134\cdots\\
&A_3 = \pm 0.3530483910\cdots, \ B_3 = \mp 1.0820295512\cdots\\
&A_8 = \pm 1.2580741376\cdots, \ A_9 = \pm 0.7598947069\cdots\\
&A_3 = \mp 1.9598517046\cdots, \ B_3 = \pm 1.1935491673\cdots\\
&A_8 = \pm 0.8583332974\cdots, \ A_9 = \pm 1.1155921675.
\end{aligned} \tag{54}$$

Under these solutions, $F_6 = F_7 = F_8 = F_9 = 0$, but $F_{10} \neq 0$. So these solutions may yield maximal number of limit cycles.

Next, we prove the sufficiency of the center conditions I, II and III. When the condition I is satisfied, system (53)$|_{\mu=0}$ can be rewritten as

$$\begin{pmatrix} \frac{\mathrm{d}\xi}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\eta + A_5\eta^4 + \frac{3A_3 - A_5 - 4A_9}{3}\eta^2\xi^2 - \frac{1}{3}A_3\xi^4 \\ -\eta^7 - 3\eta^5\xi^2 - 3\eta^3\xi^4 - \eta\xi^6, \\ \xi + \frac{3A_9 - 4A_3}{3}\eta\xi^3 - \frac{4A_5 + A_9}{3}\eta^3\xi \\ +\xi^7 + 3\xi^5\eta^2 + 3\xi^3\eta^4 + \xi\eta^6, \end{pmatrix}, & \text{for} \quad \eta > 0, \\ \begin{pmatrix} -\eta + B_5\eta^4 + \frac{3B_3 - B_5 - 4B_9}{3}\eta^2\xi^2 - \frac{1}{3}B_3\xi^4 \\ -\eta^7 - 3\eta^5\xi^2 - 3\eta^3\xi^4 - \eta\xi^6, \\ \xi + \frac{3B_9 - 4B_3}{3}\eta\xi^3 - \frac{4B_5 + B_9}{3}\eta^3\xi \\ +\xi^7 + 3\xi^5\eta^2 + 3\xi^3\eta^4 + \xi\eta^6, \end{pmatrix}, & \text{for} \quad \eta < 0. \end{cases} \tag{55}$$

Obviously, system (55) is symmetric with respect to the η-axis. So by Lemma 2.2 the origin of (55) is a center, and hence system (26)$|_{\mu=0}$ has a center at infinity.

If the condition II holds, system (53)$|_{\mu=0}$ can be rewritten as

$$\begin{pmatrix} \frac{\mathrm{d}\xi}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\eta + A_5\eta^4 + \frac{11A_3 - A_5}{3}\eta^2\xi^2 - \frac{4}{3}B_8\eta\xi^3 - \frac{1}{3}A_3\xi^4 \\ -\eta^7 - 3\eta^5\xi^2 - 3\eta^3\xi^4 - \eta\xi^6, \\ \xi + B_8\xi^4 - \frac{10A_3}{3}\eta\xi^3 - \frac{1}{3}B_8\eta^2\xi^2 - \frac{4A_5 - 2A_3}{3}\eta^3\xi \\ +\xi^7 + 3\xi^5\eta^2 + 3\xi^3\eta^4 + \xi\eta^6, \end{pmatrix}, & \text{for} \quad \eta > 0, \\ \begin{pmatrix} -\eta + B_5\eta^4 + \frac{11B_3 - B_5}{3}\eta^2\xi^2 - \frac{4}{3}B_8\eta\xi^3 - \frac{1}{3}B_3\xi^4 \\ -\eta^7 - 3\eta^5\xi^2 - 3\eta^3\xi^4 - \eta\xi^6, \\ \xi + B_8\xi^4 - \frac{10B_3}{3}\eta\xi^3 - \frac{1}{3}B_8\eta^2\xi^2 - \frac{4B_5 - 2B_3}{3}\eta^3\xi \\ +\xi^7 + 3\xi^5\eta^2 + 3\xi^3\eta^4 + \xi\eta^6, \end{pmatrix}, & \text{for} \quad \eta < 0. \end{cases} \tag{56}$$

The upper and lower systems in (56) have an integral factor,

$$M(x,y) = (\xi^2+\eta^2)^{-7},$$

and first integrals:

$$\begin{aligned} H^+(\xi,\eta) &= \frac{3-4A_5\eta^3+6\eta^6-12A_3\eta\xi^2+18\eta^4\xi^2+4A_8\xi^3+18\eta^2\xi^4+6\xi^6}{36(\xi^2+\eta^2)^6}, \\ H^-(\xi,\eta) &= \frac{3-4B_5\eta^3+6\eta^6-12B_3\eta\xi^2+18\eta^4\xi^2+4A_8\xi^3-18\eta^2\xi^4+6\xi^6}{36(\xi^2+\eta^2)^6}. \end{aligned} \tag{57}$$

Thus, the condition $H^+(\xi,0) \equiv H^-(\xi,0)$ in Lemma 2.1 is satisfied, which implies that the origin of (56) is a center. Hence, infinity of system $(26)|_{\mu=0}$ is a center.

If the condition III holds, system $(53)|_{\mu=0}$ becomes

$$\begin{pmatrix} \frac{\mathrm{d}\xi}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\eta+A_5\eta^4+\frac{3A_3-A_5-4A_9}{3}\eta^2\xi^2-\frac{4}{3}A_8\eta\xi^3-\frac{1}{3}A_3\xi^4 \\ -\eta^7-3\eta^5\xi^2-3\eta^3\xi^4-\eta\xi^6, \\ \xi+A_8\xi^4+\frac{3A_9-4A_3}{3}\eta\xi^3-\frac{1}{3}A_8\eta^2\xi^2-\frac{4A_5+A_9}{3}\eta^3\xi \\ +\xi^7+3\xi^5\eta^2+3\xi^3\eta^4+\xi\eta^6, \end{pmatrix}, & \text{for} \quad \eta>0, \\ \begin{pmatrix} -\eta-A_5\eta^4-\frac{3A_3-A_5-4A_9}{3}\eta^2\xi^2-\frac{4}{3}A_8\eta\xi^3+\frac{1}{3}A_3\xi^4 \\ -\eta^7-3\eta^5\xi^2-3\eta^3\xi^4-\eta\xi^6, \\ \xi+A_8\xi^4-\frac{3A_9-4A_3}{3}\eta\xi^3-\frac{1}{3}A_8\eta^2\xi^2+\frac{4A_5+A_9}{3}\eta^3\xi \\ +\xi^7+3\xi^5\eta^2+3\xi^3\eta^4+\xi\eta^6, \end{pmatrix}, & \text{for} \quad \eta<0. \end{cases} \tag{58}$$

It is seen that system (58) is symmetric with respect to the ξ-axis, showing that infinity of system $(26)|_{\mu=0}$ is a center.

Therefore, the conditions I, II and III are also sufficient for infinity of $(26)|_{\mu=0}$ being a center. □

Theorem 5.2. *Assume $\delta=\mu=0$. The cubic system (26) has the configuration*

$$\{4,9\} \text{ and } \{9,4\},$$

which give the maximal number of limit cycles that system (26) can have simultaneously around the origin and infinity.

Proof. We have two cases.

(1) First, we try to obtain the maximal number of limit cycles bifurcating near infinity. We take one of the real solutions from (54),

$$\begin{aligned} A_3 &= -1.1935491673\cdots, \quad & B_3 &= 1.9598517045\cdots, \\ A_8 &= -0.8583332973\cdots, \quad & A_9 &= -0.4170129068\cdots. \end{aligned} \tag{59}$$

Then under the conditions,

$$A_8-B_8=0,\ B_9=-2A_3-A_9-2B_3,\ B_5=-\frac{15A_3+2A_5+15B_3}{2}, \tag{60}$$

$A_5 = -\frac{72A_3-11A_9+86B_3}{24}$ and the solution given in (59), we have $U_{3k+1} = 0$, $k = 0, 1, 2, \ldots, 8$, but $U_{28} \neq 0$, indicating that infinity of system (26) is a 13.5-order weak focus. Further, it is easy to show that

$$\det\left[\frac{\partial(U_4, U_7, U_{10}, U_{13}, U_{16}, U_{19}, U_{22}, U_{25})}{\partial(A_3, A_5, A_8, A_9, B_3, B_5, B_8, B_9)}\right]_{(59),(60)} = -9.8120724589\cdots \times 10^{-7} \neq 0, \tag{61}$$

implying that system (26) has 9 large-amplitude limit cycles bifurcating from infinity. From the proof of Theorem 3.1, we obtain

$$V_5 = \frac{65}{768} A_8(2A_3 + A_9)^2(A_3 + B_3)\pi \neq 0$$

when $A_5 = -\frac{72A_3-11A_9+86B_3}{24}$. Hence, simultaneously, the origin of system (26) is a 2-order weak focus. In addition, we obtain that

$$\det\left[\frac{\partial(V_2, V_3, V_4)}{\partial(A_3, A_8, A_9)}\right]_{(59)} = -261.2058994389\cdots \neq 0. \tag{62}$$

Thus, system (26), in addition to 9 large-amplitude limit cycles bifurcating from infinity, has 4 small-amplitude limit cycles simultaneously bifurcating from the origin, yielding a distribution $\{4, 9\}$.

(2) Next, we first consider the maximal limit cycles bifurcating from the origin. From the results of Section 3, we know that there exist parameter values satisfying $V_1 = V_2 = \cdots = V_9 = 0$, but $V_{10} \neq 0$. This shows that 9 small-amplitude limit cycles bifurcate from the origin. Then, under the condition (60), we obtain $U_1 = U_4 = U_7 = U_{10} = 0$. Assume $A_8(2A_3 + A_9)(A_3 + B_3) \neq 0$, we have

$$U_{13} = \frac{13}{1152} A_8(2A_3 + A_9)^2(A_3 + B_3)\pi \neq 0$$

when $A_5 = -\frac{49A_3+A_9+43B_3}{12}$. In addition, we obtain that

$$\det\left[\frac{\partial(U_4, U_7, U_{10})}{\partial(A_3, A_8, A_9)}\right]_{(35)} = 6.9102089798\cdots \neq 0. \tag{63}$$

Hence, system (26), besides 9 small-amplitude limit cycles bifurcating from the origin, has 4 large-amplitude limit cycles simultaneously bifurcating from infinity, yielding the distribution $\{9, 4\}$. □

6. An example with 11 limit cycles at infinity

In this section, we present a special example of the piecewise cubic polynomial system (5) to show 11 limit cycles bifurcating from infinity. To achieve this, setting $A_3 = B_3 = A_4 = B_4 = A_9 = B_9 = A_{10} = B_{10} = 0$ and $B_6 = A_6$ in system (5), we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} A_1x + A_2y + A_5y^2 + (\mu x - y)(x^2 + y^2) \\ A_6x + A_7y + A_8x^2 + (x + \mu y)(x^2 + y^2) \end{pmatrix}, & \text{for } y > 0, \\ \begin{pmatrix} B_1x + B_2y + B_5y^2 + (\mu x - y)(x^2 + y^2) \\ A_6x + B_7y + B_8x^2 + (x + \mu y)(x^2 + y^2) \end{pmatrix}, & \text{for } y < 0. \end{cases} \tag{64}$$

Under the transformations (51) and (52), system (64) becomes

$$\begin{pmatrix}\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau}\end{pmatrix}=\begin{cases}\begin{pmatrix}-\frac{\mu}{3}\xi-\eta+A_5\eta^4-\frac{A_5}{3}\eta^2\xi^2-\frac{4A_8}{3}\eta\xi^3+A_2\eta^7+\frac{3A_1-4A_7}{3}\eta^6\xi\\ +\frac{5A_2-4A_6}{3}\eta^5\xi^2+\frac{5A_1-8A_7}{3}\eta^4\xi^3+\frac{A_2-8A_6}{3}\eta^3\xi^4\\ +\frac{A_1-4A_7}{3}\eta^2\xi^5-\frac{A_2+4A_6}{3}\eta\xi^6-\frac{A_1}{3}\xi^7,\\ \xi-\frac{\mu}{3}\eta-\frac{4A_5}{3}\eta^3\xi-\frac{A_8}{3}\eta^2\xi^2+A_8\xi^4-\frac{A_7}{3}\eta^7-\frac{4A_2+A_6}{3}\eta^6\xi\\ +\frac{A_7-4A_1}{3}\eta^5\xi^2+\frac{A_6-8A_2}{3}\eta^4\xi^3+\frac{5A_7-8A_1}{3}\eta^3\xi^4\\ +\frac{5A_6-4A_2}{3}\eta^2\xi^5+\frac{3A_7-4A_1}{3}\eta\xi^6+A_6\xi^7,\end{pmatrix}, & \text{for } \eta>0,\\ \begin{pmatrix}-\frac{\mu}{3}\xi-\eta+B_5\eta^4-\frac{B_5}{3}\eta^2\xi^2-\frac{4B_8}{3}\eta\xi^3+B_2\eta^7+\frac{3B_1-4B_7}{3}\eta^6\xi\\ +\frac{5B_2-4A_6}{3}\eta^5\xi^2+\frac{5B_1-8B_7}{3}\eta^4\xi^3+\frac{B_2-8A_6}{3}\eta^3\xi^4\\ +\frac{B_1-4B_7}{3}\eta^2\xi^5-\frac{B_2+4A_6}{3}\eta\xi^6-\frac{B_1}{3}\xi^7,\\ \xi-\frac{\mu}{3}\eta-\frac{4B_5}{3}\eta^3\xi-\frac{B_8}{3}\eta^2\xi^2+B_8\xi^4-\frac{B_7}{3}\eta^7-\frac{4B_2+A_6}{3}\eta^6\xi\\ +\frac{B_7-4B_1}{3}\eta^5\xi^2+\frac{A_6-8B_2}{3}\eta^4\xi^3+\frac{5B_7-8B_1}{3}\eta^3\xi^4\\ +\frac{5A_6-4B_2}{3}\eta^2\xi^5+\frac{3B_7-4B_1}{3}\eta\xi^6+A_6\xi^7,\end{pmatrix}, & \text{for } \eta<0.\end{cases} \tag{65}$$

Then we have the following two theorems.

Theorem 6.1. *System (64) has a center at infinity (correspondingly system (65) has a center at the origin) if and only if one of the following conditions holds:*

$$\begin{aligned}\text{IV}:&\ \ \mu=A_1+A_7=B_1+B_7=A_8B_8=0,\\ \text{V}:&\ \ \mu=A_1+B_1=A_2-B_2=A_5+B_5=A_7+B_7=A_8-B_8=0.\end{aligned} \tag{66}$$

Proof. First, we prove that the conditions IV and V are necessary. As discussed in the previous section, we have $\mu=0$ due to $U_1=0$. From the 4th Lyapunov constant, $U_4=-\frac{2}{9}(A_8-B_8)=0$, we get $B_8=A_8$. Then, we obtain $U_7=-\frac{1}{6}(A_1+A_7+B_1+B_7)\pi$. Taking $B_7=-A_7-A_1-B_1$ yields $U_7=0$, and then $U_{10}=-\frac{4}{27}(A_1+A_7)(A_5+B_5)$.

(i) Letting $A_7=-A_1$ yields $U_{10}=0$, which gives the condition IV.

(ii) If $B_5=-A_5$, for which $U_{10}=0$, and then $U_{13}=\frac{1}{12}(A_1+A_7)(-A_2+B_2)\pi$, which leads to $B_2=A_2$ from $U_{13}=0$. Consequently, we have $U_{16}=\frac{2}{15}(A_1+A_7)A_8(A_1+B_1)$.

(iia) If $B_1=-A_1$, we obtain the condition V.

(iib) If $A_8=0$, then we have

$$U_{19}=-\frac{1}{24}(A_1+A_7)(2A_1+A_7-B_1)(A_1+B_1)\pi.$$

Setting $U_{19}=0$ yields $B_1=2A_1+A_7$, and then $U_{22}=-\frac{128}{1575}A_5(A_1+A_7)^2(3A_1+A_7)$. If $A_7=-3A_1$, we get $B_1=-A_1$ which is included in condition V. Otherwise, we have $A_5=0$ from $U_{22}=0$. Further, we have $U_{25}=-\frac{1}{48}(3A_2-5A_6)(A_1+A_7)^2(3A_1+A_7)\pi$. Taking $A_2=\frac{5}{3}A_6$ yields $U_{25}=0$, leads to $U_{28} = U_{34} = U_{40} = 0$, and

$$U_{31} = \frac{1}{1152}(A_1 + A_7)^2(3A_1 + A_7)(45A_1^2 - 32A_6^2 + 126A_1A_7 + 69A_7^2)\pi,$$
$$U_{37} = -\frac{1}{1620}A_6(A_1 + A_7)^2(3A_1 + A_7)(-266A_6^2 + 333A_1A_7 + 222A_7^2)\pi.$$

We compute the resultant,

$$\begin{aligned}&\mathrm{Res}(45A_1^2 - 32A_6^2 + 126A_1A_7 + 69A_7^2, -266A_6^2 + 333A_1A_7 + 222A_7^2, A_1)\\ &= 3184020A_6^4 + 2297700A_6^2A_7^2 + 554445A_7^4,\end{aligned} \tag{67}$$

which does not have non-zero real solutions. If $A_6 = 0$, we have $U_{31} = \frac{1}{1152}(A_1 + A_7)^2(3A_1 + A_7)(45A_1^2 + 126A_1A_7 + 69A_7^2)\pi$, $U_{37} = 0$, and

$$U_{43} = \frac{373}{108000}A_7^3(A_1 + A_7)^2(3A_1 + A_7)(66A_1 + 49A_7)\pi.$$

Then, computing the resultant to obtain

$$\mathrm{Res}(45A_1^2 + 126A_1A_7 + 69A_7^2, A_7(66A_1 + 49A_7), A_1) = 1125A_7^4. \tag{68}$$

Hence, $45A_1^2 + 126A_1A_7 + 69A_7^2 = A_7(66A_1 + 49A_7) = 0$ if and only if $A_1 = A_7 = 0$.

Next, we prove the sufficiency of the conditions IV and V. When the condition IV is satisfied, system (65) can be rewritten as

$$\begin{pmatrix}\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau}\end{pmatrix} = \begin{cases}\begin{pmatrix}-\eta + A_5\eta^4 - \frac{A_5}{3}\eta^2\xi^2 - \frac{4A_8}{3}\eta\xi^3 + A_2\eta^7 + \frac{7A_1}{3}\eta^6\xi\\ + \frac{5A_2 - 4A_6}{3}\eta^5\xi^2 + \frac{13A_1}{3}\eta^4\xi^3 + \frac{A_2 - 8A_6}{3}\eta^3\xi^4\\ + \frac{5A_1}{3}\eta^2\xi^5 - \frac{A_2 + 4A_6}{3}\eta\xi^6 - \frac{A_1}{3}\xi^7,\\ \xi - \frac{4A_5}{3}\eta^3\xi - \frac{A_8}{3}\eta^2\xi^2 + A_8\xi^4 + \frac{A_1}{3}\eta^7 - \frac{4A_2 + A_6}{3}\eta^6\xi\\ - \frac{5A_1}{3}\eta^5\xi^2 + \frac{A_6 - 8A_2}{3}\eta^4\xi^3 - \frac{13A_1}{3}\eta^3\xi^4\\ + \frac{5A_6 - 4A_2}{3}\eta^2\xi^5 - \frac{7A_1}{3}\eta\xi^6 + A_6\xi^7,\end{pmatrix}, & \text{for } \eta > 0,\\ \begin{pmatrix}-\eta + B_5\eta^4 - \frac{B_5}{3}\eta^2\xi^2 - \frac{4A_8}{3}\eta\xi^3 + B_2\eta^7 + \frac{7B_1}{3}\eta^6\xi\\ + \frac{5B_2 - 4A_6}{3}\eta^5\xi^2 + \frac{13B_1}{3}\eta^4\xi^3 + \frac{B_2 - 8A_6}{3}\eta^3\xi^4\\ + \frac{5B_1}{3}\eta^2\xi^5 - \frac{B_2 + 4A_6}{3}\eta\xi^6 - \frac{B_1}{3}\xi^7,\\ \xi - \frac{4B_5}{3}\eta^3\xi - \frac{A_8}{3}\eta^2\xi^2 + A_8\xi^4 + \frac{B_1}{3}\eta^7 - \frac{4B_2 + A_6}{3}\eta^6\xi\\ - \frac{5B_1}{3}\eta^5\xi^2 + \frac{A_6 - 8B_2}{3}\eta^4\xi^3 - \frac{13B_1}{3}\eta^3\xi^4\\ + \frac{5A_6 - 4B_2}{3}\eta^2\xi^5 - \frac{7B_1}{3}\eta\xi^6 + A_6\xi^7,\end{pmatrix}, & \text{for } \eta < 0.\end{cases} \tag{69}$$

The upper and lower systems in (69) have an integral factor,

$$M(x,y)=(\xi^2+\eta^2)^{-7},$$

and first integrals:

$$\begin{aligned}
H^+(\xi,\eta)&=\frac{1}{36(\xi^2+\eta^2)^6}(3-4A_5\eta^3-6A_2\eta^6-12A_1\eta^5\xi-12A_2\eta^4\xi^2+6A_6\eta^4\xi^2\\
&\quad+4A_8\xi^3-24A_1\eta^3\xi^3-6A_2\eta^2\xi^4+12A_6\eta^2\xi^4-12A_1\eta\xi^5+6A_6\xi^6),\\
H^-(\xi,\eta)&=\frac{1}{36(\xi^2+\eta^2)^6}(3-4B_5\eta^3-6B_2\eta^6-12B_1\eta^5\xi-12B_2\eta^4\xi^2+6A_6\eta^4\xi^2\\
&\quad+4A_8\xi^3-24B_1\eta^3\xi^3-6B_2\eta^2\xi^4+12A_6\eta^2\xi^4-12B_1\eta\xi^5+6A_6\xi^6).
\end{aligned}\tag{70}$$

Thus, the condition $H^+(\xi,0)\equiv H^-(\xi,0)$ in Lemma 2.1 holds, which implies that infinity of system $(64)|_{\mu=0}$ is a center.

If the condition V holds, system (65) can be rewritten as

$$\begin{pmatrix}\frac{d\xi}{d\tau}\\ \frac{d\eta}{d\tau}\end{pmatrix}=\begin{cases}\begin{pmatrix}-\eta+A_5\eta^4-\frac{A_5}{3}\eta^2\xi^2-\frac{4A_8}{3}\eta\xi^3+A_2\eta^7+\frac{3A_1-4A_7}{3}\eta^6\xi\\ +\frac{5A_2-4A_6}{3}\eta^5\xi^2+\frac{5A_1-8A_7}{3}\eta^4\xi^3+\frac{A_2-8A_6}{3}\eta^3\xi^4\\ +\frac{A_1-4A_7}{3}\eta^2\xi^5-\frac{A_2+4A_6}{3}\eta\xi^6-\frac{A_1}{3}\xi^7,\\ \xi-\frac{4A_5}{3}\eta^3\xi-\frac{A_8}{3}\eta^2\xi^2+A_8\xi^4-\frac{A_7}{3}\eta^7-\frac{4A_2+A_6}{3}\eta^6\xi\\ +\frac{A_7-4A_1}{3}\eta^5\xi^2+\frac{A_6-8A_2}{3}\eta^4\xi^3+\frac{5A_7-8A_1}{3}\eta^3\xi^4\\ +\frac{5A_6-4A_2}{3}\eta^2\xi^5+\frac{3A_7-4A_1}{3}\eta\xi^6+A_6\xi^7,\end{pmatrix}, & \text{for }\ \eta>0,\\[2ex] \begin{pmatrix}-\eta-A_5\eta^4+\frac{A_5}{3}\eta^2\xi^2-\frac{4A_8}{3}\eta\xi^3+A_2\eta^7-\frac{3A_1-4A_7}{3}\eta^6\xi\\ +\frac{5A_2-4A_6}{3}\eta^5\xi^2-\frac{5A_1-8A_7}{3}\eta^4\xi^3+\frac{A_2-8A_6}{3}\eta^3\xi^4\\ -\frac{A_1-4A_7}{3}\eta^2\xi^5-\frac{A_2+4A_6}{3}\eta\xi^6+\frac{A_1}{3}\xi^7,\\ \xi+\frac{4A_5}{3}\eta^3\xi-\frac{A_8}{3}\eta^2\xi^2+A_8\xi^4+\frac{A_7}{3}\eta^7-\frac{4A_2+A_6}{3}\eta^6\xi\\ -\frac{A_7-4A_1}{3}\eta^5\xi^2+\frac{A_6-8A_2}{3}\eta^4\xi^3-\frac{5A_7-8A_1}{3}\eta^3\xi^4\\ +\frac{5A_6-4A_2}{3}\eta^2\xi^5-\frac{3A_7-4A_1}{3}\eta\xi^6+A_6\xi^7,\end{pmatrix}, & \text{for }\ \eta<0.\end{cases}\tag{71}$$

Obviously, system (71) is symmetric with respect to the ξ-axis. Hence, infinity of system $(64)|_{\mu=0}$ is a center. □

Theorem 6.2. *If the following conditions:*

$$\begin{aligned}&\mu=A_2=A_5=A_6=A_8=B_2=B_5=B_8=2A_1+A_7-B_1=3A_1+2A_7+B_7=0,\\ &45A_1^2+126A_1A_7+69A_7^2=0,\quad A_1+A_7\neq0,\quad A_1A_7\neq0,\end{aligned}\tag{72}$$

are satisfied, then system (64) can be perturbed to have 11 *limit cycles near infinity.*

Proof. It has been shown in Theorem 6.1 that if the conditions in (72) are satisfied, infinity of system (64) is a 21-order weak focus. Further, it can be shown that

$$\det\left[\frac{\partial(U_4,U_7,U_{10},U_{13},U_{16},U_{19},U_{22},U_{25},U_{31},U_{37})}{\partial(A_2,A_5,A_6,A_7,A_8,B_1,B_2,B_5,B_7,B_8)}\right]_{(72)}$$
$$= -\frac{37A_7(A_1+A_7)^{11}(3A_1+A_7)^5\pi^6}{16070775840000}(3A_1+2A_7)(231A_1^3+645A_1^2A_7+513A_1A_7^2+115A_7^3).$$

Let $H_0 = 45A_1^2+126A_1A_7+69A_7^2$. Then, we compute the resultants to obtain that

$$\begin{aligned}&\mathrm{Res}(H_0, 3A_1+2A_7, A_1) = 45A_7^2 \neq 0,\\ &\mathrm{Res}(H_0, 231A_1^3+645A_1^2A_7+513A_1A_7^2+115A_7^3, A_1) = 91570176A_7^6 \neq 0,\end{aligned}$$

indicating that $H_0 = 3A_1+2A_7 = 0$ and $H_0 = 231A_1^3+645A_1^2A_7+513A_1A_7^2+115A_7^3 = 0$ have no non-zero real solutions. Therefore,

$$\det\left[\frac{\partial(U_4,U_7,U_{10},U_{13},U_{16},U_{19},U_{22},U_{25},U_{31},U_{37})}{\partial(A_2,A_5,A_6,A_7,A_8,B_1,B_2,B_5,B_7,B_8)}\right]_{(72)} \neq 0,$$

which indicates that $d_\infty(r)$ has 11 simple zeros near $r=0$. This shows that 11 large-amplitude limit cycles can bifurcate from infinity of system (64). A concrete numerical example for $d_\infty(r)$ to have 11 simple zeros is given in the next section. □

7. A numerical example realization of the 11 limit cycles

Theorem 6.2 guarantees the existence of 11 large-amplitude limit cycles in system (64) under small perturbations. In order to obtain 11 large-amplitude limit cycles bifurcating from infinity, one needs to find exact 11 positive roots solved from the polynomial equation:

$$d_\infty(r) = U_1r + U_4r^4 + \cdots + U_{37}r^{37} + U_{43}r^{43} = 0. \tag{73}$$

In general, it is not an easy task to find a set of explicit parameter values to have a numerical realization. In particular, it is extremely difficult to obtain a numerical set of perturbations for the case of high multiple limit cycles. However, for our case, since the perturbations can be done one by one, it is possible to obtain parameter values such that the Eq. (73) can have 11 positive solutions. In the following, we present a concrete example for illustration. The complete set of critical values of $(\mu_c, A_{1c}, A_{2c}, \ldots, B_{7c}, B_{8c})$ are given in (72). Under these conditions, we have $U_i = 0$, $i = 1, 2, \ldots, 42$, but $U_{43} \neq 0$. Further, solving A_1 and A_7 from the equation

$$45A_1^2+126A_1A_7+69A_7^2 = 0$$

yields two real solutions. We choose one of them, given by

$$A_1 = 0.37656799869833483520\cdots, \quad A_7 = -0.50424025103273360438\cdots. \tag{74}$$

Therefore, we need perturbations such that

$$0 < U_1 \ll -U_4 \ll U_7 \ll -U_{10} \ll U_{13} \ll -U_{16} \ll U_{19} \ll -U_{22} \ll U_{25} \ll -U_{31} \ll U_{37} \ll 1. \tag{75}$$

We take perturbations in the backward order: on A_6 for U_{37}, on A_2 for U_{25}, on A_5 for U_{22}, on B_1 for U_{19}, on A_8 for U_{16}, on B_2 for U_{13}, on B_5 for U_{10}, on B_7 for U_7, on B_8 for U_4, on μ for U_1. More precisely, we choose

$$
\begin{aligned}
\mu &= \mu_c + 1.7 \times 10^{-68} = 1.7 \times 10^{-68}, \\
A_2 &= A_{2c} + 0.00033333333 = 0.00033333333, \\
A_5 &= A_{5c} + 10^{-15} = 10^{-15}, \\
A_6 &= A_{6c} + 0.0002 = 0.0002, \\
A_8 &= A_{8c} + 10^{-25} = 10^{-25}, \\
B_1 &= B_{1c} - 9 \times 10^{-21} \\
&= 0.24889574636393606601629116469159455796222099691069164950 8, \\
B_2 &= B_{2c} + 0.000333333333299999999999999999997008184186959518186004 \\
&= 0.000333333333299999999999999999997008184186959518186004, \\
B_5 &= B_{5c} - 1.000000000000000000000000528697495076723549757528 45 \times 10^{-15} \\
&= -1.00000000000000000000000052869749507672354975752845 \times 10^{-15}, \\
B_7 &= B_{7c} + 8.99999999999999999999999999981 \times 10^{-21} \\
&= -0.12122349402953729683852907185263420053122835248621, \\
B_8 &= B_{8c} + 9.99999999999999999999999999999955 \times 10^{-26} \\
&= 9.99999999999999999999999999999955 \times 10^{-26}.
\end{aligned}
\tag{76}
$$

With the above perturbed parameter values, we obtain the following Lyapunov constants:

$$
\begin{aligned}
U_1 &= 1.068141502220529701077298750315 03 \times 10^{-67}, \\
U_4 &= -10^{-57}, \\
U_7 &= 10^{-48}, \\
U_{10} &= -10^{-40}, \\
U_{13} &= 10^{-33}, \\
U_{16} &= -1.0647248678081108360917361383756436548140191531415 \times 10^{-27}, \\
U_{19} &= 9.4076207516259133434330217147167515034784700337664 \times 10^{-23}, \\
U_{22} &= -8.2856120069794437863240270300666625581487535135386 \times 10^{-19}, \\
U_{25} &= 6.6727340581550545464728040020724884052090270135 65 \times 10^{-16}, \\
U_{31} &= -3.5587914976826957581188288011053271494449005158177 \times 10^{-11}, \\
U_{37} &= 2.6829217281236385925660150873455717155374539666814 \times 10^{-8}, \\
U_{43} &= -0.20665634791709396362537493136546130186290774390368,
\end{aligned}
\tag{77}
$$

for which Eq. (73) has 11 positive roots:

$$
\begin{array}{llll}
r_1 \approx 0.0004951057, & r_2 \approx 0.0009962443, & r_3 \approx 0.0021544391, & r_4 \approx 0.0046561506, \\
r_5 \approx 0.0096891379, & r_6 \approx 0.0226219191, & r_7 \approx 0.0482546035, & r_8 \approx 0.1070254815, \\
r_9 \approx 0.1523230384, & r_{10} \approx 0.3375260483, & r_{11} \approx 0.4749986020, &
\end{array}
\tag{78}
$$

as expected. It should be noted that $U_2 = U_3 = U_5 = U_6 = \cdots = 0$ even under the perturbations.

8. Conclusion

In this paper, we have discussed the center problem and bifurcation of limit cycles for two types of piecewise cubic polynomial systems. We have developed a computationally efficient method for computing the Lyapunov constants at infinity of piecewise polynomial systems. Using our method, we consider a special piecewise cubic polynomial system to show the existence of 13 limit cycles with either $\{9,4\}$ or $\{4,9\}$ distribution at the origin and infinity. Moreover, we construct a special system to prove the existence of 11 limit cycles bifurcating from infinity, which is a new best result in the direction of this research.

We would like to develop more computationally efficient methodology to study bifurcations of limit cycles around the origin and infinity simultaneously in piecewise polynomial systems. In particular, finding the maximal number of limit cycles bifurcating at infinity for general piecewise polynomial systems with no singular point at infinity is a challenging task.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (Nos. 11371127 and 11771059), and the Natural Science and Engineering Research Council of Canada (No. R2686A02). The comments and suggestions received from the two anonymous reviewers are greatly appreciated.

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