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Center condition and bifurcation of limit cycles for quadratic switching systems with a nilpotent equilibrium point

Ting Chen [a], Lihong Huang [b], Pei Yu [c,*]

[a] School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou, 510320, China
[b] School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, 410082, China
[c] Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7, Canada

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Abstract

In this work, a new perturbation approach is developed based on Bogdanov-Takens bifurcation theory, which enables the Poincaré-Lyapunov method for switching systems with linear type centers to be applied for studying the center conditions of planar switching polynomial systems associated with a nilpotent equilibrium point. The new method is then applied to consider a class of quadratic switching nilpotent systems, and a complete classification is given on the conditions of the nilpotent equilibrium point to be a center. Moreover, based on one of the center conditions, an example is constructed to show the existence of seven small-amplitude limit cycles around the nilpotent equilibrium point, which is a new lower bound on the number of limit cycles in such systems.

MSC: 34C07; 34C15

Keywords: Switching system; Nilpotent equilibrium point; Lyapunov constant; Center; Limit cycle

* Corresponding author.
E-mail addresses: chenting0715@126.com (T. Chen), lhhuang@csust.edu.cn (L. Huang), pyu@uwo.ca (P. Yu).

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1. Introduction

In the last few years, many real complex phenomena have been modeled more accurately by dynamical systems whose vector fields are either non-differentiable or discontinuous, for instance, see [3,5,7,30,31,40]. These systems are the so-called non-smooth or discontinuous systems. Owing to extensive applications, increasing interest has been attracted to the qualitative analysis of non-smooth systems, see [1,16,41]. In this paper, we will deal with the following family of non-smooth systems,

$$(\dot{x}, \dot{y}) = \begin{cases} \left(f^+(x, y), g^+(x, y)\right), & \text{for } S(x, y) > 0, \\ \left(f^-(x, y), g^-(x, y)\right), & \text{for } S(x, y) < 0, \end{cases} \tag{1}$$

where the dot denotes differentiation with respect to time t, $S\colon \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function and $(f^\pm(x, y), g^\pm(x, y))$ are smooth vector fields. In fact, system (1) has two different regions $\Sigma^\pm = \{(x, y) \in \mathbb{R}^2 : \pm S(x, y) > 0\}$ separated by the discontinuous curve $\Sigma = S^{-1}(0)$, which is called the switching manifold. When the functions $f^\pm(x, y)$ and $g^\pm(x, y)$ are polynomials in x and y, system (1) is called a switching polynomial system or piecewise polynomial system. For convenience, we will call the system with the $+$ sign ($-$ sign) the first (second) system in the rest of the paper.

Center problem has been known as one of the very important problems in the qualitative theory of planar polynomial systems. We recall that an isolated equilibrium point of a planar vector field is monodromy if all nearby orbits rotate around the equilibrium point. The center problem is to finding the conditions under which the monodromic equilibrium point is a center, and this problem in switching systems is much more difficult and complicated than that in smooth systems. For example, an equilibrium point of system (1) on the discontinuous curve $S(x, y) = 0$ can be a center even it is not a center of either the first system or the second system of (1). On the contrary, an equilibrium point on the discontinuous curve may not be a center even it is a center for both the first and second systems of (1). Some methods have been developed for studying the center problem of the switching system (1). Gasull and Torregrosa [20] developed an efficient method for computing the Lyapunov constants of switching polynomial systems with elementary centers. By elementary center we mean that the linearized system has a pair of purely imaginary eigenvalues, while a linear type center implies that the elementary center is a center (i.e., a *nonlinear center*). By computing the Lyapunov constants, the authors of [8, 45] gave a complete classification on the center conditions of the origin in several classes of Bautin switching systems. In [24], Guo et al. studied the bi-center conditions for a family of Z_2-equivariant cubic switching systems with two symmetric elementary centers. For more results on the problem of linear type centers in switching systems, see [11,36].

Another important problem in the qualitative theory of planar polynomial systems is to determining the maximum number of limit cycles bifurcating from an equilibrium point. By the variety of non-smoothness, non-smooth systems exhibit not only Hopf bifurcation, Poincaré bifurcation and homoclinic loop bifurcation [34], but also grazing bifurcation [5], border-collision bifurcation [15,42] and sliding bifurcation [2,13], which are complicated bifurcation phenomena and do not exist in smooth systems. There have been extensive studies on the various kinds of nonstandard bifurcation phenomena in planar switching systems, see [6,17,29] and references therein. It is well known that linear smooth systems can not produce limit cycles, but switching linear systems can produce different kinds of limit cycle bifurcations. Up to now, it has been shown that there are at most three crossing limit cycles in switching linear systems with two zones separated by a straight line [18,27,39]. Li and Llibre [35] classified global phase portraits for a family of switching linear systems separated by one straight line. Recently, Bastos *et al.* [4] proved that seven is a lower bound on the number of crossing limit cycles in planar switching linear systems with two pieces divided by a cubic curve. The best lower bounds obtained so far on the crossing limit cycles for switching quadratic and cubic systems in two zones separated by a straight line are 16 [12] and 24 [22], respectively.

It should be noted that studies for switching systems have been mainly focused on the center problem and bifurcation of limit cycles associated with elementary equilibrium points. Some works consider the center problem and limit cycle bifurcations in the neighborhood of infinity of switching systems [9]. However, to the best of our knowledge, there are no results in considering the center problem and bifurcation of limit cycles in switching systems with nilpotent equilibrium points. Then a question naturally arises: What results obtained from the switching system with elementary equilibrium points can be extended to the case of non-elementary equilibrium points? How can we modify the method developed for the elementary case to study non-elementary cases?

In this paper, we will study the center conditions and bifurcation of small-amplitude limit cycles around an isolated nilpotent equilibrium point in switching systems. This is much more challenging compared to the study for switching systems with elementary equilibrium points. By an isolated nilpotent equilibrium point in planar polynomial systems, it means that both eigenvalues of the Jacobian matrix of both the first and second systems, evaluated at the equilibrium point, are zero but the Jacobian matrix is not null. Computationally efficient methods have been developed to study the center problem and the cyclicity problem for planar smooth systems with nilpotent equilibrium points, see [37,38,44,47,49]. Note that quadratic smooth polynomial systems do not have nilpotent centers, i.e., the simplest nilpotent centers must appear in cubic smooth systems [43]. In [21], a method is presented for studying the center problem in smooth systems with nilpotent critical points. García [19] developed a technique which can bound the maximum number of limit cycles for a large family of symmetric smooth polynomial systems with nilpotent equilibrium points. Recently, Li et al. [32] obtained the center conditions on two symmetric nilpotent equilibrium points in a class of Z_2-equivariant cubic smooth systems, and proved the existence of 12 limit cycles for such cubic smooth systems.

However, it is known that there are no algorithms for studying the center problem and bifurcation of limit cycles in switching nilpotent systems. Hence, in this paper, based on Bogdanov-Takens bifurcation theory we will develop a new perturbation approach so that the Poincaré-Lyapunov method for switching systems with linear type centers can be applied to study the center conditions of planar switching polynomial systems associated with a nilpotent equilibrium point, and apply it to investigate a class of quadratic switching systems with a nilpotent equilibrium point.

Consider the quadratic switching nilpotent system of the following form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}, & \text{for } x > 0, \\ \begin{pmatrix} y + A_{20}x^2 + A_{11}xy + A_{02}y^2 \\ B_{20}x^2 + B_{11}xy + B_{02}y^2 \end{pmatrix}, & \text{for } x < 0, \end{cases} \tag{2}$$

where the y-axis is the unique switching manifold, and $\lambda = (a_{20}, \dots, b_{02}, A_{20}, \dots, B_{02}) \in \mathbf{R}^{12}$, representing the parameter space. Thus, the origin (0, 0) of system (2) is a nilpotent equilibrium point. The conditions ensuring the origin of system (2) to be a center are derived under the conditions $b_{20} < 0$ and $B_{20} > 0$, i.e., the origin is a cusp of both the first and second systems in (2). Without loss of generality, we may assume that $b_{20} = -1$ and $B_{20} = 1$.

The main results of this paper are stated in the following two theorems.

Theorem 1.1. *With $b_{20} = -1$ and $B_{20} = 1$, the origin of system (2) is a center if one of the following conditions holds:*

$$\begin{aligned}
\text{I}: &\ a_{20} - A_{20} = a_{02} - A_{02} = a_{11} + A_{11} = b_{02} + B_{02} = b_{11} - B_{11} = 0;\\
\text{II}: &\ a_{20} = A_{20} = a_{02} = b_{11} = A_{02} = B_{11} = 0;\\
\text{III}: &\ a_{02} - A_{02} = a_{11} + 2b_{02} = A_{11} + 2B_{02} = b_{11} + 2a_{20} = B_{11} + 2A_{20} = 0;\\
\text{IV}: &\ a_{20} = a_{02} = A_{02} = b_{11} = A_{11} + 2B_{02} = B_{11} + 2A_{20} = 0,\ A_{20} \neq 0;\\
\text{V}: &\ a_{11} = A_{11} = a_{02} = A_{02} = b_{02} = B_{02} = b_{11} - 2A_{20} = B_{11} - 2a_{20} = 0,\ A_{20} \neq \pm a_{20};\\
\text{VI}: &\ A_{20} = a_{02} = A_{02} = a_{11} + 2b_{02} = b_{11} + 2a_{20} = B_{11} = 0,\ a_{20} \neq 0.
\end{aligned} \tag{3}$$

It should be noted that allowing $A_{20} = 0$ in the condition IV, or $a_{20} = 0$ in the condition VI yields two special cases of the condition II, and this is why the condition $A_{20} \neq 0$ is imposed in IV, and $a_{20} \neq 0$ in VI. Similarly, letting $A_{20} = a_{20}$ in the condition V yields a special case of the condition I, and $A_{20} = -a_{20}$ in V gives a special case of the condition III.

Moreover, we want to find the maximal number of limit cycles around the origin of system (2). To achieve this, we choose one of the center conditions I-VI and construct a perturbed system to prove the existence of small-amplitude limit cycles bifurcating from the perturbed nilpotent equilibrium point. As far as we are concerned with bifurcation of limit cycles, the following result is a new lower bound on cyclicity problem for such quadratic switching nilpotent systems.

Theorem 1.2. *With $b_{20} = -1$, $B_{20} = 1$, and the center condition* III *in Theorem 1.1, system (2) has at least seven limit cycles around the origin under small quadratic perturbations.*

In the next section, we present the Poincaré-Lyapunov method for computing the Lyapunov constants of switching systems with elementary centers. In Section 3, we develop a perturbation approach so that the Poincaré-Lyapunov method for switching systems with elementary centers can be applied to study system (2). Then this new method is used to prove Theorems 1.1 and 1.2 in Sections 4 and 5, respectively. Finally, conclusion is drawn in Section 6.

2. Preliminary

In this section, the Poincaré-Lyapunov method is present for studying the linear type center problem in switching systems. This method provides an algorithm which uses only a finite jet of the system at each step for calculating the Lyapunov constants, for more details see [20]. It will be generalized in the next section to develop the Poincaré-Lyapunov method for switching systems with isolated nilpotent equilibrium points.

Consider the following switching polynomial system,

$$(\dot{x}, \dot{y}) = \begin{cases} \left(\delta x - \beta_+ y + \sum_{k=2}^{n} X_k^+(x, y),\ \beta_+ x + \delta y + \sum_{k=2}^{n} Y_k^+(x, y)\right), & \text{for } y > 0, \\ \left(\delta x - \beta_- y + \sum_{k=2}^{n} X_k^-(x, y),\ \beta_- x + \delta y + \sum_{k=2}^{n} Y_k^-(x, y)\right), & \text{for } y < 0, \end{cases} \tag{4}$$

where $\delta \in \mathbf{R}$, $\beta_\pm > 0$, and $X_k^\pm(x, y)$, $Y_k^\pm(x, y)$ are homogeneous polynomials in the variables x and y of degree k, $k = 2, 3, \ldots, n$, and the x-axis is the switching manifold. The origin of (4) is a common elementary center of both the first and second systems when $\delta = 0$. Note that in next section we will transform the y-axis switching manifold of system (2) to the x-axis for our study on nilpotent switching systems.

Introducing the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ into system (4) we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \begin{cases} \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_k^+(\theta) r^k}{\beta_+ + \sum_{k=2}^{n} \Theta_k^+(\theta) r^{k-1}}, & \text{for } \theta \in (0, \pi), \\ \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_k^-(\theta) r^k}{\beta_- + \sum_{k=2}^{n} \Theta_k^-(\theta) r^{k-1}}, & \text{for } \theta \in (\pi, 2\pi), \end{cases} \tag{5}$$

where $\Upsilon_k^\pm(\theta)$ and $\Theta_k^\pm(\theta)$ are polynomials in $\cos\theta$ and $\sin\theta$ of degree $k+1$. We denote by

$$r^+(\rho, \theta) = \sum_{k\geqslant 1} v_k^+(\theta)\rho^k \quad \text{and} \quad r^-(\rho, \theta) = \sum_{k\geqslant 1} v_k^-(\theta)\rho^k,$$

the solutions of the first and second systems of (5) with $r^+(\rho, 0) = r^-(\rho, \pi) = \rho$. Further, define respectively the positive and negative half-return maps of system (5) as

$$\Delta^+(\rho) = r^+(\rho, \pi) = e^{\pi \frac{\delta}{\beta_+}}\rho + \sum_{k\geqslant 2} v_k^+ \rho^k, \quad \Delta^-(\rho) = r^-(\rho, 2\pi) = e^{\pi \frac{\delta}{\beta_-}}\rho + \sum_{k\geqslant 2} v_k^- \rho^k.$$

Then the return map, as shown in Fig. 1(a), can be constructed as

$$\Delta(\rho) = \Delta^-(\Delta^+(\rho)) = e^{\pi\delta(\frac{1}{\beta_+} + \frac{1}{\beta_-})}\rho + \sum_{k\geqslant 2} v_k \rho^k, \tag{6}$$

and the displacement function has the following form,

$$d(\rho) = \Delta(\rho) - \rho = \left[e^{\pi\delta(\frac{1}{\beta_+} + \frac{1}{\beta_-})} - 1\right]\rho + \sum_{k\geqslant 2} v_k \rho^k. \tag{7}$$

Note that it is extremely difficult to compute the composition of the two maps $\Delta^+(\rho)$ and $\Delta^-(\rho)$. However, it may follow Lemma 2.1 in [20] to develop a convenient method for computing the displacement function. By the transformation, $(x, y, t) \to (x, -y, -t)$, system (4) becomes

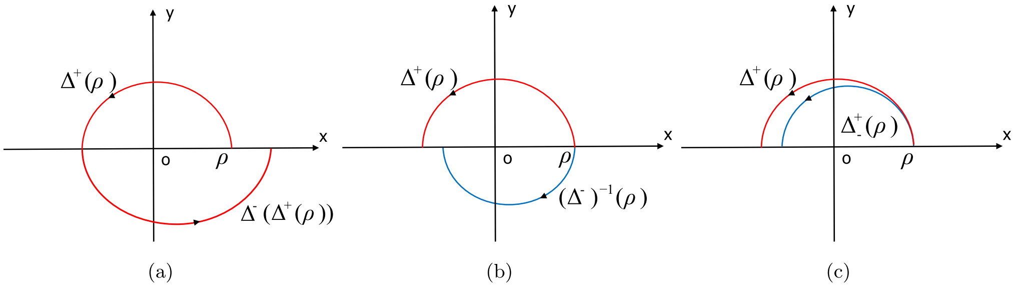

Fig. 1. (a) map $\Delta(\rho)$; (b) map $(\Delta^-)^{-1}(\rho)$; and (c) map $\Delta_-^+(\rho)$.

$$
(\dot{x}, \dot{y}) = \begin{cases} -\delta x - \beta_- y - \sum_{k=2}^{n} X_k^-(x, -y), \quad \beta_- x + \delta y + \sum_{k=2}^{n} Y_k^-(x, -y), & \text{for } \ y > 0, \\ -\delta x - \beta_+ y - \sum_{k=2}^{n} X_k^+(x, -y), \quad \beta_+ x + \delta y + \sum_{k=2}^{n} Y_k^+(x, -y), & \text{for } \ y < 0. \end{cases} \tag{8}
$$

Thus, we have the following displacement function

$$
d(\rho) = \Delta^+(\rho) - (\Delta^-)^{-1}(\rho) = \Delta^+(\rho) - \Delta_-^+(\rho), \tag{9}
$$

where $(\Delta^-)^{-1}(\rho)$ is the inverse map of $\Delta^-(\rho)$, and $\Delta_-^+(\rho)$ is the positive half-return map of system (8) (see Figs. 1(b) and 1(c), respectively).

Further, we have

$$
(\Delta^-)^{-1}(\rho) = \Delta_-^+(\rho) = e^{-\pi \frac{\delta}{\beta_-}} \rho + \sum_{k \geqslant 2} u_k^+ \rho^k,
$$

and the displacement function can be rewritten as

$$
d(\rho) = \left[\frac{e^{\pi \delta (\frac{1}{\beta_+} + \frac{1}{\beta_-})} - 1}{e^{\pi \frac{\delta}{\beta_-}}} \right] \rho + \sum_{k \geqslant 2} (v_k^+ - u_k^+) \rho^k = \sum_{k \geqslant 1} V_k \rho^k, \tag{10}
$$

where V_k is called the kth-order Lyapunov constant at the origin of the switching system (4). Hence, we need to compute v_k^+ and u_k^+ in order to get V_k ($k \geqslant 2$). Now, we show how to obtain v_k^+, and then an analogous way to get u_k^+. Letting $\delta = 0$ and $\beta_+ = \beta_- = \beta$ in (5) yields

$$
\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\sum_{k=2}^{n} \Upsilon_k^+(\theta) r^k}{\beta + \sum_{k=2}^{n} \Theta_k^+(\theta) r^{k-1}}, \quad \text{for } \ \theta \in (0, \pi), \tag{11}
$$

which can be expanded into power series in r,

$$
\frac{\mathrm{d}r}{\mathrm{d}\theta} = \sum_{k=2}^{\infty} R_k^+(\theta) r^k, \tag{12}
$$

where $R_k(\theta)$ is a polynomial in $\cos\theta$ and $\sin\theta$. We note that

$$\begin{aligned}\frac{\sum_{k=2}^{n}\Upsilon_k^+(\theta)r^k}{\beta+\sum_{k=2}^{n}\Theta_k^+(\theta)r^{k-1}} &= \frac{1}{\beta}\left[\sum_{k=2}^{n}\Upsilon_k^+(\theta)r^k\right]\left[1+\sum_{i=1}^{\infty}(-\sum_{k=2}^{n}\frac{\Theta_k^+(\theta)}{\beta}r^{k-1})^i\right] \\ &= \frac{1}{\beta}\left[\sum_{k=2}^{n}\Upsilon_k^+(\theta)r^k\right]\left[1+\sum_{k=1}^{\infty}\tilde{\Theta}_k^+(\theta)r^k\right],\end{aligned} \tag{13}$$

which is combined with the equations (11) and (12) to yield

$$R_k^+(\theta)=\frac{1}{\beta}\left[\sum_{i=2}^{k-1}\Upsilon_i^+(\theta)\tilde{\Theta}_{k-i}^+(\theta)+\Upsilon_k^+(\theta)\right].$$

Finally, we substitute the solution $r(\rho,\theta)=\sum\limits_{k\geqslant 1}v_k^+(\theta)\rho^k$ into (12) to get the differential equations:

$$\frac{\mathrm{d}v_k^+(\theta)}{\mathrm{d}\theta}=R_k^+(\theta)\Omega_{k,k}^+(\theta)+R_{k-1}^+(\theta)\Omega_{k-1,k}^+(\theta)+\cdots+R_2^+(\theta)\Omega_{2,k}^+(\theta),\quad k\geqslant 2, \tag{14}$$

where $\Omega_{k,k}^+(\theta)=(v_1^+(\theta))^k$ and $\Omega_{i,j}^+(\theta)$ are polynomials in $v_l^+(\theta)$, $1\leqslant l\leqslant j$. Consequently, we have $\frac{\mathrm{d}v_1^+(\theta)}{\mathrm{d}\theta}=0$ and thus, without loss of generality, obtain $v_1^+(\theta)=1$. Then,

$$v_k^+(\theta)=\int_0^{\theta}\left[R_k^+(\bar\theta)+R_{k-1}^+(\bar\theta)\Omega_{k-1,k}^+(\bar\theta)+\cdots+R_2^+(\bar\theta)\Omega_{2,k}^+(\bar\theta)\right]\mathrm{d}\bar\theta,\quad k\geqslant 2. \tag{15}$$

Similarly, we can compute $u_k^+(\theta)$. Then the Lyapunov constants are given by $V_k=v_k^+(\pi)-u_k^+(\pi)\equiv v_k^+-u_k^+$, $k\geqslant 2$ when $\delta=0$.

It is easy to see that the origin of system (4) is a center if and only if all the Lyapunov constants in the displacement function (10) vanish, i.e., $d(\rho)\equiv 0$ for $0<\rho\ll 1$. Further, the number of isolated zeros of $d(\rho)=0$ near $\rho=0$ (or the fixed points of $\Delta(\rho)$) corresponds to the number of limit cycles of system (4). If there exists $\lambda_*\in\Lambda$ such that

$$V_1(\lambda_*)=V_2(\lambda_*)=\cdots=V_k(\lambda_*)=0,\ \ V_{k+1}(\lambda_*)\neq 0, \tag{16}$$

then appropriately perturbing system (4) will yield at most k limit cycles around the origin. More precisely, based on Lemma 4 in [45], we have the following lemma which gives the sufficient conditions for the existence of small-amplitude limit cycles around the origin of system (4).

Lemma 2.1 *([45]). Suppose that there exists a critical point* $\lambda_*=(a_{1c},a_{2c},\ldots,a_{kc})$ *such that* $V_{i_1}(\lambda_*)=V_{i_2}(\lambda_*)=\cdots=V_{i_k}(\lambda_*)=0$, $V_{i_{k+1}}(\lambda_*)\neq 0$, *with* $1=i_1<i_2<\cdots<i_k$, *and*

$$\det\left[\frac{\partial(V_{i_1},V_{i_2},\cdots,V_{i_k})}{\partial(a_{1c},a_{2c},\cdots,a_{kc})}(\lambda_*)\right]\neq 0, \tag{17}$$

then system (4) *has exactly k limit cycles bifurcating from the origin of the system by appropriate small perturbations on* $\lambda = \lambda_*$.

3. The Poincaré-Lyapunov method for switching nilpotent systems

Now, consider the following switching system with a nilpotent critical point at the origin,

$$(\dot{x},\ \dot{y}) = \begin{cases} \left(y + f_1^+(x,y),\ f_2^+(x,y)\right), & \text{for } x > 0, \\ \left(y + f_1^-(x,y),\ f_2^-(x,y)\right), & \text{for } x < 0, \end{cases} \tag{18}$$

where $f_1^\pm(x,y)$ and $f_2^\pm(x,y)$ are real polynomials without constant and linear terms, and the origin is a common equilibrium point of both the first and second systems of (18).

Similar to the result of [8], we have the following result to prove the center conditions for system (18) at the origin.

Proposition 3.1. *Assume that the origin of system* (18) *is a monodromic equilibrium point. If the following conditions hold:*

(a) *there exist the first integrals* $H^+(x,y)$ *and* $H^-(x,y)$ *near the origin in the first half-planar system and the second one of* (18) *respectively,*
(b) $H^+(0,y)$ *and* $H^-(0,y)$ *are both even functions in* y, *or* $H^+(0,y) \equiv H^-(0,y)$,

then the origin of system (18) *is a center.*

In [33], symmetry is defined for switching systems with the unique switching manifold y-axis to prove an equilibrium point to be a center. Here, noticing that the switching manifold of system (18) is the x-axis, we thus similarly have the following result to prove the origin of system (18) to be a center.

Proposition 3.2. *Assume that the origin of system* (18) *is a monodromic equilibrium point. If the vector fields of* (18) *satisfy*

$$\begin{aligned} (f_1^+(x,y), f_2^+(x,y)) &= (-f_1^+(x,-y), f_2^+(x,-y)), \\ (f_1^-(x,y), f_2^-(x,y)) &= (-f_1^-(x,-y), f_2^-(x,-y)), \end{aligned} \tag{19}$$

or

$$(f_1^+(x,y), f_2^+(x,y)) = (f_1^-(-x,y), -f_2^-(-x,y)), \tag{20}$$

i.e., system (18) *is symmetric with respect to either the* x*-axis or the* y*-axis, then the origin of* (18) *is a center.*

In order to prove Theorems 1.1 and 1.2, we need to perturb system (18) and then establish the relation between the perturbed system and the unperturbed system. Our main idea is based on the Bogdanov-Takens (B-T) bifurcation theory. To explain the idea, we first briefly describe the

B-T bifurcation theory for smooth systems. Consider the following general nonlinear dynamical system:

$$\begin{aligned} \dot{x} &= y + f_1(x, y), \\ \dot{y} &= f_2(x, y), \end{aligned} \tag{21}$$

where f_1 and f_2 are assumed analytic and their Taylor expansions about $(x, y) = (0, 0)$ contain terms starting from second order. Thus, the origin is a nilpotent point. With a series of nonlinear transformations, the normal form of system (21) can be obtained as (for example, see [23,25,28]):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \sum_{k=2}^{\infty}(a_k x^2 + b_k x^{k-1} y), \end{aligned} \tag{22}$$

where a_k's and b_k's are called normal form coefficients, which are explicitly expressed in terms of the original system coefficients. Then the bifurcation associated with system (21) is called B-T bifurcation. Note that even for a quadratic polynomial system, its normal form (22) can have an infinite number of terms, and that's why the focus values or Lyapunov constants can go to infinite order.

To perform bifurcation analysis for system (21) in the vicinity of the origin, we need to add perturbation terms to the normal form (22), which are so-called unfolding. For example, if system (21) has a codimension-2 B-T bifurcation, then the generic normal form with unfolding is given by [23]:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \beta_1 + \beta_2\, y + a_2 x^2 + b_2 x\, y, \end{aligned} \tag{23}$$

satisfying $a_2 b_2 \neq 0$, where the term $\beta_1 + \beta_2\, y$ is called unfolding with small β_1 and β_2. The origin of this system is a nilpotent point as $\beta_1 \to 0,\ \beta_2 \to 0$. For convenience, we may call the system (23) with $\beta_1 = \beta_2 = 0$ as the *limit* of system (23).

The system (23) can have saddle-node bifurcation, Hopf bifurcation and homoclinic loop bifurcation in the vicinity of the origin (i.e., for small β_1 and β_2). In particular, the Hopf bifurcation occurs from one of the two equilibria $(x, y) = (\pm\sqrt{-\frac{\beta_1}{a_2}}, 0)$ on the bifurcation curve,

$$\beta_1 = -\frac{a_2}{b_2^2}\,\beta_2^2,$$

in the vicinity of the origin. Note that system (23) has a linear-type monodromic equilibrium point at each point on the bifurcation curve, namely, this linear-type monodromic equilibrium point holds as $\beta_1 \to 0,\ \beta_2 \to 0$. At $\beta_1 = \beta_2 = 0$, this linear-type monodromic equilibrium point is reduced to the nilpotent point. For codimension-2 B-T bifurcation, there exists one limit cycle near the Hopf bifurcation curve. A non-generic codimension-2 B-T bifurcation can have the following normal form [26]:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \beta_1\, x + \beta_2\, y + a_2\, x^2 + b_2\, x\, y, \end{aligned} \tag{24}$$

where the unfolding is $\beta_1 x + \beta_2 y$. This system does not hold symmetry under general perturbation, and it has a transcritical bifurcation, rather than a saddle-node bifurcation like that occurring in (22). Moreover, unlike system (22), this system can have a Hopf bifurcation from the equilibrium point $(x, y) = (0, 0)$ if $\beta_1 < 0$. When $a_2 b_2 = 0$, higher-order nonlinear terms are needed to determine the codimension and corresponding unfolding, for which multiple limit cycles may bifurcate. This clearly indicates that the structure of the perturbed system has changed, depending upon the added unfolding. However, it should be noted that the local dynamical behaviors around the origin such as bifurcation of limit cycles are associated with the nilpotent point.

Now we come back to our switching system (18), and add small perturbations to obtain the following perturbed system:

$$(\dot{x}, \dot{y}) = \begin{cases} \Big(y + f_1^+(x, y) + \varepsilon g_1^+(x, y, \varepsilon),\ -\varepsilon^2 x + f_2^+(x, y) + \varepsilon g_2^+(x, y, \varepsilon)\Big), & \text{for } x > 0, \\ \Big(y + f_1^-(x, y) + \varepsilon g_1^-(x, y, \varepsilon),\ -\varepsilon^2 x + f_2^-(x, y) + \varepsilon g_2^-(x, y, \varepsilon)\Big), & \text{for } x < 0, \end{cases} \tag{25}$$

where $\varepsilon > 0$ is a small perturbation parameter, and $g_{1,2}^{\pm}$ are real polynomials without constant and linear terms. These ε terms are analogous to the unfolding in system (24). Thus, it is obvious that system (25) has an *isolated* linear-type monodromic equilibrium point at the origin. Moreover, this linear-type monodromic equilibrium point is reduced to the nilpotent point as $\varepsilon \to 0$, which implies that system (18) is the limit of system (25). It should be pointed out that the ε perturbation terms in (25) are applied to the whole system, which is not restricted to the local behavior. For example, it may general global bifurcations such as homoclinic or heteroclinic loop bifurcations. But when we consider local bifurcations in the vicinity of the origin of system (25), these ε terms indeed play *weak* nonlinear influence.

Remark 3.3. Note that the perturbation terms used in (25) are in ε^2 order rather than in ε order, which can avoid $\sqrt{\varepsilon}$-order and ε^{-k}-order terms ($k > 0$) in later transformed system (see (31)).

Based on the B-T bifurcation theory and the relation established above for the two systems (18) and (25), we directly obtain the following result.

Lemma 3.4. *System* (18) *is the limit of system* (25). *Assume that the origin of system* (18) *is a monodromic equilibrium point, then the linear-type monodromic equilibrium point of system* (25) *is reduced to the nilpotent point of system* (18) *at the origin as* $\varepsilon \to 0$. *Moreover, if the linear-type monodromic equilibrium point of system* (25) *becomes a center, then the origin of system* (18) *is a nilpotent center.*

Having established Lemma 3.4, we can now apply the Poincaré-Lyapunov method and the Lyapunov constant computation, described in the previous section for switching systems with an elementary center, to study the switching polynomial system (18) with a nilpotent equilibrium point, and to prove Theorems 1.1 and 1.2. To achieve this, we compute the Lyapunov constants for system (25) since it has an isolated linear-type monodromic equilibrium point at the origin.

For proving Theorem 1.1 (to be done in the next section), instead of system (18), we use the quadratic polynomial system (2) and only add the linear perturbation term $-\varepsilon^2 x$ to this system. Then, the Lyapunov constants for this perturbed system contain ε terms. We derive the conditions under which these ε terms vanish, which implies that these conditions are necessary for the linear-type monodromic equilibrium point to become a center. By Lemma 3.4, we know that these conditions are also necessary for the nilpotent point of system (2) being a nilpotent center. We then directly use system (2) and some methods to prove that these conditions are also sufficient. By this combined approach, we obtain the six conditions, as listed in Theorem 1.1, for the origin of system (2) to be a nilpotent center.

Proving Theorem 1.2, which is given in Section 5, is different from that for proving Theorem 1.1. We actually generalize the approach for analyzing the bifurcation of limit cycles in B-T bifurcation. For a codimension-2 B-T bifurcation, there is only one limit cycle which can bifurcate from a Hopf bifurcation point. We want to find bifurcation of limit cycles as many as possible from a degenerate or generalized Hopf bifurcation by adding more ε terms, like those given in (25), to our quadratic system (2). To get more limit cycles, we use one of the center conditions in Theorem 1.1 with the linear perturbation term $-\varepsilon^2 x$ added to system (2) so that the perturbed system has an elementary center at the origin. Then, we add quadratic perturbation terms with various ε order to this system and compute the Lyapunov constants. Next, using higher ε^k-order Lyapunov constants [46,48], we can prove the existence of 7 small-amplitude limit cycles.

4. The proof of Theorem 1.1

First, we analyze the multiplicity of nilpotent equilibrium point $(0,0)$ of system (2). For the general switching polynomial system (18), we assume that

$$f^{\pm}(x) = \sum_{k=2}^{\infty} c_k^{\pm} x^k$$

are the unique solutions of the implicit function equations $y + f_1^{\pm}(x,y) = 0$ in a neighborhood of the origin, respectively, and $f^{\pm}(x)|_{x=0} = 0$. Denote that

$$\begin{aligned} f_2^{\pm}(x, f^{\pm}(x)) &= \sum_{k=2}^{\infty} \alpha_k^{\pm} x^k, \\ \left[\frac{\partial f_1^{\pm}}{\partial x} + \frac{\partial f_2^{\pm}}{\partial y}\right]_{(x, f^{\pm}(x))} &= \sum_{k=1}^{\infty} \beta_k^{\pm} x^k. \end{aligned} \tag{26}$$

For smooth polynomial systems, if $\alpha_2 = \alpha_3 = \cdots = \alpha_{k-1} = 0$ and $\alpha_k \neq 0$, then the multiplicity of the nilpotent equilibrium point is exactly k, for more detail see [38]. It follows from Theorem 3.5 in [14] that the following results can be used to determine the local behavior at a nilpotent equilibrium point of smooth polynomial systems. When $\beta_n = 0$ and $\alpha_m \neq 0$, we have

$$\begin{cases} m = 2k, \ \text{cusp}, \\ m = 2k+1, \ \begin{cases} \alpha_m > 0, \ \text{saddle}, \\ \alpha_m < 0, \ \text{center or focus}. \end{cases} \end{cases} \tag{27}$$

When $\beta_n \neq 0$, $\alpha_m \neq 0$ and $\mu = \beta_n^2 + 4(n+1)\alpha_m$, the following holds:

$$\begin{cases} m = 2k, & \begin{cases} k \leqslant n, \text{ cusp}, \\ k > n, \text{ saddle-node}, \end{cases} \\ m = 2k+1, & \begin{cases} \alpha_m > 0, \text{ saddle}, \\ \alpha_m < 0, & \begin{cases} k < n, \text{ or } k = n \text{ and } \mu < 0, \text{ center or focus}, \\ k > n, \text{ or } k = n \text{ and } \mu \geqslant 0, & \begin{cases} n \text{ even, node}, \\ n \text{ odd, H-E}, \end{cases} \end{cases} \end{cases} \end{cases} \tag{28}$$

where H-E denotes an equilibrium point consisting of one hyperbolic and one elliptic sectors.

It is easy to use the above results to obtain the following conditions for the switching polynomial system (2):

$$\begin{aligned} \beta_1^+ &= 2a_{20} + b_{11}, \quad \alpha_2^+ = b_{20}, \quad \alpha_3^+ = -a_{20}b_{11}, \\ \beta_1^- &= 2A_{20} + B_{11}, \quad \alpha_2^- = B_{20}, \quad \alpha_3^- = -A_{20}B_{11}. \end{aligned} \tag{29}$$

Proposition 4.1. *The multiplicity of the nilpotent equilibrium point* $(0, 0)$ *of the first system or the second system of* (2) *is at most equal to four.*

Proof. In fact, taking $\alpha_2^+ = \alpha_3^+ = 0$ in (29) yields

$$b_{20} = 0, \quad a_{20}b_{11} = 0.$$

Then, by assuming $a_{20} = 0$ we obtain $\alpha_4^+ = 0$. So the polynomials $y + f_1^+(x, y)$ and $f_2^+(x, y)$ have a common factor y, implying that the origin is not an isolated equilibrium point.

If $b_{11} = 0$, we have $\alpha_4^+ = a_{20}^2 b_{02} \neq 0$ ($b_{02} \neq 0$). Otherwise, we have $f_2^+(x, y) = 0$ when $b_{02} = 0$, indicating that the origin is not an isolated equilibrium point.

In summary, the multiplicity of the nilpotent equilibrium point $(0, 0)$ of the first system of (2) is at most four. Similarly, we can show that the multiplicity of the nilpotent equilibrium point $(0, 0)$ of the second system of (2) is also at most four. □

It is well known that the multiplicity of a nilpotent focus or center in smooth systems is an odd positive integer and greater than one. However, the multiplicity of a nilpotent focus or center in the first system or the second system of (2) cannot be three, because otherwise it needs

$$\alpha_2^\pm = 0, \quad \alpha_3^\pm < 0, \quad \mu^\pm = (\beta_1^\pm)^2 + 8\alpha_3^\pm < 0,$$

but we actually have $\mu^+ = (2a_{20} - b_{11})^2 \geqslant 0$ and $\mu^- = (2A_{20} - B_{11})^2 \geqslant 0$. So we can only have the center conditions of the first and second systems of (2) with a 2nd-order critical point $(0, 0)$. Therefore, assume that $b_{20} \neq 0$ and $B_{20} \neq 0$ under which $\alpha_2^+ \neq 0$ and $\alpha_2^- \neq 0$. Then, the equilibrium point $(0, 0)$ in the first system or the second system of (2) is a cusp. Since the equilibrium point $(0, 0)$ of (2) cannot be a center or a focus when $b_{20} > 0$ or $B_{20} < 0$, we only need to consider $b_{20} < 0$ and $B_{20} > 0$. Without loss of generality, we may set $b_{20} = -1$ and $B_{20} = 1$.

Example 4.2. The phase portraits of system (2), as depicted in Figs. 2(a) and 2(b), show the origin of (2) to be a cusp or a saddle.

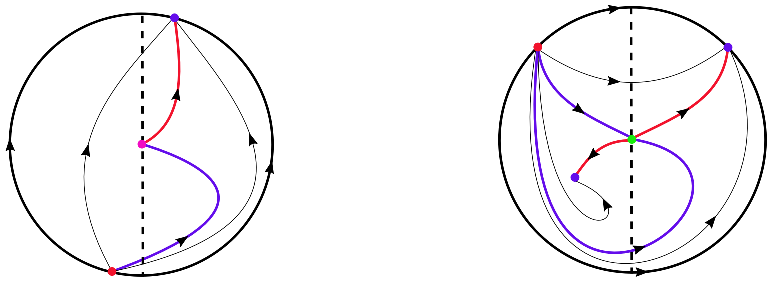

Fig. 2. The phase portraits of system (2) for (a) $a_{20} = a_{11} = a_{02} = b_{20} = b_{11} = b_{02} = A_{11} = A_{02} = B_{20} = B_{11} = B_{02} = 1$, showing a cusp; and (b) $a_{20} = a_{11} = a_{02} = b_{20} = b_{11} = b_{02} = A_{11} = A_{02} = B_{11} = B_{02} = 1$ and $B_{20} = -1$, showing a saddle.

Now, we study the center problem for the switching system (2). We add one simple perturbation term $-\varepsilon^2 x$ with $0 < \varepsilon \ll 1$ in the $\dot{y}$ equation. Thus, for simplicity, we consider the following perturbed system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ -\varepsilon^2 x - x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}, & \text{for } x > 0, \\ \begin{pmatrix} y + A_{20}x^2 + A_{11}xy + A_{02}y^2 \\ -\varepsilon^2 x + x^2 + B_{11}xy + B_{02}y^2 \end{pmatrix}, & \text{for } x < 0. \end{cases} \tag{30}$$

Then, introducing the transformation $(x, y, t) \to (\varepsilon^2 y, \varepsilon^3 x, \frac{t}{\varepsilon})$ into the above system yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + b_{02}\varepsilon^2 x^2 + b_{11}\varepsilon xy - y^2 \\ x + a_{02}\varepsilon^3 x^2 + a_{11}\varepsilon^2 xy + a_{20}\varepsilon y^2 \end{pmatrix}, & \text{for } y > 0, \\ \begin{pmatrix} -y + B_{02}\varepsilon^2 x^2 + B_{11}\varepsilon xy + y^2 \\ x + A_{02}\varepsilon^3 x^2 + A_{11}\varepsilon^2 xy + A_{20}\varepsilon y^2 \end{pmatrix}, & \text{for } y < 0. \end{cases} \tag{31}$$

Then, we can apply the method described in Section 2 to compute the Lyapunov constants associated with the origin of system (31) with the aid of a computer algebra system. It is easy to find the 1st Lyapunov constant $V_1(\lambda) = 0$ and the 2nd Lyapunov constant, given by

$$V_2(\lambda) = \frac{2}{3}\varepsilon\big[2(a_{20} - A_{20}) + b_{11} - B_{11} + (a_{02} - A_{02})\varepsilon^2\big]. \tag{32}$$

We consider two cases: $A_{20} = a_{20}$ and $A_{20} \neq a_{20}$.

4.1. Case: $A_{20} = a_{20}$

For this case, we have the following theorem.

Theorem 4.3. *Assume $b_{20} = -1$, $B_{20} = 1$ and $A_{20} = a_{20}$. The origin of system* (2) *is a center if one of the conditions* I, II and III *in* (3) *holds.*

Proof. Under the condition $A_{20} = a_{20}$, setting $V_2(\lambda) = 0$ we get the necessary center conditions $B_{11} = b_{11}$ and $A_{02} = a_{02}$. Further, we consider two sub-cases: $a_{20} = 0$ and $a_{20} \neq 0$.

(1) $a_{20} = 0$, for which we obtain the 3rd Lyapunov constant,

$$V_3(\lambda) = \frac{\pi}{8}\varepsilon^3\big\{b_{11}(b_{02} + B_{02}) - a_{02}\big[a_{11} + A_{11} + 2(b_{02} + B_{02})\big]\varepsilon^2\big\}.$$

(1a) Letting $B_{02} + b_{02} = 0$ yields $V_3(\lambda) = -\frac{\pi}{8}a_{02}(a_{11} + A_{11})\,\varepsilon^5$. Then we take $A_{11} + a_{11} = 0$, yielding $V_3(\lambda) = 0$, which is included in the condition I. Otherwise if $A_{11} + a_{11} \neq 0$, then $a_{02} = 0$ under which we have

$$V_4(\lambda) = -\frac{2}{45}\varepsilon^3\,(a_{11} + A_{11})\,b_{11}\,(2 + b_{02}\varepsilon^2).$$

Setting $V_4(\lambda) = 0$ gives $b_{11} = 0$, which is included in the condition II.

(1b) Now we consider $b_{11} = 0$, and obtain $V_3(\lambda) = -\frac{\pi}{8}a_{02}(a_{11} + A_{11} + 2b_{02} + 2B_{02})\,\varepsilon^5$. Letting $a_{02} = 0$ yields the condition II. Otherwise, $A_{11} = -a_{11} - 2b_{02} - 2B_{02}$ under which $V_4(\lambda) = -\frac{2}{3}a_{02}(a_{11} + 2b_{02})(b_{02} + B_{02})\,\varepsilon^7$. Setting $V_4(\lambda) = 0$ yields $a_{11} + 2b_{02} = 0$, which is included in the condition III.

(2) For $a_{20} \neq 0$, we have

$$V_3(\lambda) = \frac{\pi}{8}\varepsilon^3\big[b_{11}(b_{02} + B_{02}) - a_{20}(a_{11} + A_{11}) - a_{02}(a_{11} + A_{11} + 2b_{02} + 2B_{02})\varepsilon^2\big].$$

Letting $A_{11} = \frac{-a_{11}a_{20}+b_{11}(b_{02}+B_{02})}{a_{20}}$ by $b_{11}(b_{02} + B_{02}) - a_{20}(a_{11} + A_{11}) = 0$, we have $V_3(\lambda) = -\frac{\pi}{8a_{20}}a_{02}(b_{02} + B_{02})(2a_{20} + b_{11})\varepsilon^5$.

(2a) If $b_{02} + B_{02} = 0$, we obtain $A_{11} = -a_{11}$ and $V_3(\lambda) = 0$, which is already included in the condition I.

(2b) If $a_{02} = 0$, we have $V_3(\lambda) = 0$ and

$$\begin{aligned}V_4(\lambda) = &-\frac{2}{45a_{20}}\varepsilon^3(b_{02} + B_{02})\big\{2(2a_{20} + b_{11})^2 + \big[2a_{20}^2(2a_{11} + 3b_{02} - 3B_{02})\\ &- (7a_{11} + 7b_{02} - 3B_{02})a_{20}b_{11} + (4b_{02} + 3B_{02})b_{11}^2\big]\varepsilon^2\big\}.\end{aligned}$$

If $b_{11} = -2a_{20}$, we obtain $A_{11} = \frac{-a_{11}a_{20}+b_{11}(b_{02}+B_{02})}{a_{20}} = -a_{11} - 2b_{02} - 2B_{02}$ and have $V_4(\lambda) = -\frac{4}{5}a_{20}(a_{11} + 2b_{02})(b_{02} + B_{02})\varepsilon^5$. The condition $a_{11} + 2b_{02} = 0$ is included in the condition III.

(2c) If $2a_{20} + b_{11} = 0$, we have $A_{11} = -a_{11} - 2b_{02} - 2B_{02}$ and $V_4(\lambda) = -\frac{2}{15}\varepsilon^5(a_{11} + 2b_{02})(b_{02} + B_{02})(6a_{20} + 5a_{02}\varepsilon^2)$. Since $a_{20} \neq 0$ we obtain $b_{02} + B_{02} = 0$ or $a_{11} + 2b_{02} = 0$ by solving $V_4(\lambda) = 0$. These two conditions are already included in the conditions I and III, respectively.

Summarizing the above results for $A_{20} = a_{20}$, we have obtained the condition I from the cases (1a) and (2a), no matter whether a_{20} is zero or not; the condition II from the case (1a); and the condition III from the cases (1b) and (2b). These conditions yield the first few Lyapunov constants to vanish and so are necessary conditions for the origin of system (30) to be a center. Since the nilpotent center of system (2) is the limit of the center of (25) with $g^{\pm}_{1,2}(x, y, \varepsilon) = 0$, the above conditions are necessary for the origin of unperturbed system (2) to be a nilpotent center.

Next, we prove that the conditions I, II and III are also sufficient for the origin of system (2) to be a center. First, consider the condition I in (3), under which system (2) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ -x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}, & \text{for } x > 0, \\ \begin{pmatrix} y + a_{20}x^2 - a_{11}xy + a_{02}y^2 \\ x^2 + b_{11}xy - b_{02}y^2 \end{pmatrix}, & \text{for } x < 0. \end{cases} \tag{33}$$

Obviously, system (33) is symmetric with respect to the y-axis, and so the origin of system (33) is a center by Proposition 3.2.

When the condition II in (3) holds, system (2) can be simplified as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{11}xy \\ -x^2 + b_{02}y^2 \end{pmatrix}, & \text{for } x > 0, \\ \begin{pmatrix} y + A_{11}xy \\ x^2 + B_{02}y^2 \end{pmatrix}, & \text{for } x < 0. \end{cases} \tag{34}$$

Since system (34) is symmetric with respect to the x-axis, and so the origin of system (34) is a center by Proposition 3.2.

When the condition III in (3) holds, system (2) can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{20}x^2 - 2b_{02}xy + a_{02}y^2 \\ -x^2 - 2a_{20}xy + b_{02}y^2 \end{pmatrix}, & \text{for } x > 0, \\ \begin{pmatrix} y + a_{20}x^2 - 2B_{02}xy + a_{02}y^2 \\ x^2 - 2a_{20}xy + B_{02}y^2 \end{pmatrix}, & \text{for } x < 0. \end{cases} \tag{35}$$

It can be shown that the first and second systems of (35) are Hamiltonian systems, with respectively the following two Hamiltonian quantities,

$$\begin{aligned} H^+(x, y) &= \frac{1}{2}y^2 + \frac{x^3}{3} + \frac{a_{02}}{3}y^3 + a_{20}x^2y - b_{02}xy^2, \\ H^-(x, y) &= \frac{1}{2}y^2 - \frac{x^3}{3} + \frac{a_{02}}{3}y^3 + a_{20}x^2y - B_{02}xy^2. \end{aligned} \tag{36}$$

It is easy to see that $H^+(0, y) \equiv H^-(0, y)$, satisfying the condition in Proposition 3.1, which implies that the origin of system (35) is a center. □

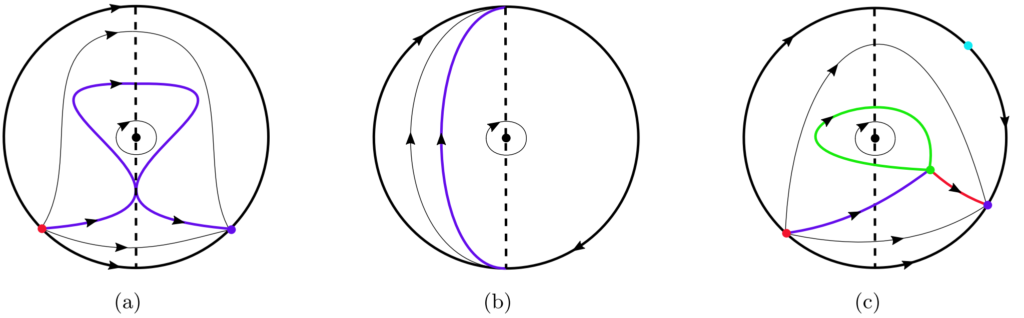

Fig. 3. The phase portraits for (a) the system (33) with $a_{20} = a_{11} = a_{02} = b_{02} = b_{11} = 1$ and $b_{02} = -1$; (b) the system (34) with $a_{11} = b_{02} = A_{11} = B_{02} = 1$; and (c) the system (35) with $a_{20} = a_{02} = b_{02} = B_{02} = 1$.

Example 4.4. The illustrated phase portraits for systems (33), (34), and (35) are shown in Figs. 3(a), 3(b) and 3(c), respectively.

4.2. *Case $A_{20} \neq a_{20}$*

For this condition, we have the following result.

Theorem 4.5. *Assume $b_{20} = -1$, $B_{20} = 1$ and $A_{20} \neq a_{20}$. The origin of system (2) is a center if one of the conditions* III, IV, V *and* VI *in* (3) *holds.*

Proof. It is seen from (32) that setting $V_2(\lambda) = 0$ yields $B_{11} = b_{11} + 2a_{20} - 2A_{20}$ and $A_{02} = a_{02}$. Further, we obtain the 3rd Lyapunov constant,

$$V_3(\lambda) = -\frac{\pi}{8}\varepsilon^3\big[a_{11}a_{20} + A_{11}A_{20} - 2a_{20}B_{02} + 2A_{20}B_{02} - b_{02}b_{11} - B_{02}b_{11} + a_{02}(a_{11} + A_{11} + 2b_{02} + 2B_{02})\varepsilon^2\big].$$

Consider two sub-cases: $a_{02} = 0$ and $A_{11} = -a_{11} - 2b_{02} - 2B_{02}$.

(1) $a_{02} = 0$, for which $V_3(\lambda)$ becomes

$$V_3(\lambda) = -\frac{\pi}{8}(a_{11}a_{20} + A_{11}A_{20} - 2a_{20}B_{02} + 2A_{20}B_{02} - b_{02}b_{11} - B_{02}b_{11})\varepsilon^3.$$

(1a) If $a_{20} = 0$ (and so $A_{20} \neq 0$), then we have

$$A_{11} = \frac{-2A_{20}B_{02} + b_{02}b_{11} + B_{02}b_{11}}{A_{20}}$$

by setting $V_3(\lambda) = 0$. Then, we obtain

$$V_4(\lambda) = -\frac{2}{45A_{20}}b_{11}\varepsilon^3\big[2a_{11}A_{20} + 12A_{20}^3 + 4A_{20}b_{02} - 6A_{20}^2b_{11} + 2b_{02}b_{11} + 2B_{02}b_{11} + (a_{11}A_{20}b_{02} - 7A_{20}b_{02}^2 - 18A_{20}b_{02}B_{02} - 9A_{20}B_{02}^2 + 4b_{02}^2b_{11} + 7b_{02}B_{02}b_{11}$$

$$+ 3B_{02}^2 b_{11})\varepsilon^2\big].$$

In order to have $V_4(\lambda) = 0$, we either set $b_{11} = 0$, leading to the condition IV, or obtain the following conditions if $b_{11} \neq 0$:

$$a_{11} = \frac{-6A_{20}^3 - 2A_{20}b_{02} + 3A_{20}^2 b_{11} - b_{02}b_{11} - B_{02}b_{11}}{A_{20}}$$

and

$$2A_{20}^3 b_{02} + 3A_{20}(b_{02} + B_{02})^2 - (A_{20}^2 b_{02} + (b_{02} + B_{02})^2)b_{11} = 0.$$

Let $M_1 = A_{20}^2 b_{02} + (b_{02} + B_{02})^2$ and $M_2 = 2A_{20}^3 b_{02} + 3A_{20}(b_{02} + B_{02})^2$.

(1a.1) If $M_1 = 0$, we have $M_2 = 0$ by $M_2 - M_1 b_{11} = 0$. Then computing the resultant of M_1 and M_2 with respect to A_{20} yields

$$\text{Resultant}[M_1, M_2, A_{20}] = b_{02}^2 (b_{02} + B_{02})^4,$$

which shows that the two polynomials M_1 and M_2 have common roots if and only if $b_{02} = B_{02} = 0$, under which we have

$$V_5(\lambda) = -\frac{\pi}{64} A_{20}^2 b_{11}(b_{11} - 2A_{20})^2 \varepsilon^5.$$

Hence, setting $V_5(\lambda) = 0$ yields $b_{11} = 2A_{20}$, which is included in the condition V.

(1a.2) If $M_1 \neq 0$, we have $b_{11} = \frac{M_2}{M_1} \neq 0$ by linearly solving $V_4(\lambda) = 0$, and further obtain

$$\begin{aligned}
V_5(\lambda) =& -\frac{\pi}{192 M_2^3}\varepsilon^5 A_{20}M_2\big[3M_3 + b_{02}(b_{02} + B_{02})^2 M_4 \varepsilon^2\big], \\
V_6(\lambda) =& \frac{1}{100800 M_2^4}\varepsilon^5 A_{20}M_2\big[-12288 M_2 M_3 + 3675\pi A_{20} M_2 M_3 \varepsilon + 1536 M_5 \varepsilon^2 \\
&+ 1225\pi A_{20} b_{02}(b_{02} + B_{02})^2 M_2 M_4 \varepsilon^3 + 256 b_{02}(b_{02} + B_{02})^2 M_6 \varepsilon^4\big],
\end{aligned}$$

where

$$\begin{aligned}
M_3 =& 2A_{20}^4 b_{02}^4 + 8A_{20}^2 b_{02}^5 + 3b_{02}^6 + 4A_{20}^6 b_{02}^2 B_{02} + 6A_{20}^4 b_{02}^3 B_{02} + 34A_{20}^2 b_{02}^4 B_{02} \\
&+ 18 b_{02}^5 B_{02} + 5A_{20}^4 b_{02}^2 B_{02}^2 + 56 A_{20}^2 b_{02}^3 B_{02}^2 + 45 b_{02}^4 B_{02}^2 + 44 A_{20}^2 b_{02}^2 B_{02}^3 \\
&+ 60 b_{02}^3 B_{02}^3 - A_{20}^4 B_{02}^4 + 16 A_{20}^2 b_{02} B_{02}^4 + 45 b_{02}^2 B_{02}^4 + 2A_{20}^2 B_{02}^5 \\
&+ 18 b_{02} B_{02}^5 + 3B_{02}^6, \\
M_4 =& 6A_{20}^4 b_{02}^2 + 20A_{20}^2 b_{02}^3 + 15 b_{02}^4 - 10A_{20}^4 b_{02} B_{02} + 26A_{20}^2 b_{02}^2 B_{02} + 60 b_{02}^3 B_{02} \\
&+ 3A_{20}^4 B_{02}^2 - 8A_{20}^2 b_{02} B_{02}^2 + 90 b_{02}^2 B_{02}^2 - 14 A_{20}^2 B_{02}^3 + 60 b_{02} B_{02}^3 + 15 B_{02}^4,
\end{aligned}$$

$$
\begin{aligned}
M_5 = {} & 12A_{20}^4 b_{02}^7 + +5A_{20}^2 b_{02}^8 - 11b_{02}^9 + 32A_{20}^8 b_{02}^4 B_{02} + 80A_{20}^6 b_{02}^5 B_{02} \\
& + 124A_{20}^4 b_{02}^6 B_{02} + 58A_{20}^2 b_{02}^7 B_{02} - 99b_{02}^8 B_{02} - 20A_{20}^8 b_{02}^3 B_{02}^2 \\
& + 117A_{20}^6 b_{02}^4 B_{02}^2 + 345A_{20}^4 b_{02}^5 B_{02}^2 + 246A_{20}^2 b_{02}^6 B_{02}^2 - 396b_{02}^7 B_{02}^2 \\
& - 4A_{20}^6 b_{02}^3 B_{02}^3 + 377A_{20}^4 b_{02}^4 B_{02}^3 + 538A_{20}^2 b_{02}^5 B_{02}^3 - 924b_{02}^6 B_{02}^3 \\
& - 38A_{20}^6 b_{02}^2 B_{02}^4 + 118A_{20}^4 b_{02}^3 B_{02}^4 + 680A_{20}^2 b_{02}^4 B_{02}^4 - 1386b_{02}^5 B_{02}^4 \\
& + 4A_{20}^6 b_{02} B_{02}^5 - 66A_{20}^4 b_{02}^2 B_{02}^5 + 510A_{20}^2 b_{02}^3 B_{02}^5 - 1386b_{02}^4 B_{02}^5 + A_{20}^6 B_{02}^6 \\
& - 43A_{20}^4 b_{02} B_{02}^6 + 218A_{20}^2 b_{02}^2 B_{02}^6 - 924b_{02}^3 B_{02}^6 - 3A_{20}^4 B_{02}^7 + 46A_{20}^2 b_{02} B_{02}^7 \\
& - 396b_{02}^2 B_{02}^7 + 3A_{20}^2 B_{02}^8 - 99b_{02} B_{02}^8 - 11B_{02}^9, \\
M_6 = {} & 96A_{20}^6 b_{02}^4 + 468A_{20}^4 b_{02}^5 + 728A_{20}^2 b_{02}^6 + 360b_{02}^7 - 252A_{20}^6 b_{02}^3 B_{02} \\
& + 147A_{20}^4 b_{02}^4 B_{02} + 2280A_{20}^2 b_{02}^5 B_{02} + 2091b_{02}^6 B_{02} + 175A_{20}^6 b_{02}^2 B_{02}^2 \\
& - 862A_{20}^4 b_{02}^3 B_{02}^2 + 1867A_{20}^2 b_{02}^4 B_{02}^2 + 4986b_{02}^5 B_{02}^2 - 12A_{20}^6 b_{02} B_{02}^3 \\
& - 275A_{20}^4 b_{02}^2 B_{02}^3 - 772A_{20}^2 b_{02}^3 B_{02}^3 + 6165b_{02}^4 B_{02}^3 - 6A_{20}^6 B_{02}^4 \\
& + 284A_{20}^4 b_{02} B_{02}^4 - 1638A_{20}^2 b_{02}^2 B_{02}^4 + 4020b_{02}^3 B_{02}^4 + 18A_{20}^4 B_{02}^5 \\
& - 524A_{20}^2 b_{02} B_{02}^5 + 1125b_{02}^2 B_{02}^5 + 27A_{20}^2 B_{02}^6 - 54b_{02} B_{02}^6 - 69B_{02}^7.
\end{aligned}
$$

If $b_{02} = 0$, we have $M_3 = -(A_{20}^2 - 3B_{02})B_{02}^4(A_{20}^2 + B_{02})$ and $M_5 = B_{02}^6(A_{20}^6 - 3A_{20}^4 B_{02} + 3A_{20}^2 B_{02}^2 - 11B_{02}^3)$. It is easy to see that it requires that $B_{02} = 0$ for $M_3 = M_5 = 0$, which contradicts with that $M_1 \neq 0$. If $B_{02} = -b_{02} \neq 0$, we have $M_3 = -4A_{20}^6 b_{02}^3 \neq 0$. Otherwise, to have common roots of M_3, M_4 and M_5, we compute the following resultants to obtain

$$
\begin{aligned}
\text{Resultant}[M_3, M_4, B_{02}] &= 9216A_{20}^{30} b_{02}^7 (6A_{20}^2 + b_{02})(54A_{20}^2 + 49b_{02}) \\
&\equiv 9216A_{20}^{30} b_{02}^7\, F_1, \\
\text{Resultant}[M_3, M_5, B_{02}] &= 256A_{20}^{56} b_{02}^{18} (9216A_{20}^{16} - 868032A_{20}^{14} b_{02} \\
&\quad + 12423360A_{20}^{12} b_{02}^2 - 65473808A_{20}^{10} b_{02}^3 \\
&\quad + 131842471A_{20}^8 b_{02}^4 - 132575168A_{20}^6 b_{02}^5 \\
&\quad + 283790520A_{20}^4 b_{02}^6 + 487738800A_{20}^2 b_{02}^7 \\
&\quad + 30337200b_{02}^8) \\
&\equiv 256A_{20}^{56} b_{02}^{18}\, F_2.
\end{aligned}
$$

Further, calculating the resultant of the polynomials F_1 and F_2 with respect to b_{02} we obtain

$$
\begin{aligned}
\text{Resultant}[F_1, F_2, b_{02}] &= -29581946441167818298632627959280000 0A_{20}^{32} \\
&\neq 0,
\end{aligned}
$$

which indicates that the three polynomials M_3, M_4 and M_5 have no common roots.

(1b) If $a_{20} \neq 0$, for which we have

$$a_{11} = \frac{-A_{11}A_{20} + 2a_{20}B_{02} - 2A_{20}B_{02} + b_{02}b_{11} + B_{02}b_{11}}{a_{20}}$$

by setting $V_3(\lambda) = 0$. Then we obtain

$$V_4(\lambda) = \frac{2}{45a_{20}}\varepsilon^3\big[-2(2a_{20}+b_{11})M_6 + M_7\varepsilon^2\big],$$

where

$$\begin{aligned} M_6 = {} & A_{11}a_{20} - A_{11}A_{20} - 6a_{20}^2A_{20} + 6a_{20}A_{20}^2 + 2a_{20}b_{02} + 4a_{20}B_{02} - 2A_{20}B_{02} \\ & + 3a_{20}^2b_{11} - 3a_{20}A_{20}b_{11} + b_{02}b_{11} + B_{02}b_{11}, \\ M_7 = {} & -4A_{11}^2a_{20}A_{20} + 4A_{11}^2A_{20}^2 + 4A_{11}a_{20}A_{20}b_{02} - 6a_{20}^2b_{02}^2 + 2A_{11}a_{20}^2B_{02} \\ & - 14A_{11}a_{20}A_{20}B_{02} + 16A_{11}A_{20}^2B_{02} - 8a_{20}^2b_{02}B_{02} + 8a_{20}A_{20}b_{02}B_{02} \\ & + 2a_{20}^2B_{02}^2 - 12a_{20}A_{20}B_{02}^2 + 16A_{20}^2B_{02}^2 - 7A_{11}A_{20}b_{02}b_{11} + 3a_{20}b_{02}^2b_{11} \\ & + A_{11}a_{20}B_{02}b_{11} - 8A_{11}A_{20}B_{02}b_{11} + 10a_{20}b_{02}B_{02}b_{11} - 14A_{20}b_{02}B_{02}b_{11} \\ & + 9a_{20}B_{02}^2b_{11} - 16A_{20}B_{02}^2b_{11} + 3b_{02}^2b_{11}^2 + 7b_{02}B_{02}b_{11}^2 + 4B_{02}^2b_{11}^2. \end{aligned}$$

(1b.1) If $b_{11} = -2a_{20}$, we obtain $M_7 = -2A_{20}(A_{11} + 2B_{02})(2A_{11}a_{20} - 2A_{11}A_{20} - 9a_{20}b_{02} - 5a_{20}B_{02} - 4A_{20}B_{02})$. Taking $A_{20} = 0$ yields $M_7 = 0$ (and so $V_4(\lambda) = 0$), and $a_{11} = -2b_{02}$, giving the condition VI; or setting $A_{11} = -2B_{02}$ again yields $M_7 = 0$ and $a_{11} = -2b_{02}$, which is included in the condition III. Otherwise, we linearly solve $M_7 = 0$ for A_{11} to obtain

$$A_{11} = \frac{9a_{20}b_{02} + 5a_{20}B_{02} + 4A_{20}B_{02}}{2(a_{20} - A_{20})}.$$

Moreover, we have $A_{11} + 2B_{02} = \frac{9a_{20}(b_{02}+B_{02})}{2(a_{20}-A_{20})} \neq 0$, under which the 5th Lyapunov constant becomes

$$\begin{aligned} V_5(\lambda) = {} & \frac{9\pi a_{20}A_{20}(b_{02}+B_{02})}{512(A_{20}-a_{20})}\varepsilon^5\big[-40a_{20}^2 + 40A_{20}^2 + 30b_{02} + 30B_{02} \\ & + 3(B_{02}^2 - b_{02}^2)\varepsilon^2\big]. \end{aligned}$$

Setting $V_5(\lambda) = 0$ yields $B_{02} = b_{02} = \frac{2a_{20}^2 - 2A_{20}^2}{3} \neq 0$. Then we obtain

$$\begin{aligned} V_6(\lambda) = {} & \frac{128a_{20}A_{20}(a_{20}-A_{20})}{4725}\varepsilon^5(a_{20}+A_{20})^2\big[108(2a_{20}^2 + 3a_{20}A_{20} + 2A_{20}^2) \\ & + (a_{20}+A_{20})^2(29a_{20}^2 + 50a_{20}A_{20} + 29A_{20}^2)\varepsilon^2\big]. \end{aligned}$$

It should be noted that $2a_{20}^2 + 3a_{20}A_{20} + 2A_{20}^2 = 0$ (and similarly $29a_{20}^2 + 50a_{20}A_{20} + 29A_{20}^2 = 0$) if and only if $A_{20} = a_{20} = 0$, which contradicts with that $a_{20} \neq A_{20}$. Hence, $V_6(\lambda) \neq 0$. Note that $a_{20} + A_{20} = 0$ does not yield $2a_{20}^2 + 3a_{20}A_{20} + 2A_{20}^2 = 0$ or $29a_{20}^2 + 50a_{20}A_{20} + 29A_{20}^2 = 0$.

(1b.2) Now assume that $b_{11} \neq -2a_{20}$. Then, we have

$$b_{02} = \frac{-1}{2a_{20} + b_{11}} \big\{ A_{11}(a_{20} - A_{20}) + 4a_{20}B_{02} + (b_{11} - 2A_{20})\big[B_{02} + 3a_{20}(a_{20} - A_{20})\big]\big\}$$

by linearly solving $M_6(\lambda) = 0$ in V_4 for b_{02}.

(1b.2.i) If $b_{11} = 2A_{20}$, we obtain $b_{11} + 2a_{20} = 2(a_{20} + A_{20}) \neq 0$ and $M_7 = (a_{20} + A_{20})(A_{11} + 2B_{02})^2$. So setting $M_7 = 0$ (and so $V_4(\lambda) = 0$) yields $A_{11} = -2B_{02}$. Then we obtain $V_5(\lambda) = -\frac{\pi}{8} B_{02}(a_{20} - A_{20})(a_{20} + A_{20})^2 \varepsilon^5$. Setting $V_5(\lambda) = 0$ we obtain $B_{02} = 0$, which leads to the condition included in condition V.

(1b.2.ii) If $b_{11} \neq 2A_{20}$, then M_7 is given as $-\frac{3}{2a_{20}+b_{11}} a_{20}(a_{20} - A_{20})\widetilde{M}_7$, where

$$\begin{aligned}
\widetilde{M}_7 = {} & A_{11}^2 a_{20} + 3A_{11}^2 A_{20} - 12A_{11}a_{20}^2 A_{20} + 4A_{11}a_{20}A_{20}^2 \\
& + 36a_{20}^3 A_{20}^2 - 36a_{20}^2 A_{20}^3 + 4A_{11}a_{20}B_{02} + 12A_{11}A_{20}B_{02} \\
& - 32a_{20}^2 A_{20}B_{02} + 8a_{20}A_{20}^2 B_{02} + 4a_{20}B_{02}^2 + 12A_{20}B_{02}^2 \\
& - A_{11}^2 b_{11} + 6A_{11}a_{20}^2 b_{11} + 10A_{11}a_{20}A_{20}b_{11} - 36a_{20}^3 A_{20}b_{11} \\
& + 2A_{11}A_{20}^2 b_{11} + 36a_{20}A_{20}^3 b_{11} - 4A_{11}B_{02}b_{11} + 16a_{20}^2 B_{02}b_{11} \\
& + 12a_{20}A_{20}B_{02}b_{11} + 4A_{20}^2 B_{02}b_{11} - 4B_{02}^2 b_{11} - 6A_{11}a_{20}b_{11}^2 \\
& + 9a_{20}^3 b_{11}^2 - A_{11}A_{20}b_{11}^2 + 27a_{20}^2 A_{20}b_{11}^2 - 36a_{20}A_{20}^2 b_{11}^2 \\
& - 8a_{20}B_{02}b_{11}^2 - 4A_{20}B_{02}b_{11}^2 - 9a_{20}^2 b_{11}^3 + 9a_{20}A_{20}b_{11}^3 \\
& + B_{02}b_{11}^3.
\end{aligned}$$

Moreover, we obtain the 5th and 6th Lyapunov constants given as follows:

$$V_5(\lambda) = \frac{1}{960(2a_{20} + b_{11})^2} \varepsilon^5 (a_{20} - A_{20})\big[5(2a_{20} + b_{11})\pi M_8 - 256(2a_{20} + b_{11})^2 \widetilde{M}_7 \varepsilon + 5\pi M_9 \varepsilon^2\big],$$

and

$$\begin{aligned}
V_6(\lambda) = {} & -\frac{1}{302400(2a_{20} + b_{11})^3} \varepsilon^5 (a_{20} - A_{20}) \\
& \times \big[-12288(2a_{20} + b_{11})^2 M_{10} \\
& \quad - 735(2a_{20} + b_{11})^3 \pi M_{11}\varepsilon + 256(2a_{20} + b_{11})M_{12}\varepsilon^2
\end{aligned}$$

$$- 735(2a_{20} + b_{11})^2 \pi M_{13}\varepsilon^3 + 384M_{14}\varepsilon^4\big],$$

where

$$\begin{aligned}
M_8 = {} & 4A_{11}^2 a_{20} - 24A_{11}a_{20}^3 + 30A_{11}^2 A_{20} + 144a_{20}^4 A_{20} + 4A_{11}a_{20}A_{20}^2 \\
& + 96a_{20}^3 A_{20}^2 - 96a_{20}^2 A_{20}^3 + 16A_{11}a_{20}B_{02} - 96a_{20}^3 B_{02} \\
& + 120A_{11}A_{20}B_{02} - 248a_{20}^2 A_{20}B_{02} + 8a_{20}A_{20}^2 B_{02} \\
& + 16a_{20}B_{02}^2 + 120A_{20}B_{02}^2 - 13A_{11}^2 b_{11} + 36A_{11}a_{20}^2 b_{11} \\
& - 72a_{20}^4 b_{11} + 202A_{11}a_{20}A_{20}b_{11} - 216a_{20}^3 A_{20}b_{11} + 2A_{11}A_{20}^2 b_{11} \\
& - 420a_{20}^2 A_{20}^2 b_{11} + 516a_{20}A_{20}^3 b_{11} - 52A_{11}B_{02}b_{11} \\
& + 124a_{20}^2 B_{02}b_{11} + 156a_{20}A_{20}B_{02}b_{11} + 4A_{20}^2 B_{02}b_{11} - 52B_{02}^2 b_{11} \\
& - 78A_{11}a_{20}b_{11}^2 + 84a_{20}^3 b_{11}^2 + 11A_{11}A_{20}b_{11}^2 + 444a_{20}^2 A_{20}b_{11}^2 \\
& - 528a_{20}A_{20}^2 b_{11}^2 + 12A_{20}^3 b_{11}^2 - 68a_{20}B_{02}b_{11}^2 - 40A_{20}B_{02}b_{11}^2 \\
& - 6A_{11}b_{11}^3 - 105a_{20}^2 b_{11}^3 + 165a_{20}A_{20}b_{11}^3 - 12A_{20}^2 b_{11}^3 \\
& + 13B_{02}b_{11}^3 - 15a_{20}b_{11}^4 + 3A_{20}b_{11}^4, \\
M_9 = {} & 12A_{11}^3 a_{20}^2 - 44A_{11}^3 a_{20}A_{20} - 216A_{11}^2 a_{20}^3 A_{20} - 30A_{11}^3 A_{20}^2 \\
& + 464A_{11}^2 a_{20}^2 A_{20}^2 + 1296A_{11}a_{20}^4 A_{20}^2 - 84A_{11}^2 a_{20}A_{20}^3 \\
& - 2688A_{11}a_{20}^3 A_{20}^3 - 2592a_{20}^5 A_{20}^3 + 1392A_{11}a_{20}^2 A_{20}^4 + \cdots, \\
M_{10} = {} & A_{11}^2 a_{20} + 24A_{11}a_{20}^3 - 15A_{11}^2 A_{20} - 60A_{11}a_{20}^2 A_{20} \\
& - 144a_{20}^4 A_{20} + 16A_{11}a_{20}A_{20}^2 + 84a_{20}^3 A_{20}^2 - 84a_{20}^2 A_{20}^3 \\
& + 4A_{11}a_{20}B_{02} + 96a_{20}^3 B_{02} + \cdots, \\
M_{11} = {} & 44A_{11}^2 a_{20} - 120A_{11}a_{20}^3 + 222A_{11}^2 A_{20} - 288A_{11}a_{20}^2 A_{20} \\
& + 720a_{20}^4 A_{20} + 116A_{11}a_{20}A_{20}^2 + 1344a_{20}^3 A_{20}^2 \\
& - 1344a_{20}^2 A_{20}^3 + 176A_{11}a_{20}B_{02} - 480a_{20}^3 B_{02} + \cdots, \\
M_{12} = {} & 330A_{11}^3 a_{20}^2 + 2800A_{11}^2 a_{20}^4 - 846A_{11}^3 a_{20}A_{20} \\
& + 5916A_{11}^2 a_{20}^3 A_{20} - 33600A_{11}a_{20}^5 A_{20} \\
& - 900A_{11}^3 A_{20}^2 + 8448A_{11}^2 a_{20}^2 A_{20}^2 + 34744A_{11}a_{20}^4 A_{20}^2 \\
& + 100800a_{20}^6 A_{20}^2 + 324A_{11}^2 a_{20}A_{20}^3 + \cdots, \\
M_{13} = {} & 60A_{11}^3 a_{20}^2 - 244A_{11}^3 a_{20}A_{20} - 1080A_{11}^2 a_{20}^3 A_{20} \\
& - 222A_{11}^3 A_{20}^2 + 2608A_{11}^2 a_{20}^2 A_{20}^2 + 6480A_{11}a_{20}^4 A_{20}^2 \\
& - 516A_{11}^2 a_{20}A_{20}^3 - 14304A_{11}a_{20}^3 A_{20}^3 - 12960a_{20}^5 A_{20}^3 \\
& + 7824A_{11}a_{20}^2 A_{20}^4 + \cdots,
\end{aligned}$$

$$\begin{aligned}M_{14} = {} & 195A_{11}^4a_{20}^3 - 645A_{11}^4a_{20}^2A_{20} - 4680A_{11}^3a_{20}^4A_{20} \\ & + 901A_{11}^4a_{20}A_{20}^2 + 18064A_{11}^3a_{20}^3A_{20}^2 + 42120A_{11}^2a_{20}^5A_{20}^2 \\ & + 525A_{11}^4A_{20}^3 - 18552A_{11}^3a_{20}^2A_{20}^3 - 150504A_{11}^2a_{20}^4A_{20}^3 \\ & - 168480A_{11}a_{20}^6A_{20}^3 + \cdots .\end{aligned}$$

Assume $B_{02} = 0$. For solving $\widetilde{M}_7 = M_8 = M_9 = \cdots = M_{14} = 0$, we compute the following resultants:

$$\begin{aligned}\text{Resultant}[\widetilde{M}_7, M_8, A_{11}] &= 9(2A_{20} - b_{11})^2(2a_{20} + b_{11})^3\, G_1, \\ \text{Resultant}[\widetilde{M}_7, M_9, A_{11}] &= 27a_{20}(a_{20} - A_{20})^2A_{20}^2(a_{20} - b_{11}) \\ & \quad \times (2A_{20} - b_{11})^6(2a_{20} + b_{11})^2\, G_2, \\ \text{Resultant}[\widetilde{M}_7, M_{10}, A_{11}] &= 9(2A_{20} - b_{11})^2(2a_{20} + b_{11})^3\, G_1, \\ \text{Resultant}[\widetilde{M}_7, M_{11}, A_{11}] &= 225(2A_{20} - b_{11})^2(2a_{20} + b_{11})^3\, G_1, \\ \text{Resultant}[\widetilde{M}_7, M_{12}, A_{11}] &= 324(a_{20} - A_{20})(2A_{20} - b_{11})^4 \\ & \quad \times (2a_{20} + b_{11})^2\, G_3, \\ \text{Resultant}[\widetilde{M}_7, M_{13}, A_{11}] &= 675a_{20}(a_{20} - A_{20})^2A_{20}^2(a_{20} - b_{11}) \\ & \quad \times (2A_{20} - b_{11})^6(2a_{20} + b_{11})^2\, G_2, \\ \text{Resultant}[\widetilde{M}_7, M_{14}, A_{11}] &= 864a_{20}(a_{20} - A_{20})^3A_{20}^2(a_{20} - b_{11}) \\ & \quad \times (2A_{20} - b_{11})^8(2a_{20} + b_{11})^3\, G_4,\end{aligned}$$

where

$$\begin{aligned}G_1 = {} & 192a_{20}^6A_{20} + 808a_{20}^5A_{20}^2 + 320a_{20}^4A_{20}^3 + 1792a_{20}^3A_{20}^4 \\ & - 3280a_{20}^2A_{20}^5 + 2088a_{20}A_{20}^6 - 696a_{20}^5A_{20}b_{11} \\ & - 2212a_{20}^4A_{20}^2b_{11} - 2704a_{20}^3A_{20}^3b_{11} + 3740a_{20}^2A_{20}^4b_{11} \\ & - 4504a_{20}A_{20}^5b_{11} + 72A_{20}^6b_{11} + 2a_{20}^5b_{11}^2 + 1048a_{20}^4A_{20}b_{11}^2 \\ & + 2528a_{20}^3A_{20}^2b_{11}^2 - 356a_{20}^2A_{20}^3b_{11}^2 + 3986a_{20}A_{20}^4b_{11}^2 \\ & - 136A_{20}^5b_{11}^2 - 3a_{20}^4b_{11}^3 - 724a_{20}^3A_{20}b_{11}^3 - 815a_{20}^2A_{20}^2b_{11}^3 \\ & - 1840a_{20}A_{20}^3b_{11}^3 + 94A_{20}^4b_{11}^3 + 224a_{20}^2A_{20}b_{11}^4 \\ & + 444a_{20}A_{20}^2b_{11}^4 - 28A_{20}^3b_{11}^4 + a_{20}^2b_{11}^5 - 44a_{20}A_{20}b_{11}^5 \\ & + 3A_{20}^2b_{11}^5, \\ G_2 = {} & 2028a_{20}^5 + 3820a_{20}^4A_{20} - 5396a_{20}^3A_{20}^2 - 12a_{20}^2A_{20}^3 \\ & - 2028a_{20}^4b_{11} + 2468a_{20}^3A_{20}b_{11} + 5062a_{20}^2A_{20}^2b_{11} \\ & - 1662a_{20}A_{20}^3b_{11} - 1521a_{20}^3b_{11}^2 - 4105a_{20}^2A_{20}b_{11}^2\end{aligned}$$

$$
\begin{aligned}
&+3094a_{20}A_{20}^2b_{11}^2-18A_{20}^3b_{11}^2+1014a_{20}^2b_{11}^3\\
&-2186a_{20}A_{20}b_{11}^3+12A_{20}^2b_{11}^3+507a_{20}b_{11}^4+3A_{20}b_{11}^4,\\
G_3=\;&123904a_{20}^{10}A_{20}^2+58368a_{20}^9A_{20}^3+1092416a_{20}^8A_{20}^4\\
&+176128a_{20}^7A_{20}^5+1129856a_{20}^6A_{20}^6-373760a_{20}^5A_{20}^7\\
&-179392a_{20}^4A_{20}^8+2816a_{20}^{10}A_{20}b_{11}-160704a_{20}^9A_{20}^2b_{11}\\
&-720960a_{20}^8A_{20}^3b_{11}+\cdots,\\
G_4=\;&25392a_{20}^8+455952a_{20}^7A_{20}+719728a_{20}^6A_{20}^2\\
&-1490448a_{20}^5A_{20}^3+310748a_{20}^4A_{20}^4-10092a_{20}^3A_{20}^5\\
&-38088a_{20}^7b_{11}-744648a_{20}^6A_{20}b_{11}-200224a_{20}^5A_{20}^2b_{11}\\
&+2072980a_{20}^4A_{20}^3b_{11}+\cdots.
\end{aligned}
$$

Since $a_{20}(a_{20}-A_{20})(2A_{20}-b_{11})(2a_{20}+b_{11})\neq 0$ we discuss the following three subcases. First, considering the last two resultants to be zero we have two subcases.
(1) If $A_{20}=0$, under which $\widetilde{M}_7=(a_{20}-b_{11})(A_{11}+3a_{20}b_{11})^2$. Taking $b_{11}=a_{20}$ yields $\widetilde{M}_7=0$, and then we obtain $M_8=-9a_{20}(A_{11}+2a_{20}^2)(A_{11}+6a_{20}^2)$. If taking $A_{11}=-2a_{20}^2$, we have $M_8=M_9=M_{10}=M_{11}=0$ but $M_{12}=-36a_{20}^8\neq 0$. Similarly, we have $M_{12}\neq 0$ when $A_{11}=-6a_{20}^2$. If taking another factor in $\widetilde{M}_7$ and setting $A_{11}=-3a_{20}b_{11}$, we obtain $\widetilde{M}_7=0$, but $M_8=3a_{20}b_{11}^2(2a_{20}+b_{11})^2\neq 0$.
(2) If $b_{11}=a_{20}$, which yields the last two resultants to be zero, we obtain $\widetilde{M}_7=3A_{11}A_{20}(A_{11}-a_{20}^2+2a_{20}A_{20})$. Taking $A_{11}=a_{20}^2-2a_{20}A_{20}$ we have $\widetilde{M}_7=0$, and $M_8=-9a_{20}^2(a_{20}-2A_{20})(21a_{20}^2-65a_{20}A_{20}+30A_{20}^2)$. First, consider $a_{20}=2A_{20}$, under which we have $M_8=M_9=\cdots=M_{14}=0$, leading to $b_{02}=A_{11}=0$. These conditions belong to the condition V. Next, consider $21a_{20}^2-65a_{20}A_{20}+30A_{20}^2=0$, under which we have $M_9=-3a_{20}^3(77a_{20}-47A_{20})(a_{20}-2A_{20})^3A_{20}\neq 0$.
(3) Besides the above two special cases, in general we calculate the resultants of the polynomials G_1, G_2 and G_3 with respect to A_{20} to obtain

$$
\begin{aligned}
\text{Resultant}[G_1,G_2,A_{20}]&=12288a_{20}^2(a_{20}-b_{11})^4(2a_{20}+b_{11})^2\,G_{12},\\
\text{Resultant}[G_1,G_3,A_{20}]&=4194304a_{20}^{10}(a_{20}-b_{11})^8b_{11}^2\\
&\quad\times(2a_{20}+b_{11})^5\,G_{13},
\end{aligned}
$$

where G_{12} and G_{13} are respectively 25th- and 35th-degree polynomials in a_{20} and b_{11}. Further, we obtain $\text{Resultant}[G_{12},G_{13},b_{11}]=C_1a_{20}^{1375}\neq 0$, where C_1 is a nonzero constant. This implies that the polynomials $\widetilde{M}_7, M_8,\cdots,M_{12}$ have no common roots when $B_{02}=0$.

For $B_{02} \neq 0$, we similarly only need to consider the following resultants:

$$\begin{aligned}
\text{Resultant}[\widetilde{M}_7, M_8, A_{11}] &= 9(2a_{20}+b_{11})^3\, \widetilde{G}_1,\\
\text{Resultant}[\widetilde{M}_7, M_9, A_{11}] &= (2A_{20}-b_{11})^2(2a_{20}+b_{11})^2\, \widetilde{G}_2,\\
\text{Resultant}[\widetilde{M}_7, M_{12}, A_{11}] &= 324(2a_{20}+b_{11})^2\, \widetilde{G}_3,\\
\text{Resultant}[M_7', M_{14}, A_{11}] &= 4(2A_{20}-b_{11})^2(2a_{20}+b_{11})^3\, \widetilde{G}_4,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{G}_1 &= 768a_{20}^6A_{20}^3 + 3232a_{20}^5A_{20}^4 + 1280a_{20}^4A_{20}^5 + 7168a_{20}^3A_{20}^6 - 13120a_{20}^2A_{20}^7\\
&\quad + 8352a_{20}A_{20}^8 - 64a_{20}^6A_{20}B_{02} - 896a_{20}^5A_{20}^2B_{02} - 6112a_{20}^4A_{20}^3B_{02}\\
&\quad - 12192a_{20}^3A_{20}^4B_{02} + \cdots,\\
\widetilde{G}_2 &= 876096a_{20}^9A_{20}^6 - 101952a_{20}^8A_{20}^7 - 4755456a_{20}^7A_{20}^8 + 6307200a_{20}^6A_{20}^9\\
&\quad - 2320704a_{20}^5A_{20}^{10} - 5184a_{20}^4A_{20}^{11} - 44928a_{20}^9A_{20}^4B_{02} - 2865600a_{20}^8A_{20}^5B_{02}\\
&\quad + 2813184a_{20}^7A_{20}^6B_{02} + 6021792a_{20}^6A_{20}^7B_{02} + \cdots,\\
\widetilde{G}_3 &= 1982464a_{20}^{11}A_{20}^6 - 1048576a_{20}^{10}A_{20}^7 + 16544768a_{20}^9A_{20}^8 - 14660608a_{20}^8A_{20}^9\\
&\quad + 15259648a_{20}^7A_{20}^{10} - 24057856a_{20}^6A_{20}^{11} + 3109888a_{20}^5A_{20}^{12} + 2870272a_{20}^4A_{20}^{13}\\
&\quad - 450560a_{20}^{11}A_{20}^4B_{02} - 8114176a_{20}^{10}A_{20}^5B_{02} + \cdots,\\
\widetilde{G}_4 &= 351019008a_{20}^{13}A_{20}^8 + 5250023424a_{20}^{12}A_{20}^9 - 7906664448a_{20}^{11}A_{20}^{10}\\
&\quad - 31894290432a_{20}^{10}A_{20}^{11} + 89653118976a_{20}^9A_{20}^{12} - 84788232192a_{20}^8A_{20}^{13}\\
&\quad + 33909829632a_{20}^7A_{20}^{14} - 4714315776a_{20}^6A_{20}^{15} + 139511808a_{20}^5A_{20}^{16}\\
&\quad - 5087232a_{20}^{13}A_{20}^6B_{02} + \cdots.
\end{aligned}$$

Since $B_{02} \neq 0$, without loss of generality, we set $B_{02}=1$. Then we compute the resultants of $\widetilde{G}_1$ with $\widetilde{G}_2$, $\widetilde{G}_3$ and $\widetilde{G}_4$, respectively, with respect to A_{20} to obtain

$$\begin{aligned}
\text{Resultant}[\widetilde{G}_1, \widetilde{G}_2, A_{20}] &= 5699868278390784(2a_{20}+b_{11})^{17}\, \widetilde{G}_{121}\, \widetilde{G}_{122}\, \widetilde{G}_{123},\\
\text{Resultant}[\widetilde{G}_1, \widetilde{G}_3, A_{20}] &= -288230376151711744(2a_{20}+b_{11})^{23}\, \widetilde{G}_{131}\, \widetilde{G}_{132},\\
\text{Resultant}[\widetilde{G}_1, \widetilde{G}_4, A_{20}] &= -1494186269970473680896(2a_{20}+b_{11})^{22}\, \widetilde{G}_{141}\, \widetilde{G}_{142},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{G}_{121} &= -121a_{20}^2 + 60a_{20}^4 + 110a_{20}b_{11} - 141a_{20}^3b_{11} - 25b_{11}^2 + 123a_{20}^2b_{11}^2\\
&\quad - 51a_{20}b_{11}^3 + 9b_{11}^4,\\
\widetilde{G}_{122} &= 28961669805094733414 4a_{20}^{14} - 1073649770932154190336 0a_{20}^{16}\\
&\quad + 5493125388894302737612 8a_{20}^{18} + 8709077835725235978 24a_{20}^{13}b_{11}
\end{aligned}$$

$$
\begin{aligned}
&- 6245225035669890385920a_{20}^{15}b_{11} + \cdots ,\\
\widetilde{G}_{123} = &- 154971533932943410202093599850496a_{20}^{22}\\
&+ 2451279371191048485514748512614481 92a_{20}^{24}\\
&- 9332393517869767622873084556225596 6208a_{20}^{26}\\
&- 39431789612954077644652991112 8064a_{20}^{21}b_{11}\\
&+ 514963260106823723915161085178544128a_{20}^{23}b_{11} + \cdots ,\\
G'_{131} = &\,26440988381408194117632 0000a_{20}^{18} + 2775459686442265516223692800a_{20}^{20}\\
&- 1122685149931517476385862 4512a_{20}^{22}\\
&+ 5042989378848829685760000 00a_{20}^{17}b_{11}\\
&+ 3709268522401291247016345600a_{20}^{19}b_{11} + \cdots ,\\
\widetilde{G}_{132} = &- 150662772779102167346332468844955697152a_{20}^{24}\\
&+ 956884173013998040990691912523523817472 00a_{20}^{26}\\
&- 138413910133838366701911819492302509666467 84a_{20}^{28}\\
&- 565929774858769568739347763104241942528a_{20}^{23}b_{11}\\
&+ 223616561258050719911933750549824678133760a_{20}^{25}b_{11} + \cdots ,\\
\widetilde{G}_{141} = &- 5176594054662117051392522188339087171584 0000a_{20}^{21}\\
&+ 33565519020957556043782430498922552057102336 00a_{20}^{23}\\
&- 275856419544348938752911567049611218186653532 16a_{20}^{25}\\
&- 18727852738168040865293957291932568039424 0000a_{20}^{20}b_{11}\\
&+ 4822081570260602331816220332062131667946700800a_{20}^{22}b_{11} + \cdots ,\\
\widetilde{G}_{142} = &\,36269543629580214815625172703429864019143598916 23936a_{20}^{29}\\
&- 94790228501886865101011155277475314681278213626636 53376a_{20}^{31}\\
&+ 576747026108613239600274510501305645649823003778889 6264192a_{20}^{33}\\
&+ 1213390077778692903021859978054522523350574153859 0720a_{20}^{28}b_{11}\\
&- 395062993995819309905653079156119913515675727311206 48192a_{20}^{30}b_{11}\\
&+ \cdots .
\end{aligned}
$$

Further, we compute the resultants $\widetilde{G}_{121}$ with $\widetilde{G}_{131}$ and $\widetilde{G}_{141}$, respectively, with respect to b_{11} to obtain

$$
\begin{aligned}
\text{Resultant}[\widetilde{G}_{121}, \widetilde{G}_{131}, b_{11}] &= -5719198113740 2500a_{20}^{36}(361 - 246a_{20}^{2}\\
&\quad + 252a_{20}^{4})\,\widetilde{G}_{1213},\\
\text{Resultant}[\widetilde{G}_{121}, \widetilde{G}_{141}, b_{11}] &= C_2a_{20}^{42}(361 - 246a_{20}^{2} + 252a_{20}^{4})\,\widetilde{G}_{1214},
\end{aligned}
$$

where C_2 is a nonzero integer, $\widetilde{G}_{1213}$ and $\widetilde{G}_{1214}$ are respectively 132th- and 166th-degree polynomials in a_{20}. Since $a_{20} \neq 0$ and $361 - 246a_{20}^2 + 252a_{20}^4 \neq 0$, and moreover it can be shown that Resultant$[\widetilde{G}_{1213}, \widetilde{G}_{1214}, a_{20}] \neq 0$, implying that the polynomials $\widetilde{G}_{121}$, $\widetilde{G}_{131}$ and $\widetilde{G}_{141}$ have no common roots. Similarly, we can show that the polynomials $\widetilde{G}_{12i}$, $\widetilde{G}_{13j}$ and $\widetilde{G}_{14k}$ $(i = 1, 2, 3,\ j = 1, 2,\ k = 1, 2)$ do not have common roots. Hence, we conclude that the polynomial equations $\widetilde{M}_7 = M_8 = M_9 = M_{12} = M_{14} = 0$ have no common zeros when $a_{20} \neq 0$ and $B_{02} \neq 0$.

(2) Now we consider the second case: $A_{11} = -a_{11} - 2b_{02} - 2B_{02}$. Solving $V_3(\lambda) = 0$ for a_{11} we have

$$a_{11} = \frac{2A_{20}b_{02} + 2a_{20}B_{02} + b_{02}b_{11} + B_{02}b_{11}}{a_{20} - A_{20}}.$$

Then we obtain the 4th Lyapunov constant,

$$V_4(\lambda) = -\frac{2}{15(a_{20} - A_{20})}\,\varepsilon^3(2a_{20} + b_{11})\big[-2(a_{20} - A_{20})^2(2A_{20} - b_{11}) + \widetilde{M}_8\varepsilon^2 + 5a_{02}(b_{02} + B_{02})^2\varepsilon^4\big],$$

where

$$\widetilde{M}_8 = -2a_{02}a_{20}^2 + 4a_{02}a_{20}A_{20} - 2a_{02}A_{20}^2 + a_{20}b_{02}^2 + 3A_{20}b_{02}^2 + 2a_{20}b_{02}B_{02} + 6A_{20}b_{02}B_{02} + a_{20}B_{02}^2 + 3A_{20}B_{02}^2 - b_{02}^2b_{11} - 2b_{02}B_{02}b_{11} - B_{02}^2b_{11}.$$

To have $V_4(\lambda) = 0$, we take $b_{11} = -2a_{20}$ which yields $a_{11} = -2b_{02}$, leading to the condition III. Otherwise, if $A_{20} = \frac{b_{11}}{2}$, then the ε^3-order term in $V_4(\lambda)$ is zero. So we obtain $a_{20} - A_{20} = \frac{2a_{20} - b_{11}}{2} \neq 0$, and thus

$$a_{11} = \frac{2(2a_{20}B_{02} + 2b_{02}b_{11} + B_{02}b_{11})}{2a_{20} - b_{11}}.$$

For the ε^7-order term in $V_4(\lambda)$, let $B_{02} = -b_{02}$. Then we have $\widetilde{M}_8 = -\frac{1}{2}a_{02}(2a_{20} - b_{11})^2$. The solution for $M_8 = 0$ is $a_{02} = 0$, under which the 5th Lyapunov constant becomes

$$V_5(\lambda) = \frac{\pi}{64}\,b_{02}(2a_{20} - b_{11})(2a_{20} + b_{11})^2\,\varepsilon^5.$$

Setting $V_5(\lambda) = 0$ yields $b_{02} = 0$, leading to the condition V. Note that letting $A_{20} + a_{20} = 0$ in the condition V yields a special case of III.

Therefore, the conditions obtained above, belonging to the cases III, IV, V and VI, are the necessary conditions for the origin of the unperturbed system (30) to be a center.

Next, we prove that these conditions are also sufficient. When the condition III in (3) holds, system (2) becomes

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}y+a_{20}x^2-2b_{02}xy+a_{02}y^2\\ -x^2-2a_{20}xy+b_{02}y^2\end{pmatrix}, & \text{for } x>0,\\ \begin{pmatrix}y+A_{20}x^2-2B_{02}xy+a_{02}y^2\\ x^2-2A_{20}xy+B_{02}y^2\end{pmatrix}, & \text{for } x<0.\end{cases} \tag{37}$$

It can be shown that the first and second systems in (35) are Hamiltonian systems, with respectively the Hamiltonian quantities,

$$\begin{aligned} H^+(x,y)&=\frac{1}{2}y^2+\frac{x^3}{3}+\frac{a_{02}}{3}y^3+a_{20}x^2y-b_{02}xy^2,\\ H^-(x,y)&=\frac{1}{2}y^2-\frac{x^3}{3}+\frac{a_{02}}{3}y^3+A_{20}x^2y-B_{02}xy^2. \end{aligned} \tag{38}$$

Further, noticing that the condition $H^+(0,y)\equiv H^-(0,y)$ in Proposition 3.1 holds, which implies that the origin of system (37) is a center.

When the condition IV in (3) is satisfied, system (2) is reduced to

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}y+a_{11}xy\\ -x^2+b_{02}y^2\end{pmatrix}, & \text{for } x>0,\\ \begin{pmatrix}y+A_{20}x^2-2B_{02}xy\\ x^2-2A_{20}xy+B_{02}y^2\end{pmatrix}, & \text{for } x<0.\end{cases} \tag{39}$$

The first system in (39) has the following first integrals,

$$\begin{aligned}
H^+(x,y)&=\frac{1}{b_{02}(a_{11}-b_{02})(a_{11}-2b_{02})}(1+a_{11}x)^{-\frac{2b_{02}}{a_{11}}}(1+2b_{02}x-a_{11}b_{02}x^2\\
&\quad+2b_{02}^2x^2-a_{11}^2b_{02}y^2+3a_{11}b_{02}^2y^2-2b_{02}^3y^2), \quad \text{if } a_{11}(a_{11}-b_{02})(a_{11}-2b_{02})\neq 0,\\
H^+(x,y)&=\frac{1}{2}y^2+\frac{x^3}{3}, \quad \text{if } a_{11}=b_{02}=0,\\
H^+(x,y)&=\frac{e^{-2b_{02}x}}{2b_{02}^3}(-1-2b_{02}x-2b_{02}^2x^2+2b_{02}^3y^2), \quad \text{if } a_{11}=0,\ b_{02}\neq 0,\\
H^+(x,y)&=\frac{1}{2a_{11}^3}(-2a_{11}x+a_{11}^2x^2+a_{11}^3y^2+\ln(1+a_{11}x), \quad \text{if } a_{11}\neq 0,\ b_{02}=0,\\
H^+(x,y)&=\frac{1}{b_{02}^3(1+b_{02}x)^2}\big[3+4b_{02}x+b_{02}^3y^2+2\ln(1+b_{02}x)\\
&\quad+4b_{02}x\ln(1+b_{02}x)+2b_{02}^2x^2\ln(1+b_{02}x)\big], \quad \text{if } a_{11}=b_{02}\neq 0,\\
H^+(x,y)&=\frac{1}{4b_{02}^3(1+b_{02}x)^2}\big[-1+2b_{02}x+4b_{02}^2x^2+4b_{02}^3y^2\\
&\quad-2\ln(1+2b_{02}x)-4b_{02}x\ln(1+2b_{02}x)\big], \quad \text{if } a_{11}=2b_{02}\neq 0.
\end{aligned} \tag{40}$$

The second system in (39) has the first integral,

$$H^-(x,y) = -\frac{2}{3}x^3 + 2A_{20}x^2y + 3y^2 - 3B_{02}xy^2.$$

It is easy to verify that $H^+(0,y)$ and $H^-(0,y)$ are even functions in y, which implies that the origin of system (39) is a center by Proposition 3.1.

If the condition V in (3) holds, system (2) has the form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{20}x^2 \\ -x^2 + 2A_{20}xy \end{pmatrix}, & \text{for } x > 0, \\ \begin{pmatrix} y + A_{20}x^2 \\ x^2 + 2a_{20}xy \end{pmatrix}, & \text{for } x < 0. \end{cases} \tag{41}$$

This case is hard to prove because the approaches used above are not applicable for this case. It is noted that system (41) is invariant under the transformation $(x, t, a_{20}, A_{20}) \to (-x, -t, A_{20}, a_{20})$, but we cannot apply the time-invertible system theory here to prove that the origin of the system is a center since A_{20} and a_{20} have been exchanged. Hence, we inspect the property in the Lyapunov constants, and by using systems (30) and (31), we obtain the perturbed system of (41) under the transformation $(x, y, t) \to (\varepsilon^2 y, \varepsilon^3 x, \frac{t}{\varepsilon})$ as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + 2A_{20}\varepsilon xy - y^2 \\ x + a_{20}\varepsilon y^2 \end{pmatrix}, & \text{for } y > 0, \\ \begin{pmatrix} -y + 2a_{20}\varepsilon xy + y^2 \\ x + A_{20}\varepsilon y^2 \end{pmatrix}, & \text{for } y < 0. \end{cases} \tag{42}$$

It is easy to see that the perturbed system (42) is also invariant under the transformation $(y, t, a_{20}, A_{20}) \to (-y, -t, A_{20}, a_{20})$. For convenience and clear, let the Lyapunov constants for the first and second systems be $v_k^+(a_{20}, A_{20})$ and $u_k^+(a_{20}, A_{20})$, respectively. Then, $u_k^+(A_{20}, a_{20}) = v_k^+(a_{20}, A_{20})$, and so $u_k^+(a_{20}, A_{20}) = v_k^+(A_{20}, a_{20})$. Moreover, we have found that $v_k^+(a_{20}, A_{20})$ is invariant if exchanging a_{20} and A_{20}, i.e., $v_k^+(a_{20}, A_{20}) = v_k^+(A_{20}, a_{20})$ at least up to $k = 10$. This implies that

$$u_k^+(a_{20}, A_{20}) = v_k^+(A_{20}, a_{20}) = v_k^+(a_{20}, A_{20}),$$

and so $V_k(\lambda) = v_k^+(a_{20}, A_{20}) - u_k^+(a_{20}, A_{20}) = 0$ at least for $k = 2, 3, \ldots, 10$. If we can show that this is true for any $k \geqslant 2$, then we can conclude that the origin of the switching system (41) is a center since the origin of both the first and second systems is a cusp. However, this is still very hard to prove. Thus, we use simulations for the right-half plane system ($x > 0$) of (41) to demonstrate that this is true. In Fig. 4(a), we show two simulated trajectories starting from the same initial point $(x, y) = (0.0.0025)$ (on the positive y-axis), one is for $(a_{20}, A_{20}) = (1, 5)$, which converges to the stable equilibrium point $(x, y) = (-0.1, -0.01)$ (see the red curve), and the other is for exchanging a_{20} and A_{20} with $(a_{20}, A_{20}) = (5, 1)$, which converges to the stable equilibrium point $(x, y) = (-0.1, -0.05)$ (see the blue curve). Note that these two curves intersect at the same point on the negative y-axis, implying that our above conjecture is true, i.e., v_k^+ is invariant under the exchange of a_{20} and A_{20}. Then, we reverse the blue trajectory in Fig. 4(a) about the y-axis (which thus becomes a solution of the left-half plane system of (41)) to form a closed orbit with the red trajectory, as shown in Fig. 4(b). In order to further numerically confirm that the origin of system (41) is a center, we use an iterative integral scheme to simulate the switching system (41) with time step 0.0001 and 3×10^{11} iterations. The results for $(a_{20}, A_{20}) = (1, 5)$ are depicted in Fig. 4(c), where the red orbit is obtained after 5×10^6 iterations while the blue orbit is obtained after 3×10^{11} iterations. It is clear that these two closed orbits are coincide, indicating that the origin of the system is a center. Similarly, the results for $(a_{20}, A_{20}) = (1, -5)$ are shown in Fig. 4(d), again indicating that the origin of the system is a center.

The simulated phase portraits of system (41) for $(a_{20}, A_{20}) = (1, 5)$ and $(1, -5)$ are shown in Figs. 5(a) and 5(b), respectively. The above simulation results indicate that the origin of system (41) is a center, which is though not a rigorous mathematical proof.

When the condition VI in (3) holds, system (2) becomes

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases}\begin{pmatrix}y + a_{20}x^2 - 2b_{02}xy\\ -x^2 - 2a_{20}xy + b_{02}y^2\end{pmatrix}, & \text{for } x>0,\\ \begin{pmatrix}y + A_{11}xy\\ x^2 + B_{02}y^2\end{pmatrix}, & \text{for } x<0.\end{cases} \tag{43}$$

Similarly, we can find the first integral for the first system in (35), given by

$$H^+(x,y) = \frac{2}{3}x^3 + 2a_{20}x^2y + 3y^2 - 3b_{02}xy^2, \tag{44}$$

and the first integrals for the second system in (43) as given below:

$$\begin{aligned}
H^-(x,y) = {} & \frac{1}{B_{02}(A_{11}-B_{02})(A_{11}-2B_{02})}(1+A_{11}x)^{-\frac{2B_{02}}{A_{11}}}(1+2B_{02}x - A_{11}B_{02}x^2\\
& +2B_{02}^2x^2 + A_{11}^2B_{02}y^2 - 3A_{11}B_{02}^2y^2 + 2B_{02}^3y^2),\\
& \qquad\qquad \text{if } A_{11}B_{02}(A_{11}-B_{02})(a_{11}-2B_{02}) \neq 0,\\
H^-(x,y) = {} & \frac{1}{2}y^2 - \frac{x^3}{3}, \qquad \text{if } A_{11}=B_{02}=0,\\
H^-(x,y) = {} & \frac{e^{-2B_{02}x}}{2B_{02}^3}(1+2B_{02}x + 2B_{02}^2x^2 + 2B_{02}^3y^2), \qquad \text{if } A_{11}=0,\ B_{02}\neq 0,\\
H^-(x,y) = {} & \frac{1}{2A_{11}^3}(2A_{11}x - A_{11}^2x^2 + A_{11}^3y^2 - \ln(1+A_{11}x), \qquad \text{if } A_{11}\neq 0,\ B_{02}=0,\\
H^-(x,y) = {} & \frac{1}{B_{02}^3(1+B_{02}x)^2}\big[-3 - 4B_{02}x + B_{02}^3y^2 - 2\ln(1+B_{02}x)\\
& -4B_{02}x\ln(1+B_{02}x) - 2B_{02}^2x^2\ln(1+B_{02}x)\big], \qquad \text{if } A_{11}=B_{02}\neq 0,\\
H^-(x,y) = {} & \frac{1}{4B_{02}^3(1+B_{02}x)^2}\big[-1 - 2B_{02}x - 4B_{02}^2x^2 + 4B_{02}^3y^2\\
& -2\ln(1+2B_{02}x) + 4B_{02}x\ln(1+2B_{02}x)\big], \qquad \text{if } A_{11}=2B_{02}\neq 0.
\end{aligned} \tag{45}$$

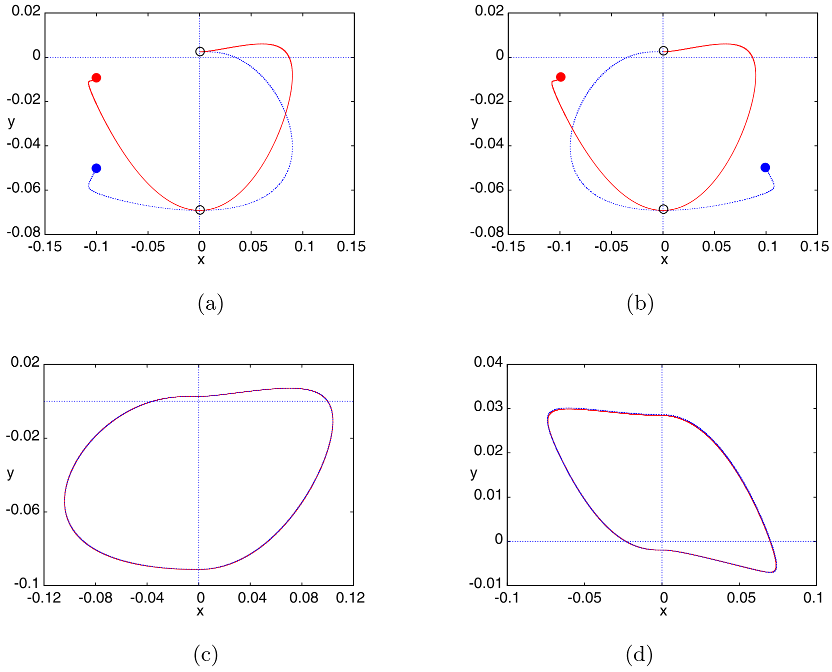

Fig. 4. Simulations of system (41) demonstrating that the origin of system (41) is a center: (a) two trajectories from simulating the right-half plane system of (41), with the red one for $(a_{20}, A_{20}) = (1, 5)$ converging to the equilibrium point $(x, y) = (-0.1, -0.01)$, and the blue one for $(a_{20}, A_{20}) = (5, 1)$ converging to the equilibrium point $(x, y) = (-0.1, -0.05)$, which start from a same initial point on the positive y-axis and end at a same point on the negative y-axis; (b) the blue trajectory in part (a) is reversed about the y-axis to form a closed orbit with the red trajectory; (c) two simulated closed orbits of system (41) for $(a_{20}, A_{20}) = (1, 5)$, using time step 0.0001 and 3×10^{11} iterations, coincide with the red one obtained after 5×10^6 iterations and the blue one obtained after 3×10^{11} iterations; and (d) two simulated closed orbits of system (41) for $(a_{20}, A_{20}) = (1, -5)$, using time step 0.0001 and 3×10^{11} iterations, coincide with the red one obtained after 5×10^6 iterations and the blue one obtained after 3×10^{11} iterations. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

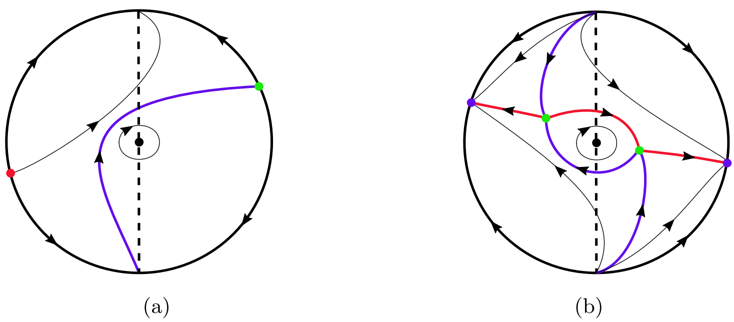

Fig. 5. The phase portraits of system (41): (a) for $(a_{20}, A_{20}) = (1, 5)$; and (b) for $(a_{20}, A_{20}) = (1, -5)$.

It is seen that $H^+(0, y)$ and $H^-(0, y)$ are even functions in y, satisfying the conditions in Proposition 3.1, and so the origin of system (43) is a center. □

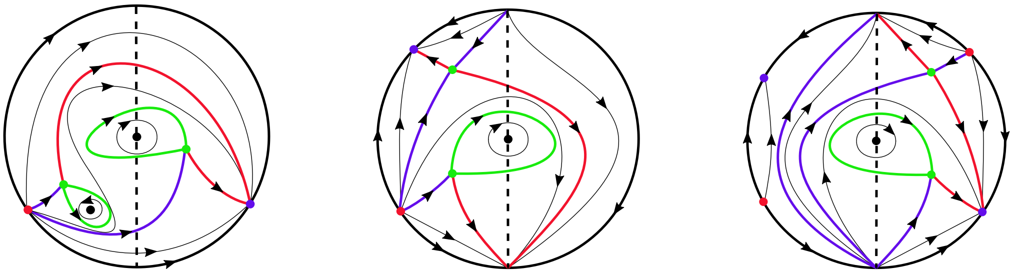

Fig. 6. The phase portraits for (a) the system (37) with $a_{20} = a_{02} = b_{02} = B_{02} = 1$ and $A_{20} = 1.5$; (b) the system (39) with $a_{11} = b_{02} = A_{20} = 1$ and $B_{02} = -1$; and (c) the system (43) with $a_{20} = b_{02} = A_{11} = 1$ and $B_{02} = 0$.

Example 4.6. For illustration, the phase portraits for systems (37), (39) and (43) are shown in Figs. 6(a), (b) and (c), respectively.

5. The proof of Theorem 1.2

In this section, we will perturb system (2) with the center condition III in (3) to prove the existence of small-amplitude limit cycles around the origin $(0, 0)$. It will be shown that the existence of 7 limit cycles needs perturbations up to ε^{12} order.

Proof. We add quadratic perturbations to system (37) to obtain the following perturbed system (noticing that $b_{20} = -1,\ B_{20} = 1$):

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + a_{20}x^2 - 2b_{02}xy + a_{02}y^2 \\ \quad + \sum_{k=1}^{12} \varepsilon^k (\varepsilon\,\delta_k x + p_{20k}x^2 + p_{11k}xy + p_{02k}y^2), \\ -\varepsilon^2 x - x^2 - 2a_{20}xy + b_{02}y^2 \\ \quad + \sum_{k=1}^{12} \varepsilon^k (\varepsilon\,\delta_k y + q_{20k}x^2 + q_{11k}xy + q_{02k}y^2), \end{pmatrix} & \text{for } x > 0, \\ \begin{pmatrix} y + A_{20}x^2 - 2B_{02}xy + a_{02}y^2 \\ \quad + \sum_{k=1}^{12} \varepsilon^k (\varepsilon\,\delta_k x + P_{20k}x^2 + P_{11k}xy + P_{02k}y^2), \\ -\varepsilon^2 x + x^2 - 2A_{20}xy + B_{02}y^2 \\ \quad + \sum_{k=1}^{12} \varepsilon^k (\varepsilon\,\delta_k y + Q_{20k}x^2 + Q_{11k}xy + Q_{02k}y^2), \end{pmatrix} & \text{for } x < 0, \end{cases} \tag{46}$$

where $0 < \varepsilon \ll 1$, and δ_k, p_{ijk}, q_{ijk}, P_{ijk} and Q_{ijk} $(i + j = 2)$ are real perturbation parameters. It will be seen in the following coefficients reduction procedure that the perturbing coefficients q_{20k}, q_{11k}, q_{02k} and Q_{20k}, Q_{11k}, Q_{02k} can be set zero. Note that when we compute the Lyapunov constants of the above system we set the linear perturbation terms zero, i.e. $\delta_k = 0$. The Lyapunov constants are calculated according to the ε^k-order, starting from $k = 1$. We use the ε-order Lyapunov constants to determine the number of limit cycles, and in addition perturbing the linear coefficient δ_1 to get one more small-amplitude limit cycle. However, if we think we may obtain more limit cycles from ε^2-order terms, then we need to find the conditions on parameters such that all the ε-order Lyapunov constants vanish, in other words, the origin of the system is a center up to ε-order, and the conditions are called the center conditions for that ε-order [46]. This process can go further until we reach a value of k such that no center conditions can be found and so maximal number of limit cycles may be obtained.

First, system (46) contains many perturbation parameters, which yields difficulty in calculating the Lyapunov constants and solving the polynomial equations resulted from the Lyapunov constants. In order to reduce the number of parameters, we follow the approach described in [46] to make the following near-identity transformations:

$$\begin{aligned} & x \to x + d_1(\varepsilon)x + d_2(\varepsilon)y, \\ & y \to y + d_3(\varepsilon)x + d_4(\varepsilon)y, \\ & t \to t + d_5(\varepsilon)t, \end{aligned} \tag{47}$$

for the upper system, where

$$d_i(\varepsilon) = d_{i1}\varepsilon + d_{i2}\varepsilon^2 + \cdots + d_{in}\varepsilon^n, \quad i = 1, 2, \ldots, 5. \tag{48}$$

A similar transformation can be formed for the lower system. Note that $(47)|_{\varepsilon=0}$ is an identity map, and thus keeps the unperturbed system of (46) unchanged. Moreover, the new system under the transformation (47) can be still written in the same form of (46). Therefore, we may find proper $d_i(\varepsilon)$'s to simplify the perturbations without loss of generality. To illustrate the procedure, we show how to simplify the ε-order terms, that is, consider $n = 1$. We only consider the upper system, and the same process can be applied to the lower system. Using (47) and substituting it into system (46) and taking the ε-order terms, we obtain

$$\begin{aligned} \dot{x} =& d_{31}\, x + (-d_{11} + d_{41} + d_{51})\, y + \left[d_{21} + a_{20}(d_{11} + d_{51}) - 2b_{02}d_{31} + p_{201}\right] x^2 \\ & + \left[2a_{20}d_{21} + a_{02}d_{31} - b_{02}(d_{41} + d_{51}) + \tfrac{1}{2}p_{111}\right] x\, y \\ & + \left[a_{02}(-d_{11} + 2d_{41} + d_{51}) - 3b_{02}d_{21} + p_{021})\right] y^2, \\ \dot{y} =& - d_{31}\, y - (2d_{11} + 3a_{20}d_{31} - d_{41} + d_{51} - q_{201})\, x^2 \\ & + \left[a_{20}(d_{11} - d_{51}) + d_{21} - 2b_{02}d_{31} - \tfrac{1}{2}q_{111}\right] x\, y \\ & - \left[2a_{20}d_{21} + a_{02}d_{31} - b_{02}(d_{41} + d_{51}) - q_{021}\right] y^2. \end{aligned} \tag{49}$$

Now, simplify setting

$$d_{11} = -\frac{a_{20}^2 q_{201} - a_{20}q_{111} + q_{021}}{(a_{20}^2 + b_{02})},$$

$$
\begin{aligned}
d_{21} &= -\frac{a_{20}b_{02}q_{201} - b_{02}q_{111} - a_{20}q_{021}}{2(a_{20}^2 + b_{02})}, \\
d_{31} &= 0, \\
d_{41} &= -\frac{(4a_{20}^2 + b_{02})q_{201} - 3a_{20}q_{111} + 3q_{021}}{2(a_{20}^2 + b_{02})}, \\
d_{51} &= \frac{(2a_{20}^2 + b_{02})q_{201} - a_{20}q_{111} + q_{021}}{2(a_{20}^2 + b_{02})},
\end{aligned}
$$

eliminates the term $d_{31}x$ in the $\dot{x}$ equation and all the terms in the $\dot{y}$ equation. Thus, we may assume that the perturbations are only needed for the first equations of both upper and lower systems. This implies that the perturbing terms $\sum_{k=1}^{12} \varepsilon^k (q_{20k}x^2 + q_{11k}xy + q_{02k}y^2)$ and $\sum_{k=1}^{12} \varepsilon^k (Q_{20k}x^2 + Q_{11k}xy + Q_{02k}y^2)$ in (46) are redundant, and can be removed. So in the following we assume $q_{20k} = q_{11k} = q_{02k} = Q_{20k} = Q_{11k} = Q_{02k} = 0$.

Further, similarly introducing the scaling $(x, y, t) \to \left(\varepsilon^2 y, \varepsilon^3 x, \frac{t}{\varepsilon}\right)$ into system (46), we finally obtain the following system up to ε^{12}-order terms,

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases}
\begin{pmatrix}
-y - y^2 - 2\varepsilon a_{20}xy + \varepsilon^2 b_{02}x^2 + \sum_{k=1}^{12} \varepsilon^k \delta_k x, \\
x + \varepsilon a_{20}y^2 + \varepsilon^2(-2b_{02}xy + p_{201}y^2) + \varepsilon^3(a_{02}x^2 + p_{111}xy + p_{202}y^2) \\
\quad + \sum_{k=1}^{12} \varepsilon^k \delta_k y + \sum_{k=4}^{12} \varepsilon^k (p_{02(k-3)}x^2 + p_{11(k-2)}xy + p_{20(k-1)}y^2),
\end{pmatrix}, \\
\text{for } y > 0, \\
\begin{pmatrix}
-y + y^2 - 2\varepsilon A_{20}xy + \varepsilon^2 B_{02}x^2 + \sum_{k=1}^{12} \varepsilon^k \delta_k x, \\
x + \varepsilon A_{20}y^2 + \varepsilon^2(-2B_{02}xy + P_{201}y^2) + \varepsilon^3(a_{02}x^2 + P_{111}xy + P_{202}y^2) \\
\quad + \sum_{k=1}^{12} \varepsilon^k \delta_k y + \sum_{k=4}^{12} \varepsilon^k (P_{02(k-3)}x^2 + P_{11(k-2)}xy + p_{20(k-1)}y^2),
\end{pmatrix}, \\
\text{for } y < 0.
\end{cases} \tag{50}
$$

To prove the existence of small-amplitude limit cycles around the origin, we compute the Lyapunov constants V_{jk} for $j = 1, 2, \ldots, 8$ and $k = 1, 2, \ldots, 12$, where k indicates that the Lyapunov constants correspond to the ε^k terms. Then, based on V_{jk} we solve the polynomial equations to determine the number of limit cycles. We start from $k = 1$. For a fixed k, we choose appropriate parameter values such that as many as Lyapunov constants vanish. As a matter of fact, this solution procedure is similar to that in the Hopf bifurcation analysis near a Bogdanov-Takens critical point, but here we deal with a more degenerate (or generalized) Hopf bifurcation.

First, for $k = 0$, it is easy to use Proposition 3.2 (but note that it needs to exchange the variables x and y since now the switch line of system (50) is the y-axis) to show that the origin of system (50) is a center when $\varepsilon = 0$. Also note that $V_{1k} = 0$ when $\delta_k = 0$ for any k.

Next, consider $k = 1$. We obtain that all Lyapunov constants vanish when $\delta_1 = 0$, i.e., $V_{j1} = 0$, $j = 1, 2, \ldots$. Moreover, through the computation of the Lyapunov constants up to ε^{12} order, we observe that the following parameters are not used and can be set zero. (Of course, if we continue to go to higher-order computations, these parameters should not be set zero.)

$$\begin{aligned} &p_{20k} = 0,\ k = 3, 4, \ldots, 11, \qquad &&p_{02k} = 0,\ k = 5, 6, \ldots, 9, \\ &p_{11k} = 0,\ k = 8, 9, 10, &&P_{119} = P_{1110} = 0. \end{aligned} \tag{51}$$

Now, for $k = 2$, $V_{12} = 0$ with $\delta_2 = 0$. Then, $V_{22} = \frac{4}{3}(p_{201} - P_{201})$. Setting

$$P_{201} = p_{201}$$

yields $V_{j2} = 0$ for all j. So, for $k = 2$, we can obtain 1 small-amplitude limit by perturbing δ_2 with $V_{22} \neq 0$.

For $k = 3$, again we obtain

$$P_{202} = p_{202}$$

by solving $V_{23} = 0$, leading to $V_{j3} = 0$ for all j, which implies that for $k = 3$, again 1 small-amplitude limit can be obtained by perturbing δ_3 with $V_{23} \neq 0$.

For $k = 4$, solving $V_{24} = 0$ we have

$$P_{203} = \frac{1}{2}(p_{021} - P_{021}),$$

and letting $V_{34} = 0$ yields

$$P_{021} = p_{021} + 2(B_{02} + b_{02})p_{201} - (A_{20}P_{111} + a_{20}p_{111}).$$

Then, we obtain

$$V_{44} = -\frac{16}{15}\, p_{201}(A_{20}^2 - a_{20}^2).$$

Setting $V_{44} = 0$ has three choices. Choosing $p_{201} = 0$ will result in less number of limit cycles. Hence, we choose

$$A_{20} = a_{20},$$

under which $V_{j4} = 0$ for all j, which indicates that for $k = 4$ we can obtain 3 small-amplitude limit cycles if we choose $V_{44} \neq 0$ and perturb P_{021}, P_{203} and δ_4.

For $k = 5$, we similarly obtain

$$P_{204} = \frac{1}{2}(p_{022} - P_{022})$$

by solving $V_{25} = 0$, and

$$P_{022} = p_{022} - (P_{111} + p_{111})p_{201} + 2(B_{02} + b_{02})p_{202} - a_{20}(P_{112} + p_{112})$$

by solving $V_{35} = 0$. Then V_{45} becomes $V_{45} = -\frac{8}{45} p_{201}(P_{111} + p_{111})$. For the same reason that setting $p_{201} = 0$ gives less number of limit cycles, we choose

$$P_{111} = -p_{111},$$

under which $V_{j5} = 0$ for all j. This implies that 3 small-amplitude limit cycles can be obtained for $k = 4$ if choosing $V_{44} \neq 0$ and perturbing P_{022}, P_{204} and δ_5.

For $k = 6$, similarly by solving $V_{26} = V_{36} = 0$ we obtain

$$P_{205} = \frac{1}{2}(p_{023} - P_{023}),$$
$$P_{023} = p_{023} - \big[2B_{02}(B_{02} + b_{02}) + (P_{112} + p_{112})\big]p_{201} - a_{20}(P_{113} + p_{113}).$$

Then, we solve $V_{46} = 0$ for p_{111} to obtain

$$p_{111} = -\frac{p_{201}}{9a_{20}(B_{02} + b_{02})}\big\{2(P_{112} + p_{112}) + 3(B_{02} + b_{02})\big[3(B_{02} - b_{02}) - 4a_{20}^2\big]\big\},$$

under the assumption (combined with the above assumption $p_{201} \neq 0$):

$$a_{20}(B_{02}^2 - b_{02}^2)\, p_{201} \neq 0, \tag{52}$$

in which the condition $B_{02} - b_{02} \neq 0$ will be needed later. Then we have

$$V_{56} = \frac{5\pi p_{201}}{96}\big[P_{112} + p_{112} + 12(B_{02} + b_{02})a_{20}^2\big],$$

which yields

$$P_{112} = -p_{112} - 12a_{20}^2(B_{02} + b_{02})$$

by setting $V_{56} = 0$. With the above solutions, $V_{j6} = 0$ for all j. Therefore, for $k = 6$, we can obtain 4 small-amplitude limit cycles for choosing $V_{56} \neq 0$ and perturbing p_{111}, P_{023}, P_{025} and δ_6.

For $k = 7$, under the condition in (52), we solve $V_{27} = 0$ for P_{206}, $V_{37} = 0$ for P_{024}, $V_{47} = 0$ for p_{112}, and $V_{57} = 0$ for P_{113} to obtain

$$P_{206} = \frac{1}{2}(p_{024} - P_{024}),$$
$$\begin{aligned} P_{024} = p_{024} &- p_{201}(P_{113} + p_{113}) - a_{20}(P_{114} + p_{114}) \\ &+ 2(B_{02} + b_{02})\big[(6a_{20}^2 - B_{02})p_{202} + 2a_{20}p_{201}^2 + 6a_{20}^2(a_{02} - a_{20}B_{02})\big] \\ &- \frac{1}{a_{20}}(B_{02}^2 - b_{02}^2)p_{201}^2, \end{aligned}$$
$$p_{112} = -\frac{1}{3a_{20}^2}\big[3a_{20}(B_{02} - b_{02} - 4a_{20}^2)\, p_{202} - 2(B_{02} - b_{02} - 2a_{20}^2)\, p_{201}^2$$

$$+12a_{20}^3(3B_{02}a_{20}-2a_{20}^3-a_{02})\big]-\frac{2}{9a_{20}(B_{02}+b_{02})}(P_{113}+p_{113})\,p_{201},$$

$$P_{113}=-p_{113}-\frac{3}{5a_{20}}(B_{02}+b_{02})(B_{02}-b_{02}+10a_{20}^2)\,p_{201},$$

under which $V_{j7}=0$ for all j. This indicates that for $k=7$, we can also get 4 small-amplitude limit cycles for choosing $V_{57}\neq 0$ and perturbing p_{112}, P_{024}, P_{026} and δ_7.

For $k=8$, with the condition in (52), we similarly solve $V_{28}=0$ for P_{207}, $V_{38}=0$ for P_{025}, $V_{48}=0$ for p_{113}, and $V_{58}=0$ for P_{114} to obtain

$$P_{207}=-\frac{1}{2}P_{025},$$

$$\begin{aligned}P_{025}=&-p_{201}(P_{114}+p_{114})-a_{20}(P_{115}+p_{115})+\frac{1}{5a_{20}^2}(B_{02}+b_{02})\big\{4(B_{02}-b_{02})p_{201}^3\\&+a_{20}\big[10a_{20}^3(16a_{20}^2+9b_{02})+B_{02}a_{20}(7B_{02}+3b_{02}-120a_{20}^2)\\&+a_{02}\big(50a_{20}^2+3(B_{02}-b_{02})\big)\big]p_{201}+60a_{20}^4p_{021}+7a_{20}(10a_{20}^2-B_{02}+b_{02})p_{201}p_{202}\big\},\end{aligned}$$

$$\begin{aligned}p_{113}=&-\frac{2}{9a_{20}(B_{02}+b_{02})}(P_{114}+p_{114})p_{201}\\&+\frac{1}{90a_{20}^3}\big\{360a_{20}^4p_{021}+12a_{20}\big[11(B_{02}-b_{02})-10a_{20}^2\big]p_{201}p_{202}-48(B_{02}-b_{02})p_{201}^3\\&+a_{20}\big[360a_{20}^5-12(32B_{02}-17b_{02})a_{20}^3+18B_{02}a_{20}(2B_{02}+3b_{02})\\&-300a_{02}a_{20}^2+93(B_{02}-b_{02})a_{02}\big]p_{201}\big\},\end{aligned}$$

$$\begin{aligned}P_{114}=&-p_{114}+\frac{3}{25a_{20}^2}(B_{02}+b_{02})\big[10a_{20}^3(38a_{20}^3+2a_{02}-5p_{202})\\&+25(B_{02}^2-4B_{02}b_{02}+b_{02}^2)a_{20}^2-(B_{02}-b_{02})(250a_{20}^4+25a_{02}a_{20}+5a_{20}p_{202}-4p_{201}^2)\big].\end{aligned}$$

Then, solving $V_{68}=0$ yields

$$a_{02}=\frac{a_{20}}{B_{02}-b_{02}}\big[(B_{02}-b_{02})^2-2(B_{02}-b_{02})(a_{20}^2+b_{02})-2b_{02}^2\big],$$

under which $V_{j8}=0$ for all j. Thus, for $k=8$, we can obtain 5 small-amplitude limit cycles for choosing $V_{58}\neq 0$ and perturbing P_{114}, p_{113}, P_{025}, P_{027} and δ_8.

For $k=9$, we can similarly obtain 5 small-amplitude limit cycles by solving $V_{29}=0$ (with P_{208}), $V_{39}=0$ (with P_{026}), $V_{49}=0$ (with p_{114}), and $V_{59}=0$ (with P_{115}), yielding the following solutions (with $\delta_9=0$):

$$P_{208}=-\frac{1}{2}P_{026},$$

$$\begin{aligned}P_{026}=&-p_{201}(P_{115}+p_{115})-a_{20}(P_{116}+p_{116})+\frac{2(B_{02}+b_{02})}{5a_{20}}(25a_{20}^2-B_{02}+b_{02})p_{202}^2\\&+\frac{48(B_{02}^2-b_{02}^2)}{25a_{20}^2}p_{201}^2p_{202}+12(B_{02}+b_{02})a_{20}^2p_{022}\end{aligned}$$

$$
\begin{aligned}
&+\frac{(B_{02}+b_{02})}{5a_{20}}\big[3(B_{02}-b_{02})+50a_{20}^2\big]p_{021}p_{201}\\
&+\frac{B_{02}+b_{02}}{5(B_{02}-b_{02})}\big[10(B_{02}-b_{02})^3+(B_{02}-b_{02})^2(32a_{20}^2+11b_{02})\\
&-2(B_{02}-b_{02})(72a_{20}^4+53a_{20}^2b_{02}-2b_{02}^2)-76a_{20}^2b_{02}^2\big]p_{202}-\frac{16(B_{02}^2-b_{02}^2)}{25a_{20}^3}\,p_{201}^4\\
&+\frac{(B_{02}+b_{02})}{75a_{20}(B_{02}-b_{02})}\big[235(B_{02}-b_{02})^3-(B_{02}-b_{02})^2(1118a_{20}^2-253b_{02})\\
&+4(B_{02}-b_{02})(705a_{20}^4+70a_{20}^2b_{02}-2b_{02}^2)+580a_{20}^2b_{02}^2\big]p_{201}^2\\
&+\frac{12a_{20}^3(B_{02}+b_{02})}{5(B_{02}-b_{02})}\big[2(B_{02}-b_{02})^2(a_{20}^2-4b_{02})+(B_{02}-b_{02})(34a_{20}^4\\
&+67a_{20}^2b_{02}+13b_{02}^2)+2b_{02}^2(5a_{20}^2+6b_{02})\big]\\
&-\frac{48}{5(B_{02}-b_{02})^2}a_{20}^3(B_{02}+b_{02})B_{02}^3b_{02},
\end{aligned}
$$

$$
\begin{aligned}
p_{114}=\;&4a_{20}p_{022}+\frac{4(B_{02}-b_{02})}{5a_{20}^2}\,p_{202}^2+\frac{1}{30a_{20}^2}\big[31(B_{02}-b_{02})-100a_{20}^2\big]p_{201}p_{021}\\
&-\frac{128(B_{02}-b_{02})}{75a_{20}^3}\,p_{201}^2p_{202}+\frac{1}{30a_{20}(B_{02}-b_{02})}\big[43(B_{02}-b_{02})^3\\
&-2(B_{02}-b_{02})^2(73a_{20}^2+10b_{02})+4(B_{02}-b_{02})(12a_{20}^4+43a_{20}^2b_{02}-8b_{02}^2)\\
&+232a_{20}^2b_{02}^2\big]p_{202}-\frac{100}{450a_{20}(B_{02}+b_{02})}(P_{115}+p_{115})p_{201}\\
&+\frac{32(B_{02}-b_{02})}{75a_{20}^4}\,p_{201}^4\\
&-\frac{1}{225a_{20}^2(B_{02}-b_{02})}\big[128(B_{02}-b_{02})^3+5(B_{02}-b_{02})^2(208a_{20}^2-155b_{02})\\
&-4(B_{02}-b_{02})(240a_{20}^4-440a_{20}^2b_{02}+133b_{02}^2)+560a_{20}^2b_{02}^2\big]p_{201}^2\\
&-\frac{2a_{20}^2}{15(B_{02}-b_{02})^2}\big[88(B_{02}-b_{02})^4-4(B_{02}-b_{02})^3(111a_{20}^2-26b_{02})\\
&+(B_{02}-b_{02})^2(180a_{20}^4-570a_{20}^2b_{02}+147b_{02}^2)\\
&-4(B_{02}-b_{02})b_{02}^2(21a_{20}^2-71b_{02})+124b_{02}^4\big],
\end{aligned}
$$

$$
\begin{aligned}
P_{115}=\;&-p_{115}+\frac{3(B_{02}+b_{02})}{5a_{20}}(4a_{20}^2-5(B_{02}-b_{02}))p_{021}+\frac{24(B_{02}^2-b_{02}^2)}{25a_{20}^2}\,p_{201}p_{202}\\
&-\frac{48(B_{02}^2-b_{02}^2)}{125a_{20}^3}\,p_{201}^3-\frac{(B_{02}+b_{02})}{50a_{20}(B_{02}-b_{02})}\big[139(B_{02}-b_{02})^3\\
&-2(B_{02}-b_{02})^2(48a_{20}^2-271b_{02})-4(B_{02}-b_{02})(297a_{20}^4+163a_{20}^2b_{02}-68b_{02}^2)
\end{aligned}
$$

$$- 112a_{20}^2 b_{02}^2\big] p_{201} - \frac{12}{(B_{02}-b_{02})^2\, p_{201}} a_{20}^3 (B_{02}+b_{02})^3 (2B_{02}-b_{02})(B_{02}-2b_{02}).$$

Then, solving $V_{69}=0$ for p_{021} we obtain

$$\begin{aligned} p_{021} = &-\frac{2}{45(B_{02}-b_{02})}\big[26(B_{02}-b_{02})^2 - (B_{02}-b_{02})(44a_{20}^2 - 81b_{02}) + 36b_{02}^2\big] p_{201} \\ &- \frac{4}{(B_{02}-b_{02})^3\, p_{201}} (b_{02}+B_{02})^2 a_{20}^4 (2B_{02}-b_{02})(B_{02}-2b_{02}), \end{aligned}$$

under which $V_{j9}=0$ for all j.

For $k=10$, we obtain 6 small-amplitude limit cycles. We solve $V_{210}=0$ to obtain $P_{209} = -\frac{1}{2}P_{027}$, and $V_{310}=0$ for P_{027}, $V_{410}=0$ for p_{115}, $V_{510}=0$ for P_{116}, and $V_{610}=0$ for p_{022}. These lengthy parameter solutions are not listed for brevity. Then V_{710} and V_{810} are given by

$$V_{710} = \frac{245\pi(B_{02}+b_{02})}{9216}\, p_{201}\, G_{7810}, \quad V_{810} = -\frac{64(B_{02}+b_{02})}{243}\, p_{201}\, G_{7810}, \tag{53}$$

where

$$G_{7810} = 16(B_{02}-b_{02})\Big[a_{20}^4 + \frac{5B_{02}b_{02}}{2(B_{02}-b_{02})}\, a_{20}^2 + \frac{5}{16}\, (B_{02}+b_{02})^2\Big]. \tag{54}$$

Thus, with the condition in (52), $V_{710}\neq 0$ if $G_{7810}\neq 0$, leading to 6 small-amplitude limit cycles. To get possible more limit cycles from higher ε order Lyapunov constants, we set $G_{7810}=0$ which yields $V_{j10}=0$ for all j.

To solve $G_{7810}=0$, we note that the term in the square bracket is a quadratic polynomial in a_{20}^2. To have positive solutions for a_{20}^2, it needs $B_{02}b_{02}(B_{02}-b_{02})<0$ and its discriminate must be non-negative, for which we have

$$a_{20}^2 = \frac{-5B_{02}b_{02}}{4(B_{02}-b_{02})}\left[1 \pm \sqrt{1 - \frac{(B_{02}^2-b_{02}^2)^2}{5B_{02}^2 b_{02}^2}}\,\right],$$

showing that a_{20}^2 can have one or two positive solutions provided that

$$\frac{B_{02}b_{02}}{B_{02}-b_{02}} < 0, \quad 1 - \frac{(B_{02}^2-b_{02}^2)^2}{5B_{02}^2 b_{02}^2} \geqslant 0 \quad \Longrightarrow \quad \begin{cases} 0 < B_{02} < b_{02}, \\ \dfrac{3+\sqrt{5}}{2} b_{02} \leqslant B_{02} < b_{02} < 0, \\ 0 < \dfrac{-3+\sqrt{5}}{2} b_{02} \leqslant B_{02} \leqslant \dfrac{-3-\sqrt{5}}{2} b_{02}. \end{cases} \tag{55}$$

For $k=11$, we solve $V_{211}=0$ to obtain $P_{2010} = -\frac{1}{2}P_{028}$. Then, using $V_{j11},\ j=3,4,5,6$ to solve for P_{028}, p_{116}, P_{117} and p_{023}, respectively. Again we do not list these lengthy parameter solutions here. With these solutions, we obtain

$$V_{711} = \frac{\pi(B_{02}+b_{02})}{82944a_{20}(B_{02}-b_{02})^2}\, G_{7811}, \quad V_{811} = -\frac{64(B_{02}+b_{02})}{535815a_{20}(B_{02}-b_{02})^2}\, G_{7811}, \tag{56}$$

where

$$
\begin{aligned}
G_{7811} &= 2205\, G_{7810}\, a_{20}(B_{02}-b_{02})^2 p_{202} - 2(B_{02}-b_{02})^2\, G_1\, p_{201}^2 + 5040 a_{20}^6\, G_2, \\
G_1 &= -31604(B_{02}-b_{02})a_{20}^4 + 8(683B_{02}^2 a_{20}^2 - 12112B_{02}b_{02} + 683b_{02}^2)a_{20}^2 \\
&\quad + (B_{02}-b_{02})(3253B_{02}^2 + 7282B_{02}b_{02} + 3253b_{02}^2), \\
G_2 &= -36(B_{02}-b_{02})^2 a_{20}^4 + 42(B_{02}-b_{02})(B_{02}^2 - 4B_{02}b_{02} + b_{02}^2)\, a_{20}^2 \\
&\quad 35(B_{02}^4 + 2B_{02}^3 b_{02} - 10B_{02}^2 b_{02}^2 + 2B_{02}b_{02}^3 + b_{02}^4).
\end{aligned}
\tag{57}
$$

Thus, for $k=11$, we also obtain 6 small-amplitude limit cycles if $G_{7811}\neq 0$.

Finally, we continue to consider $k=12$, for which we need $G_{7810}=G_{7811}=0$ so that $V_{j10}=V_{j11}=0$ for all j. To achieve this, we may use a_{20}^2 to solve $G_{7810}=0$, and p_{201}^2 to solve $G_{7811}=0$ provided that $G_1G_2>0$. Before solving there two equations, we consider the Lyapunov constants $V_{j12},\ j=2,3,\ldots,8$. Similarly, we solve $V_{212}=0$ to obtain $P_{2011}=-\frac{1}{2}P_{029}$. Then we use P_{029} to solve for $V_{312}=0$, p_{117} for $V_{412}=0$, P_{118} for $V_{512}=0$, p_{024} for $V_{612}=0$, and p_{202} for $V_{712}=0$, respectively. Then, we obtain

$$
\begin{aligned}
V_{812} = {}& -\frac{(B_{02}+b_{02})p_{201}}{1451520(B_{02}-b_{02})}\Big\{-231525\pi(B_{02}-b_{02})\,G_{7810}\,p_{201} + 16384\big[108(B_{02}-b_{02})^5 \\
& + (B_{02}-b_{02})^4(40a_{20}^2 + 459b_{02}) + (B_{02}-b_{02})^3(430a_{20}^2 b_{02} + 567b_{02}^2) \\
& + 2(B_{02}-b_{02})^2(280a_{20}^6 + 700a_{20}^4 b_{02} + 755a_{20}^2 b_{02}^2 + 108b_{02}^3) \\
& + 4(B_{02}-b_{02})(350a_{20}^4 b_{02}^2 + 540a_{20}^2 b_{02}^3 + 27b_{02}^4) + 1080a_{20}^2 b_{02}^4\big]\Big\}.
\end{aligned}
\tag{58}
$$

Summarizing the above results, we can conclude that 7 small-amplitude limit cycles can be obtained from the ε^{12}-order terms as long as we can find solutions such that $G_{7810}=G_{7811}=0$ but $V_{812}\neq 0$. There exist an infinite number of solutions, one of which is given below:

$$
b_{02} = -\frac{1}{20},
$$

$$
B_{02} = -\frac{1}{10} \implies \begin{cases} a_{20} &= \frac{1}{4}\sqrt{2-\frac{1}{5}\sqrt{55}} = 0.17971510\cdots, \\ p_{201} &= \frac{\sqrt{42}(10-\sqrt{55})}{48}\sqrt{\frac{286065628-36326077\sqrt{55}}{3869989561}} = 0.02289181\cdots, \end{cases}
$$

under which $G_{7810}=G_{7811}=0$, yielding

$$
\begin{aligned}
V_{j10} &= V_{j11} = 0 \quad \text{for all } \ j, \\
V_{812} &= -\frac{9(42-5\sqrt{55})}{866877661664000000}\sqrt{812697807810(16859771408-2270370899\sqrt{55})} \\
&= -0.21716684\cdots\times 10^{-6}\neq 0.
\end{aligned}
$$

The other parameter values are computed based on the obtained formulas. Denote these parameter values as a critical point C. Moreover, a direct computation shows that

Table 5.1
Bifurcation of limit cycles in system (2) under the center condition III.

ε^k-order Lyapunov Constants	1	2	3	4	5	6	7	8	9	10	11	12
Number of Limit Cycles	0	1	1	3	3	4	4	5	5	6	6	7

$$\begin{aligned}&\det\left[\frac{\partial(V_{212},V_{312},V_{412},V_{512},V_{612},V_{712})}{\partial(P_{2011},P_{029},p_{117},P_{118},p_{024},p_{202})}\right]_C\\&=\frac{\pi(-563381+8452\sqrt{55})}{158544691200000000\sqrt{2-\frac{55}{5}}}\left[\frac{35(16859771408-2270370899\sqrt{55})}{23219937366}\right]^{\frac{3}{2}}\\&=-0.83664488\cdots\times10^{-12}\neq0,\end{aligned}$$

which implies that 6 small-amplitude limit cycles can be obtained by perturbing the parameters, in backward, p_{202}, p_{024}, P_{118}, p_{117}, P_{029} and P_{2011}. Finally, perturbing δ_{12} to get one more small-amplitude limit cycle, giving a total 7 small-amplitude limit cycles.

Note that the parameter values of b_{02} and B_{02} can be easily adjusted to change the values of V_{812} and the above determinant.

This completes the proof for Theorem 1.2. □

Remark 5.1.

(i) It is seen from the above proof for Theorem 1.2 that the number of small-amplitude limit cycles bifurcating in the quadratic system (2) under the center condition III, corresponding to the ε^k-order Lyapunov constants, are obtained as listed in Table 5.1.
A natural question arises: Is 7 the maximal number of limit cycles which can be obtained for this case? Moreover, using other center conditions I, II, IV, V and VI, can more limit cycles be obtained? These questions are left for future study.

(ii) It should be noted from the above proof for Theorem 1.2 that for each set of ε^k-order Lyapunov constants V_{jk}, $2\leqslant k\leqslant 11$, we verify $V_{jk}=0$ for an enough large number j and then assume that $V_{jk}=0$ for all $j\geqslant 1$. The proof for Theorem 1.2 is different from that for Theorem 1.1. In proving Theorem 1.1, we first obtain all possible necessary conditions by eliminating the ε^k-order terms in the Lyapunov constants as many as possible, and then use other approaches to prove the sufficiency. Thus, when we derive the necessary conditions, there is no need to worry about if they are sufficient, that is, if the Lyapunov constants are truly vanish up to infinite order, which certainly cannot be verified by computation. To prove the existence of limit cycles using the ε^{k+1}-order Lyapunov constants, theoretically we need to show that all the Lyapunov constants vanish up to ε^k-order, namely, the origin of the system is a center up to ε^k order. But this is extremely difficult or impossible to give such a proof especially if the order is high.
In our proof for Theorem 1.2, based on system (46) which is a perturbed quadratic system of (37) (noticing that (37) is the original quadratic system (2) under the center condition III), we use the ε^{12}-order terms to prove the existence of 7 small-amplitude limit cycles. That is, theoretically we assume that all the Lyapunov coefficients vanish up to ε^{11} order, and thus the ideal normal form for the analysis of limit cycle bifurcation can be written as

$$d(r) = \varepsilon^{12} r \left[V_{112} + V_{212} r + V_{312} r^2 + \cdots + V_{712} r^6 + V_{812} r^7 + O(r^8)\right] + \varepsilon^{13} h(r, \varepsilon), \quad (59)$$

with $V_{112} = e^{\varepsilon\pi\delta_{12}} - e^{-\varepsilon\pi\delta_{12}} \approx 2\varepsilon\pi\delta_{12}$. All the coefficients $V_{j12},\ j = 2, 3, \ldots, 8$ and the tailed terms are functions of $p_{202},\ p_{024},\ P_{118},\ p_{117},\ P_{029}$ and P_{201}. According to Theorem 2.4 in [10] (p. 108), the tailed term $\varepsilon^{13} h(r, \varepsilon)$ can be ignored for $\varepsilon \ll r$.
However, without assuming that all the Lyapunov coefficients vanish up to ε^{11} order, it is better to write the $d(r)$ as follows:

$$d(r) = r\left[V_1(\varepsilon) + V_2(\varepsilon) r + V_3(\varepsilon) r^2 + \cdots + V_7(\varepsilon) r^6 + V_8(\varepsilon) r^7 + O(r^8)\right], \quad (60)$$

where V_k are named by the kth-order Lyapunov constant and $V_k(\varepsilon) = \sum_{j=12}^{\infty} V_{kj} \varepsilon^j$. For any given ε, when perturbing the parameters from the critical pint C, it is easy to obtain 7 zeros for r under the condition that r is small enough, which correspond to 7 small-amplitude bifurcating limit cycles. A concrete example has been constructed to verify this conclusion.

6. Conclusion

In this paper, we have discussed the center problem and bifurcation of limit cycles for planar switching systems with a nilpotent equilibrium point. We have developed a perturbation approach so that the Poincaré-Lyapunov method can be used to compute the Lyapunov constants of switching nilpotent systems. Using this new method, we have obtained the center conditions for a class of quadratic switching systems with a nilpotent equilibrium point. Moreover, we have constructed a perturbed system with one of the center conditions to show the existence of seven small-amplitude limit cycles around the origin of the system. This is a new result on the lower bound of maximal number of limit cycles for such quadratic switching systems with a nilpotent equilibrium point. Future works in this direction may focus on the following two unsolved problems in this paper: (i) Is 7 the maximal number of small-amplitude limit cycles which can bifurcate in system (2) under the center condition (III)? (ii) Using all center conditions (I)-(VI), what is the maximal number of limit cycles that the system (2) can have under appropriate perturbations?

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